WRIGHT STATE UNIVERSITY

Cross sectional Shape Selection

- **Materials have properties**
  - **Strength, stiffness, electrical conductivity, etc.**
- **A component or structure is a material made into a particular shape**
- **Different shapes are more or less efficient for carrying a particular type of loading**
- **An efficient shape is one that uses the least amount of material for a given strength or stiffness**

More info: "**Materials Selection in Mechanical Design**", **Chapters 11 and 12**

Mechanical loading and associated components

- Axial Loading
  - Tension – ties or tie rods

  - Compression – columns

- Bending – beams

- Torsion – shafts

- Each type of loading has a different failure mode, and some shapes are more efficient than others for that loading

Ties or Tie rods

- **Tensile axial loading**

$$\delta = \frac{FL}{AE}$$

$$S = \frac{F}{\delta} = \frac{AE}{L}$$

$$\sigma = \frac{F}{A}$$

- **The stiffness of a tie rod for a given material depends only on the cross sectional area A and not the shape**
- **The strength of a tie rod depends only on the cross sectional area A and not the shape**
- **Therefore in tensile loading all shapes of the same cross-sectional area are equivalent**

Elastic Bending

- **Appendix A-3 gives the deflection of beams as a function of the type of loading. Generally**

$$\delta = \frac{FL^3}{C_1 EI}$$

$$or$$

$$\delta = \frac{ML^2}{C_1 EI}$$

- **The stiffness of a beam S is defined as the ratio of load to displacement**

$$S = \frac{F}{\delta}$$

$$or$$

$$S = \frac{M}{\delta}$$

- **Using either definition, S is proportional to EI**
  - **E = elastic modulus of the material**
  - **I = moment of inertia of the cross section**

Elastic Bending

- I = Moment of inertia of the cross section

$$I = \int_{Cross-section} y^2 dA$$

- Table 11.2 gives the section properties of different shapes
- For a circular cross section

$$A = \pi r^2$$

$$I_O = \frac{\pi r^4}{4} = \frac{A^2}{4\pi}$$

- If S is the stiffness for another shape with the same cross sectional area made of the same material and subject to the same loading, then the shape factor for elastic bending is defined as

$$\phi_B^e = \frac{S}{S_O} = \frac{C_1 EI}{C_1 EI_O} = 4\pi \frac{I}{A^2}$$

Elastic Bending

- Derive shape factor for elastic bending of
  - Square cross-section of side a
  - Hollow tube of radius r and thickness t where $r >> t$
- For a square cross section

$$\phi_B^e = \frac{S_{sq}}{S_O} = \frac{EI_{sq}}{EI_O} = \frac{4\pi}{12} = 1.05$$

- For a hollow tube

$$\phi_B^e = \frac{S}{S_O} = \frac{EI}{EI_O} = 4\pi \frac{I}{A^2} = \frac{r}{t}$$

Elastic Bending - Square cross-section beam

- For a square cross-section of side a

$$A = a^2$$

$$I_{sq} = \frac{a^4}{12} = \frac{A^2}{12}$$

- Compare with a circle with the same area A

$$I_{sq} = \frac{4\pi}{12} I_O$$

- Shape factor during elastic bending of a square cross-section relative to a circular cross section of the same area is:

$$\phi_B^e = \frac{S_{sq}}{S_O} = \frac{I_{sq}}{I_O} = \frac{\left(A^2/12\right)}{\left(A^2/4\pi\right)} = \frac{4\pi}{12} = 1.05$$

- Therefore, a square cross-section is about 5% stiffer than a circular cross-section

Elastic Bending – Tubular beam

- For a tubular beam with radius r and wall thickness t where $r >> t$

$$A = 2\pi r t$$

$$I_{tube} = \pi r^3 t$$

- Shape factor during elastic bending of a tubular beam relative to a circular cross-section of the same area is:

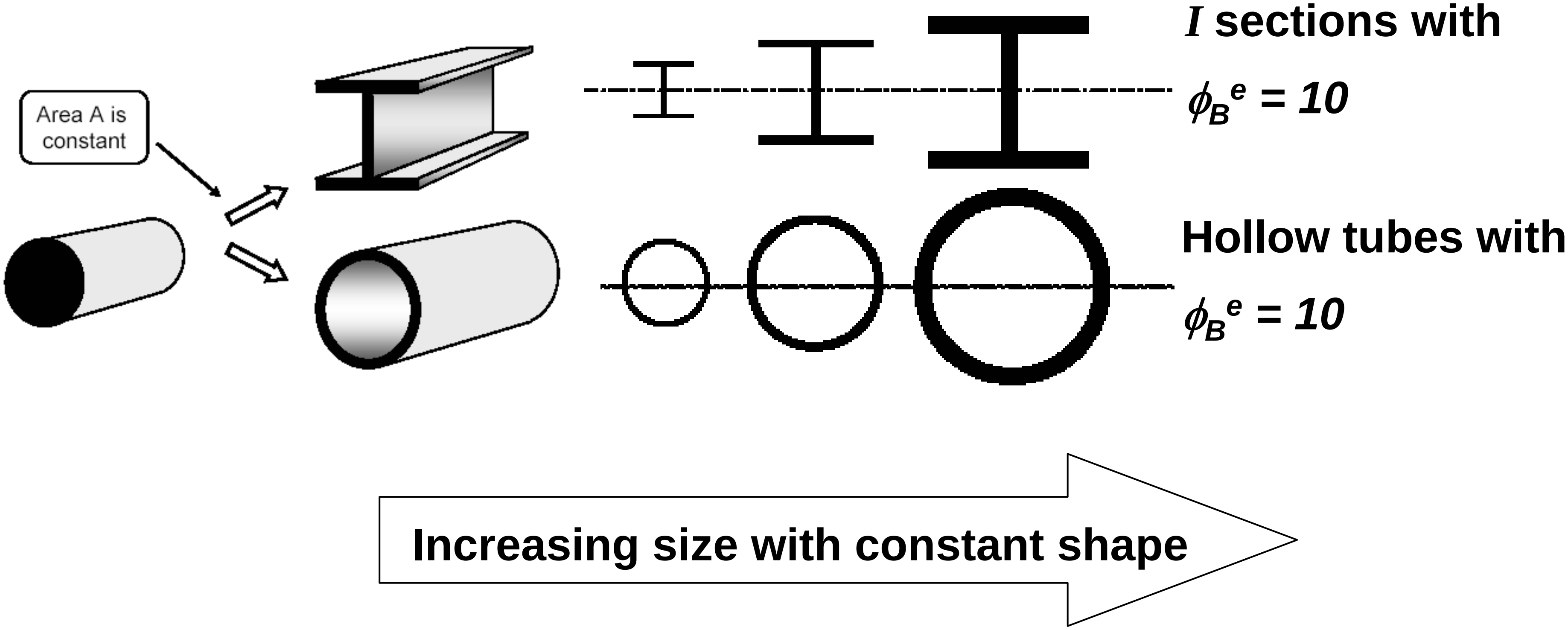
$$\phi_B^e = \frac{S_{tube}}{S_O} = \frac{I_{tube}}{I_O} = \frac{I_{tube}}{\left(A^2 / 4\pi\right)} = 4\pi \frac{\pi r^3 t}{(2\pi r t)^2} = \frac{r}{t}$$

- Therefore, a thin walled tubular beam with $r = 10t$ is 10 times as stiff as a circular cross-section beam of the same area

Please note that the derivations here assume a circle as the reference shape. The text book assumes the reference shape to be a square.

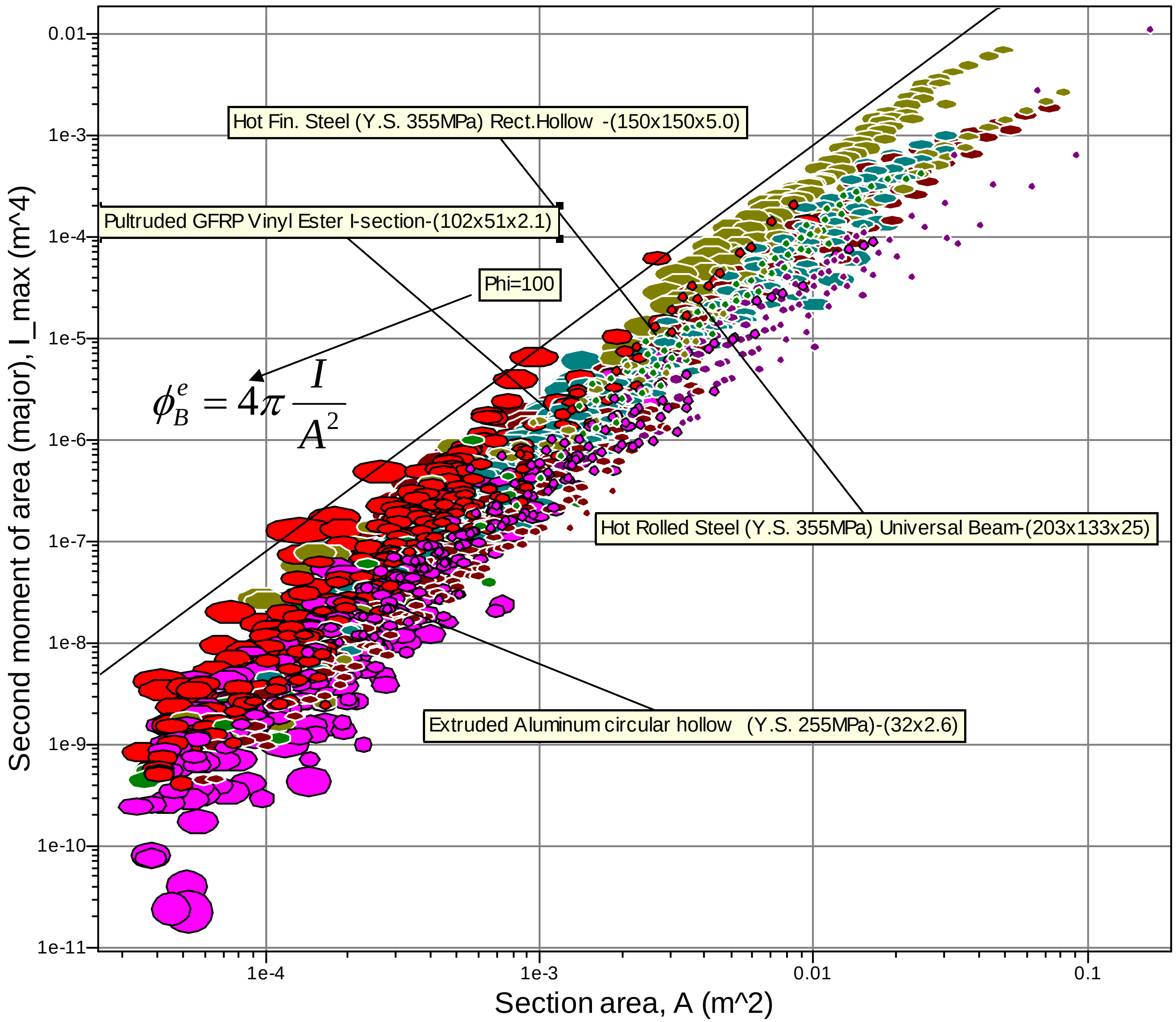
Elastic Bending

- The shape factor ϕ_B^e is dimensionless, i.e. it is a pure number that characterizes the cross-sectional shape relative to a circular cross-section



Elastic Bending

- EduPack Level 3 includes most of the commercially available structural shapes made from different materials.
- Using a plot of the moment of inertia versus section area one can compare different structural shapes



Failure in Bending

- Failure in bending can be defined as the initiation of plastic deformation in the beam.
- The stress on the top and bottom surfaces of a symmetric beam is given by

$$\sigma = \frac{Mc}{I} = \frac{M}{Z}$$

where

$$Z = \frac{I}{c}$$

- Where c is the distance of the top or bottom surface from the neutral surface
- At yield,

$$\sigma = \sigma_f = \text{yield stress}$$

Failure in Bending

- **For a circular cross-section**

$$I = \frac{\pi}{4} r^4$$

$$c = r$$

$$Therefore$$

$$Z_O = \frac{\pi r^3}{4} = \frac{A^{3/2}}{4\sqrt{\pi}}$$

Failure in Bending

- Define the shape factor for failure in bending as

$$\phi_B^f = \frac{Z}{Z_O}$$

- Derive
  - For a square of side a

$$\phi_B^f = \frac{Z_{sq}}{Z_O} = \frac{2\sqrt{\pi}}{3} = 1.18$$

  - For a hollow cylinder of radius r and wall thickness t, where $r >> t$

$$\phi_B^f = \frac{Z}{Z_O} = \sqrt{8}\sqrt{\frac{r}{t}}$$

Failure in Bending – Square cross section beam

- For a square of side a

$$A = a^2$$

$$I = \frac{a^4}{12}$$

$$Z_{sq} = \frac{I}{(a/2)} = \frac{a^3}{6} = \frac{A^{(3/2)}}{6}$$

- The shape factor for a square cross section is

$$\phi_B^f = \frac{Z_{sq}}{Z_O} = \frac{\left(A^{(3/2)}/6\right)}{\left(A^{(3/2)}/4\sqrt{\pi}\right)} = \frac{2\sqrt{\pi}}{3} = 1.18$$

- The square cross section almost 20% stronger than a circular cross-section

Failure in Bending – Tubular beam

- For a tubular beam with radius r and wall thickness t where $r >> t$

$$A = 2\pi r t$$

$$I = \pi r^3 t$$

$$Z_{tube} = \frac{I}{r} = \pi r^2 t$$

- The shape factor for a tubular beam is

$$\phi_B^f = \frac{Z_{tube}}{Z_O} = \frac{\pi r^2 t}{\left(A^{(3/2)} / 4\sqrt{\pi}\right)} = 4\sqrt{\pi}\,\frac{\pi r^2 t}{(2\pi r t)^{(3/2)}} = \sqrt{8}\sqrt{\frac{r}{t}}$$

- The tubular beam with $r = 10t$ has a shape factor of 8.9, i.e., the tubular beam is almost 9 times as strong as a circular cross-section beam

Elastic Torsion

- During elastic torsion, the angle of twist per unit length is $\theta = \frac{T}{JG}$
- Where T is the torque, J is the polar moment of inertia, and G is the shear modulus of the material.

$$A = \pi r^2$$

$$J = \frac{\pi}{2} r^4 = \frac{A^2}{2\pi}$$

- The stiffness of a solid circular shaft in torsion S_T is defined as the ratio of load to angle of twist per unit length

$$S_{T_O} = \frac{T}{\theta} = J_O G$$

- The shape factor for a different cross section is defined as

$$\phi_T^e = \frac{S_T}{S_{T_O}} = \frac{GJ}{GJ_O} = \frac{J}{\left(A^2 / 2\pi\right)}$$

Elastic Torsion

- For a hollow shaft with radius r and wall thickness t where $r >> t$

$$A = 2\pi rt$$
$$J \approx 2\pi r^3 t$$

- Shape factor for elastic torsion is

$$\phi_T^e = \frac{2\pi r^3 t}{\left(A^2 / 2\pi\right)} = \frac{4\pi^2 r^3 t}{4\pi^2 r^2 t^2} = \frac{r}{t}$$

- Therefore, a thin walled shaft with $r = 10t$ is 10 times as stiff as a circular cross-section shaft of the same area

Failure by plastic deformation during Torsion

- The shear stress at the surface of a cylindrical shaft subject to a torque T is

$$\tau = \frac{Tr}{J_O} = \frac{T}{(J_O / r)} = \frac{T}{Q_O}$$

where

$$J_O = \frac{\pi}{2} r^4$$

and

$$Q_O = \frac{\pi}{2} r^3 = \frac{A^{3/2}}{2\sqrt{\pi}}$$

- Failure occurs when the stress reaches the shear yield stress, or one-half of the tensile yield stress

$$\tau_f = \frac{\sigma_f}{2} = \frac{T}{Q_O}$$

- The shape factor for a shaft of a different cross-section can be defined as

$$\phi_T^f = \frac{Q}{Q_O} = 2\sqrt{\pi} \frac{Q}{A^{3/2}}$$

Shape factor for failure in Torsion of a Hollow Shaft

- For a hollow shaft with radius r and wall thickness t where $r >> t$

$$A = 2\pi r t$$

$$J \approx 2\pi r^3 t$$

$$Q = \frac{J}{r} = 2\pi r^2 t$$

- Shape factor is

$$\phi_T^f = 2\sqrt{\pi}\,\frac{Q}{A^{3/2}} = 2\sqrt{\pi}\,\frac{2\pi r^2 t}{(2\pi r t)^{3/2}} = \sqrt{\frac{2r}{t}}$$

- Therefore, a thin walled shaft with $r = 10t$ is 4.5 times as strong as a circular cross-section shaft of the same area

Homework Assignment

- **Show that the shape factor of elastic buckling is the same as that for elastic bending**

Empirical upper limits for the different shape factors

- The limits to the different shape factors derived above based on manufacturing considerations, as well as competing failure mechanisms is given in Table 11.4

Material	$\left(\phi_B^e\right)_{max}$	$\left(\phi_T^e\right)_{max}$	$\left(\phi_B^f\right)_{max}$	$\left(\phi_T^f\right)_{max}$
Structural Steel	65	25	13	7
AA 6061	44	31	10	8
GFRP and CFRP	39	26	9	7
Polymers (nylon)	12	8	5	4
Wood (solid section)	5	1	3	1
Elastomers	<6	3		

Co-selecting shape and material for stiff beams

- Suppose it is desired to make a beam with a stiffness of S_B and length L with a minimum mass.
- This problem can be translated as

Function	Beam
Objective	Minimize mass $m = \rho LA$ is
Constraints	•Length L is specified •Stiffness S_B is specified
Free variables:	Material Size and Shape of cross section

Co-selecting shape and material for stiff beams

- The stiffness in bending is given by

$$S_B = C_1 \frac{EI}{L^3}$$

where C_1 depends upon exactly how the load is distributed

- If we replace the moment of inertia I by

$$I = \phi_B^E I_O = \frac{\phi_B^E A^2}{4\pi}$$

$$since$$

$$I_O = \frac{A^2}{4\pi}$$

- Then

$$S_B = \frac{C_1}{4\pi} \frac{E}{L^3} \phi_B^e A^2$$

Co-selecting shape and material for stiff beams

- Eliminating A from the equation for mass m we get

$$m = \left(\frac{4\pi S_B}{C_1} \right)^{1/2} L^{5/2} \left[\frac{\rho}{\left(\phi_B^e E \right)^{1/2}} \right]$$

- The material index to be maximized is therefore

$$M = \frac{\left(\phi_B^e E \right)^{1/2}}{\rho} = \frac{\left(E / \phi_B^e \right)^{1/2}}{\left(\rho / \phi_B^e \right)} = \frac{\left(E^* \right)^{1/2}}{\rho^*}$$

- So if we want to co-select both shape and material for a stiff beam, the basis for comparison is the material index M, above.
- Graphically, we can co-select materials and shape by assuming new material properties of E^* and ρ^*

Co-selecting shape and material for other loading

- By a similar analysis, for elastic torsion, the material index is

$$M = \frac{\left(\phi_T^e E\right)^{1/2}}{\rho}$$

- For failure in bending it is

$$M = \frac{\left(\phi_B^f E\right)^{2/2}}{\rho}$$

- And for failure in torsion it is

$$M = \frac{\left(\phi_T^f E\right)^{3/2}}{\rho}$$

Example: The wing-spar of a human powered plane

- See example 12.1 in the book
- The human powered plane is basically a large model airplane capable of flying under the power of a human being
- The design requirement is that the weight (or mass) of the plane be minimized

Function	Wing spar
Objective	Minimize mass
Constraints	•Length L is specified •Stiffness S_B is specified
Free variables:	Material Size and Shape of cross section

Example: The wing-spar of a human powered plane

Material	Modulus E (GPa)	Density ρ (Mg/m^3)	Material Index ($E^{1/2}/\rho$)	Shape factor ϕ_B^e	Modified Material Index ($(E\phi_B^e)^{1/2}/\rho$)
Balsa Wood	4.2 – 5.2	0.17 – 0.24	**10**	2	15
Spruce	9.8 – 11.9	0.36 – 0.44	8	2	12
Steel	200 – 210	7.82 – 7.84	1.8	25	9
AA-7075-T6	71 – 73	2.8 – 2.82	3	20	14
CFRP	100 – 160	1.5 – 1.6	7	10	**23**

- Without taking shape into account, Balsa wood appears to have the best properties, and aluminum has a relatively poor performance
- If we take shape into account, using typical values of the shape factor for beams of different materials, CFRP is best, while AA-7075 performs as well as the woods
- Early planes were made of balsa, but later designs used aluminum or CFRP (if cost was not an issue).