Derivation of Formulas for the Area or Volume of Various Geometric Shapes

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Introduction

It is the purpose of this paper to investigate the origins of the formulas that yield the area or volume of sundry geometric shapes. Included is a discussion of a fundamental conception of area and volume that will serve as the springboard for the methods of derivation employed throughout. The following are the actual area and volume formulas that it shall presently be endeavored to derive. The formula for the area A of a square of length l and height h is A = lh. The area A of a circle of radius r is $A = \pi r^2$. The formula for the volume V of a right circular cone of height h and radius r is $V = (\pi h r^2)/3$.[α]

The Nature of Area and the Area of a Square

It is helpful to begin this investigation with a discussion of the concept of area. The area of a geometric shape that lies on a two-dimensional plane may be conceived of as the sum of all of the infinitely thin, one-dimensional line segments, parallel to each other, which may be drawn across it.

This idea may be easily illustrated with a square (Figure 1). Imagine that all of the small, horizontal line segments in Figure 1 are infinitely thin. The top of the square shows but a few of the unlimited number of one-dimensional line segments that may be drawn across. However, the line segments become more densely concentrated farther down. The solid black portion at the square's bottom shows the sum of all of the horizontal line segments possible. Thus, the result of the combination of all of the parallel, infinitely thin, one-dimensional line segments in a figure is a two-dimensional product: area.

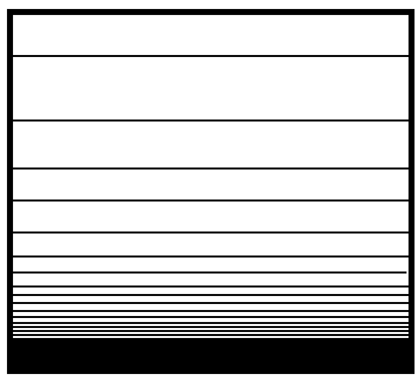


Figure 1. A square's area may be pictured as the sum of all of the parallel, one-dimensional line segments that can be drawn.

The task that now arises is to represent the sum of an infinite number of line segments. This may be accomplished with a one-dimensional line segment that runs perpendicularly to all of the other line segments.[β] Because it runs the length of all of the infinitely thin sides of the other line segments put together, the Perpendicular Line Segment, in a way, represents the number of horizontal line segments present; a doubling of the length of the Perpendicular Line Segment would imply a doubling of the infinite number of horizontal segments, for example.

It is obvious that the product of the number of items present and the magnitude of one item—when the value of each item is identical—is equal to the sum of the magnitudes of all of the items. The value of three one-dollar bills, for instance, may be determined either by addition of the worth of each bill ($1 + $1 + $1 = $3) or by multiplication of the value of one bill by the number of bills present ($1/bill ·3 bills = $3). In the same way, the sum of the horizontal line segments may be represented as the product of the magnitude (i.e., length) of one horizontal segment and the number of segments present. It was previously stated that the latter is equivalent to the length of the Perpendicular Line Segment, which in Figure 1 has the same distance as the square's height. Accordingly, the sum of all of the horizontal, one-dimensional line segments in a square (i.e., the area of the square) is equal to the length of one of the horizontal segments times the height of the square. Thence, the area A of a square of length l and height h is given by A = lh.

[α] All cones considered in this work shall be right circular cones.

[β] This one-dimensional line segment, which serves to "add up" all of the other line segments contained in a shape's area, shall hereafter be referred to as "the Perpendicular Line Segment." See the Glossary at the end for a complete list of such terms contrived specifically for this piece.

Inasmuch as the derivations of the formulas for the areas of such geometric shapes as parallelograms, rectangles, rhombi, trapezoids, and triangles are nothing more than simple extensions of the process utilized above, they will not be explored further in the present disquisition.

This discourse on one conception of the nature of area was intended to serve as a preface to the discussion of the derivation of the formula for the area of a circle that follows.

The Area of a Circle

The principles explicated above may now be applied to circles. Just as a square's area can be considered to be the sum of an infinite number of one-dimensional horizontal line segments, the area of a circle can similarly be imagined as the sum of an infinite number of one-dimensional radii (see Figure 2, drawn in the same pattern as Figure 1).

The objective now is to identify the distance that can be multiplied by the length of a radius to yield the circle's area. It is apparent that this distance will be the length of a circumference. What is less readily apparent, however, is that the desired circumference is not that of the circle itself, but rather the circumference of a second, concentric circle with half the radius of the first.[χ]

The reason for this lies in the visualization of the Perpendicular Circumference, which must pass through the centers of the one-dimensional Larger Radii that are to be combined into the circle's area because it must represent all points on those Larger Radii with balanced emphasis.

A line segment is essentially just the sum of an infinite number of infinitely miniscule, collinear points. Its one-dimensional length is a sort of representation of the number of points it contains in the same way that a Perpendicular Line Segment indicates the number of parallel lines in a square's area. In order for one single line, the Perpendicular Circumference, to represent all of the points on the Larger Radius at once, it is necessary to find the average location of all the points.

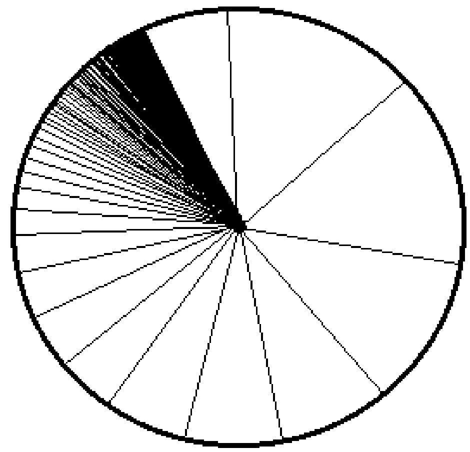
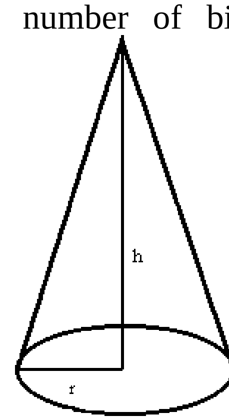


Figure 2. A circle's area may be conceived of as the sum of all of the infinitely thin radii that may be drawn to its center. This diagram illustrates a trend of increasing radii concentration as one moves in the clockwise direction until a region of solid area—representing the combination of all of the possible radii—is reached.

This concept may be illustrated with an analogy. Suppose that a person has two bills: a ten-dollar bill and a twenty-dollar bill. Their total worth may be determined through addition of the worth of each bill ($10 + $20 = $30) or through multiplication of their average ... the number of bills ($15/bill ·2 bills = ...

In the case of a circle, the "number of bills" is the Perpendicular Circumference. The "ten-dollar bill" might be all of the points, collectively, to the left of the Average Point, while the "twenty-dollar bill" could represent the collective total of all of the points to the right of the same. The average of the points on either side is the point directly in the center of the Larger

Figure 3. This diagram illustrates a cone of radius r and height h.

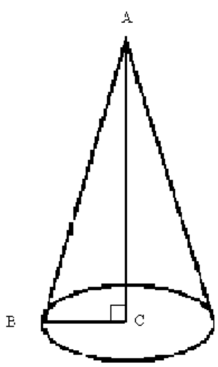


χ The circumference by which the length of the radius is multiplied to yield the circle's area will henceforth be called the "Perpendicular Circumference," in order to suggest its relation to the function served by the Perpendicular Line Segment in squares. The circle formed by the Perpendicular Circumference shall be known as the "Smaller Circle," and its radius, the "Smaller Radius." The original circle, the determination of the area of which was the reason for creating the Smaller Circle, will be called the "Larger Circle," and its radius, the "Larger Radius." The Larger Circle and Smaller Circle will always be concentric with each other. The point at which the Perpendicular Circumference crosses the Larger Radius will be called the "Average Point"; the distance between the Average Point and the center of the circles will always be the length of the Smaller Radius.

Radius.[δ]

Pursuant to this premise, the proper Smaller Radius will be exactly half the length of the Larger Radius. If the variable r is assigned to the Larger Radius, the length of the Smaller Radius will be r/2. The formula for the circumference C of a circle of radius r is $C = 2\pi r$. Wherefore, the distance around the Smaller Circle is $C = 2\pi(r/2) = \pi r$. This is the length of the Perpendicular Circumference, which indicates the number of radii. Thence, the sum of the of the circle) is $rC = r(\pi r) = \pi r^2$.

Figure 4. Triangle ABC, a Volume Triangle, is contained within this cone. A represents the tip of the cone, B is any point on the outer circle of the cone's base, and C is the center of that outer circle. Line segment AC is the triangle's height, and line segment BC is the triangle's base; it is also the Larger Radius.



The concept of "Balance of Points" encountered in this investigation of the area of a circle will resurface in a more intricate form in the subsequent derivation.

The Nature of Volume and the Volume of a Cone

The process for determining volume is really little different from that for determining area. The Perpendicular Line Segment and Perpendicular Circumference are no longer multiplied by the one-dimensional line segments that make up area; rather, they are multiplied by the two-dimensional areas that create volume.

As with area determination, a Balance of Points on either side of the Average Point must always be achieved. It should be kept in mind that both the number of points and their horizontal distances from the Average Point are factors influencing that Balance of Points.

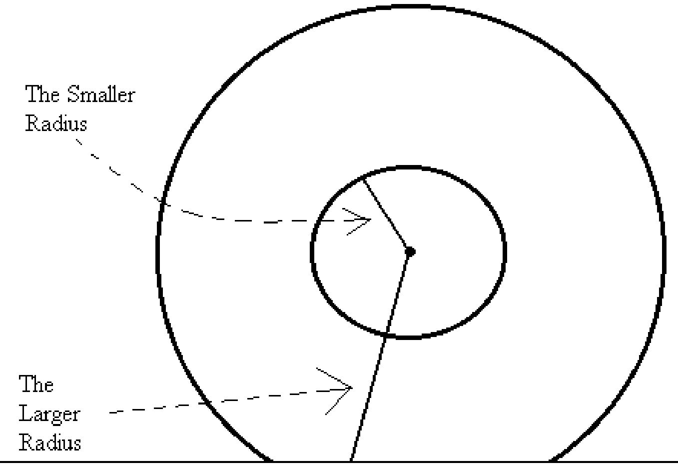


Figure 5. The Larger Circle in this diagram represents the outer circle of the bottom edge of the cone. The Larger Radius indicated above is equivalent to length r in Figure 3, as well as to line segment BC in Figure 4. The Smaller Circle shows the Perpendicular Circumference. The ratio of the Smaller Radius to the Larger Radius is presently unknown, but it has been hypothesized to be less than ½ because a substantial portion of the area of the Volume Triangle is clustered into the region in close proximity to the center of the circle.

A problem arises when determining the volume of a cone like that shown in Figure 3. It is quickly apparent that the cone's volume may be thought of as the product of the area of a Volume Triangle[ε] and a Perpendicular Circumference. The difficulty lies in discovering the radius of that Perpendicular Circumference (i.e., the length of the Smaller Radius) such that a Balance of Points is achieved.

Obviously, the Smaller Radius will no longer be half of the Larger Radius, as was the case in the previous section, forasmuch as the Volume Triangle's area is not evenly distributed on either side of the midpoint of its base; instead, the ratio of the Smaller Radius to the Larger Radius will be less than ½ because most of the Volume Triangle's area is concentrated near the c Larger Circles (see Figure obvious is that the Average represented by line segme not split the Volume Tri equal area, as might be exp

The wherefore lie "Balance of Points," which the number

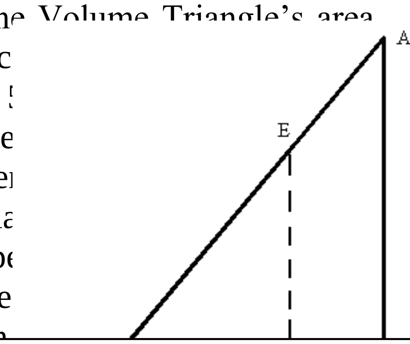


Figure 6. This is an expanded version of ΔABC as it is drawn in Figure 4. Line segment BC represents the radius of the Larger Circle, while line segment CD is the radius of the Smaller Circle. Point D indicates the Average Point, and line segment DE is the Average Line Segment. It is important to realize that DE does *not* divide ΔABC into two sections of equal area. In other words, the area of ΔBDE is not equal to the area of trapezoid ACDE. This is a consequence of another factor—the horizontal distance of points from DE—which will be elucidated shortly.

[δ] The Perpendicular Circumference must always attain what will hereafter be referred to as a "Balance of Points." In two-dimensional area calculations, this means that every point to the left of the Average Point must be balanced by an equidistant point to the right of the same. It will be seen in later volume determinations that both the number of points and their relative horizontal distances from the Average Point are factors affecting the Balance of Points.

[ε] The right trian Perpendicular hereafter be ref Figure 4 is an e points weighted by their relative horizontal distance from the Average Point. One point two units to the left of the Average Point, for example, would carry the same significance as two points that are each one unit to the right of the same. The Average Point must attain a center position that "compromises" the points on either side, which means that a point farther from the compromise stance is more "implacable" than one closer thereto.

The situation is quite analogous to a lever. Torque, which produces rotational acceleration of a rigid body, depends upon both the magnitude of the applied force and the distance of that force from the axis of rotation. The Average Point may be pictured as an axis of rotation, which, in the case of a lever, is a fulcrum. The Significance of Points on either side of that fulcrum can be thought of as a form of torque. The number of points (i.e., the area) is comparable to the magnitude of the applied force; for example, it is easy to imagine each point of the area as one atom in a block of iron, contributing an infinitesimal fraction of the block's total weight.[ϕ] In order to determine the distance of the applied force—caused by the weight of the block—from the fulcrum, it is necessary to identify the block's center of gravity. In the same way, a Center of Area line segment must be drawn on either side of the Average Point. Just as the torque contributed by the iron block is equal to the product of the block's weight and the horizontal distance from the fulcrum to its center of gravity, the Significance of Points on one side of the Average Point may be obtained through multiplication of the area on that side by the horizontal distance between the Center of Area line and the Average Point.

Rotational equilibrium exists when the net torque on a stationary object is zero; this condition results from the equality of the magnitudes of the net torques acting in the clockwise and counterclockwise directions. Similarly, a Balance of Points is attained when the Significance of Points on the right side of the Average Point is equal to the Significance of Points to the left of the same.

A Balance of Points must exist if the Perpendicular Circumference is to be the correct length. Wherefore, it may be stated that

(Significance of Points to the left of the Average Point) = (Significance of Points to the right of the Average Point).

Substitution of the product equivalent to Significance of Points yields the following:

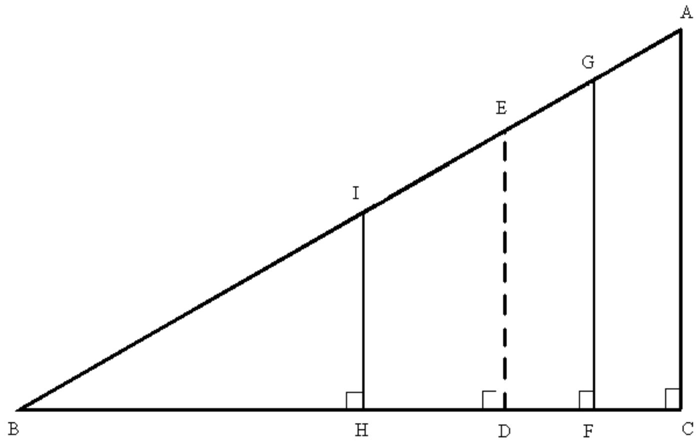
(Area to the left of the Average Line Segment)(Horizontal distance between the Center of Area on the left and the Average Line Segment) = (Area to the right of the Average Line Segment)(Horizontal distance between the Center of Area on the right and the Average Line Segment).

These values may be expressed with the tangible line segments drawn in Figure 7. Line segment DE is the Average Line Segment, and point D is the Average Point. Line segment FG represents the Center of Area on the right, while line segment HI illustrates the Center of Area on the left. As before, line segment BC is the Larger Radius, and line segment CD is the Smaller Radius, the radius of the Perpendicular Circumference. It should be borne in mind that the present objective is to discover the ratio of the Smaller Radius to the Larger Radius—of line segment CD to line segment BC—in order that the length of the Perpendicular Circumference may be expressed in terms of the Larger Radius.

Figure 7. This is the same ΔABC as is illustrated in Figures 4 and 6, with accretions.

[ϕ] This is, of course, merely a conceptual illustration, for it is obvious that the number of atoms in a block is finite, while the number of points in an area is infinite.

Utilizing the vertex labels in Figure 7, the above equation may be written thus:



(Area of ΔBDE)(DH) = (Area of trapezoid ACDE)(DF)

[(1/2)(BD)(DE)](DH) = [(1/2)(DE + AC)(CD)](DF)

(BD)(DE)(DH) = (DE + AC)(CD)(DF)

Angle B has the same measure in both ΔBDE and ΔABC. Thence, the following equality may be stated:

tan B = tan B

(DE)/(BD) = (AC)/(BC)

(DE)/(BD) = (AC)/(BD + CD)

To do:

-add volume of sphere

-adjust everything for the simplified vocab terms, which are able to simplify much of the explanations!!!!!!!!

-make sure that all line segments don't say just "lines"

-make sure that all triangles, etc are in alphabetical order

-once finished, go back at least once to ensure that all vocab terms, explanations, etc. are consistent and in the proper order to make sense

Glossary

The following list of definitions includes only those words invented specifically for ease of explanation within this piece. Definitions for other words may be found in a dictionary or reference book.

Average Line Segment: In three-dimensional cones, the Average Line Segment is the line segment that extends upward from the Average Point to the outside of the cone. The Average Line Segment is perpendicular to the Larger and Smaller Radii.

Average Point: In two-dimensional circles and three-dimensional cones, the infinitely small location at which the Perpendicular Circumference crosses a Larger Radius is the Average Point, so called because it represents the average of all the points on either side of it. The distance between the Average Point and the center of the Smaller or Larger Circles is the Smaller Radius.

Balance of Points: A Balance of Points is a condition wherein the Significance of Points on either side of the Average Point is equal. The Significance of Points on one side of the Average Point is equivalent to the product of the number of points on that side—as is indicated by the length of a line segment in two-dimensional circles or by the area of a triangle or trapezoid in three-dimensional cones—and the horizontal distance to the vertical line segment which divides the points on that one side into two sections, each containing an equal number of points irrespective of their horizontal distances from that vertical line segment. This vertical line segment is called the Center of Area in three-dimensional cones.

Center of Area: In three-dimensional cones, the Center of Area is the line segment—perpendicular to the Larger Radius—that divides into two sections of equal area the polygon (either a triangle or trapezoid) formed as a result of the partitionment of the Volume Triangle by the Average Line Segment.

Larger Circle: In two-dimensional circles, the Larger Circle is the original circle, the derivation of the formula for the area of which is the objective. The Larger Circle is always concentric with the Smaller Circle. In three-dimensional cones, the Larger Circle is the outer circle that constitutes the base circle of the cone. (The area of the Larger Circle is a portion of the surface area of the cone.)

Again, the Larger Circle always has the same center as the Smaller Circle.

Larger Radius: In two-dimensional circles and three-dimensional cones, the radius of the Larger Circle is the Larger Radius.

Perpendicular Circumference: In two-dimensional circles, the Perpendicular Circumference is the distance by which the length of the Larger Radius can be multiplied to give the area of the Larger Circle. The Perpendicular Circumference is always the circumference of the Smaller Circle. In three-dimensional cones, the Perpendicular Circumference is the distance by which the area of the Volume Triangle is multiplied to yield the volume of the cone. Again, the circumference of the Smaller Circle is the Perpendicular Circumference.

Perpendicular Line Segment: The area of a square can be pictured as the sum of the infinite number of parallel line segments that can be drawn across. The Perpendicular Line Segment runs at a right angle to all of those line segments. Because it measures the height of all of the parallel line segments stacked contiguously on top of each other, it indicates their relative number and can therefore be multiplied by the length of one of the parallel segments to yield the square's area. This definition applies solely to two-dimensional squares.

Significance of Points: Analogous to torque, the Significance of Points is the amount of "weight" or impact a group of points has. The Significance of Points on one side of the Average Point is equivalent to the product of the size of the line segment (in two-dimensional circles) or area (in three-dimensional cones) on that side and the horizontal distance between the Average Point and the midpoint of that line segment (in two-dimensional circles) or the Center of Area (in three-dimensional cones). In order to achieve a Balance of Points, the Significance of Points on either side of the Average Point must be equal.

Smaller Circle: In two-dimensional circles and three-dimensional cones, the circumference of the Smaller Circle is the Perpendicular Circumference. The Smaller Circle always has the same center as the Larger Circle.

Smaller Radius: In two-dimensional circles and three-dimensional cones, the Smaller Radius is the radius of the Smaller Circle.

Volume Triangle: In three-dimensional cones, the right triangle whose hypotenuse is a slanted edge of the cone, whose base is the radius of the cone's outer circle (i.e., the Larger Radius), and whose height is the altitude of the cone is the Volume Triangle. The area of a Volume Triangle, when multiplied by the length of the Perpendicular Circumference, results in the volume of the cone.

Saved scraps:

The challenge is to position the fulcrum so that the lever will balance.