Integration Lecture Notes [1]

1 Area Under a Curve

Let $f(x) = x^2$. We wish to find the area under the graph $y = x^2$ above the x-axis between $x = 0$ and $x = 1$. We can see from a graph that this area should be less than 1/2.

To do this we divide the unit interval $[0, 1]$ into n segments of equal length for some positive integer n. Let $x_i = i/n$ for $i = 0$ to n. That is $x_0 = 0$, $x_1 = 1/n$, $x_2 = 2/n$, $x_3 = 3/n$, ..., $x_n = n/n = 1$. We then compute the area of the thin rectangles with base $[x_{i-1}, x_i]$ and height $f(x_i) = (x_i)^2$. If we let $\Delta x = 1/n$ then the area of the i-th rectangle is $f(x_i)\Delta x$. Figure 1 shows the case for $n = 10$.

Since the rectangles are constructed by evaluating the function at the right end point of each subinterval $[x_{i-1}, x_i]$, $i = 1, ..., n$, this method is called finding the approximate area under the curve using **right end points**.

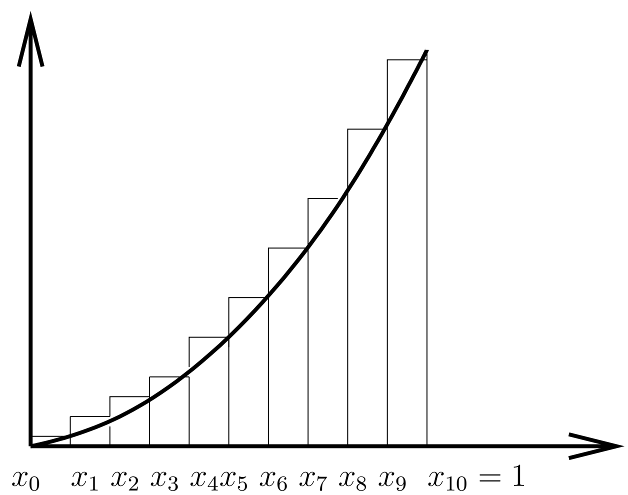

Figure 1: Thin rectangles approximating area under $y = x^2$

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Add these for $i = 1, \ldots, n$ and call the sum A_n. Thus,

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2.$$

But, we have a formula for $\sum_{i=1}^{n} i^2$. We get

$$A_n = \frac{n(n+1)(2n+1)}{6n^3}.$$

For any n, A_n will over estimate the area. A few computations give, $A_{10} = 0.38500$, $A_{15} \approx 0.36741$, $A_{100} = 0.33835$. As n gets larger the closer A_n gets to the "true" area. Although the theory of limits we developed was for functions of a continuous variable, taking the limit as an integer increases without bound is done in essentially the same way. This leads us to write

$$A = \lim_{n\to\infty} A_n.$$

In this example we get

$$A = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = 1/3.$$

We notice that for each n the value of A_n is larger than 1/3. So, how do we know for sure that the area really is 1/3 and not some value a bit below this? To see that this is not the case we will redo our calculations using **left end points**. The height of the thin rectangles will be determined by evaluating the function at the left end point of each subinterval $[x_{i-1}, x_i]$, $i = 1, ..., n$. See Figure 2.

This gives

$$A_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{1=0}^{n-1} i^2 = \frac{(n-1)(n-2)(2n-1)}{6n^3}.$$

Thus we have $A_{10} = 0.228$ and $A_{100} = 0.321783$. These under estimate the area. The reader can check that

$$\lim_{n\to\infty} A_n = \frac{1}{3}.$$

We conclude that the area "really is" 1/3.

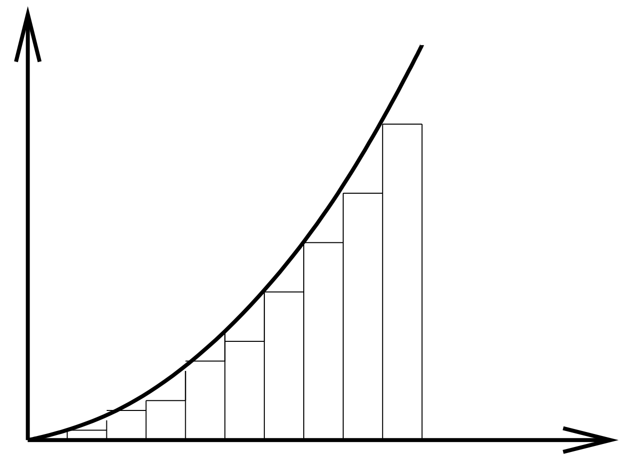

Figure 2: Using left end pint to estimate area under $y = x^2$

Example 1. We shall generalize this. Let's find the area under $y = x^2$ from $x = 0$ to $x = a > 0$.

Solution. We will use right end points. The result comes out the same if we use left end points instead. The only difference is that the size of the sub-intervals will be $\Delta x = a/n$. This means $x_0 = 0$, $x_1 = a/n$, $x_2 = 2a/n$, and so on until $x_n = na/n = a$. The formula for x_i is ia/n. Hence

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{ia}{n}\right)^2 \frac{a}{n} = a^3 \cdot \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{a^3}{n^3} \cdot \sum_{i=1}^{n} i^2 = \frac{a^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

Thus the area is given by

$$A = \lim_{n\to\infty} a^3 \cdot \frac{n(n+1)(2n+1)}{6n^3} = \frac{a^3}{3}.$$

□

Example 2. One last generalization. Let's find the area under the curve $y = x^2$ from $x = a$ to $x = b$, with $0 \leq a < b$. The area from 0 to b is $b^3/3$ while the area from 0 to a is $a^3/3$. Thus,

$$\text{the area under } y = x^2 \text{ between } a \text{ and } b \text{ is } \frac{b^3}{3} - \frac{a^3}{3}.$$

In fact using symmetry it is easy to check that this formula works as long as $a < b$, even if they are not positive.

Example 3. Find a formula for the area under the curve $y = x^3$ and above the x-axis between $x = a$ and $x = b$ where $0 \leq a < b$.

Solution. First we will find the area from 0 to a assuming $a > 0$.

$$A_n = \sum_{i=1}^{n} \left(\frac{ai}{n}\right)^3 \frac{a}{n} = \frac{a^4}{n^4} \cdot \sum_{i=1}^{n} i^3 = \frac{a^4}{n^4} \cdot \frac{1}{4} n^2(n+1)^2.$$

Then

$$A = \lim_{n\to\infty} a^4 \cdot \frac{n^2(n+1)^2}{4n^4} = \frac{a^4}{4}.$$

It is now easy to generalize to the case $0 \leq a < b$ to get that the

$$\text{area under } y = x^3 \text{ between } a \text{ and } b \text{ is } \frac{b^4}{4} - \frac{a^4}{4}.$$

□

Signed Area. When a function is negative between a and b its graph is below the x-axis and we define the area to be -1 times the area between $|f(x)|$ and the x-axis. Thus the signed area under $y = x^3$ from $a = -2$ to $b = -1$ is $1/4 - 2^4/4 = -3.75$. Using this convention and the symmetry of $y = x^3$ through the origin one can show that for and a and b with $a < b$ we have that

$$\text{area under } y = x^3 \text{ between } a \text{ and } b \text{ is } \frac{b^4}{4} - \frac{a^4}{4}.$$

Using the formulas for summations of integers powers one can find the signed area under $y = x^n$ from $x = a$ to $x = b$ for other positive integers n.

$$\text{Area under } y = x^4 \text{ between } a \text{ and } b \text{ is } \frac{b^5}{5} - \frac{a^5}{5}.$$

$$\text{Area under } y = x^5 \text{ between } a \text{ and } b \text{ is } \frac{b^6}{6} - \frac{a^6}{6}.$$

$$\text{Area under } y = x^6 \text{ between } a \text{ and } b \text{ is } \frac{b^7}{7} - \frac{a^7}{7}.$$

The pattern should be clear. In fact for $n = 1$ simple geometry shows that the signed area under $y = x$ from a to b is $b^2/2 - a^2/2$. See Problem 1.4. It is also easy to see that for $n = 0$ the ares under $y = x^0 = 1$ from a to b is just $b - a$. Based on it we make the following conjecture.

Conjecture 1. For any integer $n \geq 0$,

$$\text{the area under } y = x^n \text{ between } a \text{ and } b \text{ is } \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

You may have noticed that

$$\left(\frac{x^{n+1}}{n+1}\right)' = x^n.$$

This suggests a possible relationship between areas and anti-derivatives. Before we explore this further we develop some notation.

Definition 2 (Simple Integral)**.** Let $f(x)$ be a function defined over the closed interval $[a, b]$. Then the **simple integral with right end points** of $f(x)$ over $[a, b]$ is defined by

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, \ldots, n$. When this limit exist (and is finite) we define it to be the **signed area** between the graph of $y = f(x)$ and the x-axis. We often refer to this as the "area under the curve" as a less formal shorthand. To define the **simple integral with left end points** just take the sum to be from $i = 0$ to $n - 1$.

It is convenient to define $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$ and $\int_a^a f(x)\, dx = 0$.

Theorem 3. *Let $f(x)$ be a continuous function over a closed bounded interval $[a, b]$. Then both simple integrals exist and are equal. Thus the notation in 2 is not ambiguous. From now on we will just refer to the simple integral. The result also holds functions with a finite number of jump discontinuities.*

We will not do the proof. It is rather subtle. It is however straight forward to show that if the left and right simple integrals exist they must be equal.

$$\begin{aligned}
\left(\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x\right) - \left(\lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i)\Delta x\right) &= \lim_{n \to \infty} \left(f(x_n) - f(x_0)\right)\Delta x \\
&= \lim_{n \to \infty} \left(f(b) - f(a)\right)\Delta x \\
&= \left(f(b) - f(a)\right) \lim_{n \to \infty} \Delta x = 0.
\end{aligned}$$

Theorem 4 (Properties of Simple Integrals)**.** *Let f and g be continuous functions over $[a, b]$. Let c be any number in (a, b) and let k be be any real number. Then the following hold.*

1. $\displaystyle\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$ *for* $n = 0, 1, 2, 3, 4, \cdots$.

2. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4. $\displaystyle\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Part 1 is Conjecture 1 which we know holds for low values of n and will later show is valid for all integers $n \geq 0$; in fact it will hold for all real numbers $n \neq -1$. (Notice this formula cannot be valid for $n = -1$.) For the other parts the proofs follow easily from the properties of sums and limits. One last bit of notation, for any function $F(x)$ we define

$$F(x)|_a^b = F(b) - F(a).$$

Now we are ready to do business.

Example 4.

$$\int_0^1 x^3 - 2x^2 + 3x - 1\,dx = \left(\frac{x^4}{4} - 2\cdot\frac{x^3}{3} + 3\cdot\frac{x^2}{2} - x\right)\Bigg|_0^1$$

$$= (1/4 - 2/3 + 3/2 - 1) - (0) = 1/12.$$

Example 5.

$$\int_{-3}^2 x^4 - 3x^2\,dx = \left(\frac{x^5}{5} - x^3\right)\Bigg|_{-3}^2 = \left(2^5/5 - 2^3\right) - \left(-3^5/4 + 3^3\right) = 51.35.$$

Example 6. In this example we use Property 5 of Theorem 4. Watch closely.

$$\int_0^4 |x^2-2x|\,dx = \int_0^2 |x^2-2x|\,dx + \int_2^4 |x^2-2x|\,dx = \int_0^2 -(x^2-2x)\,dx + \int_2^4 x^2-2x\,dx =$$

$$[(-2/3)-(0)] + [(64/3-4)-(2/3)] = 16.$$

Next we investigate the area under the curve $y=\sqrt{x}$.

Example 7. Find a formula for the area under $y=\sqrt{x}$ from $x=a$ to b where $0 \le a < b$.

Solution. We shall start with $a=0$ and $b=1$. As Figure 3 shows the area between the curve $y=\sqrt{x}$ and the x-axis is just 1 minus the area between our curve and the y-axis. But this area is just the same as we found in Example 1, that is 1/3. Thus the area under $y=\sqrt{x}$ from 0 to 1 is 2/3.

Next we do the case where $a=0$ but b is any positive number. The larger rectangle in Figure 3 is $b\cdot\sqrt{b}$. The area between the curve are the y-axis is the same as the area under $y=x^2$ from 0 to $\sqrt{b}$. Therefore

$$\int_0^b \sqrt{x}\,dx = b\sqrt{b} - \frac{(\sqrt{b})^3}{3} = \frac{2}{3}b^{\frac{3}{2}}.$$

The more general formula is then

$$\int_a^b \sqrt{x}\,dx = \frac{2}{3}b^{\frac{3}{2}} - \frac{2}{3}a^{\frac{3}{2}}.$$

But, we should be cautious in saying the simple integral limit will in fact converge to this value. In fact this is quite difficult to compute this limit. All we have really shown is that if our definition of the simple integral is 'correct' then it should converge to this value. This will involve shifting to a most robust integral which we will do later in Section 3. □

The reader may have noticed that $\left(\frac{2}{3}x^{\frac{3}{2}}\right)' = \sqrt{x}$. The connection between area and anti-derivatives is now even more compelling.

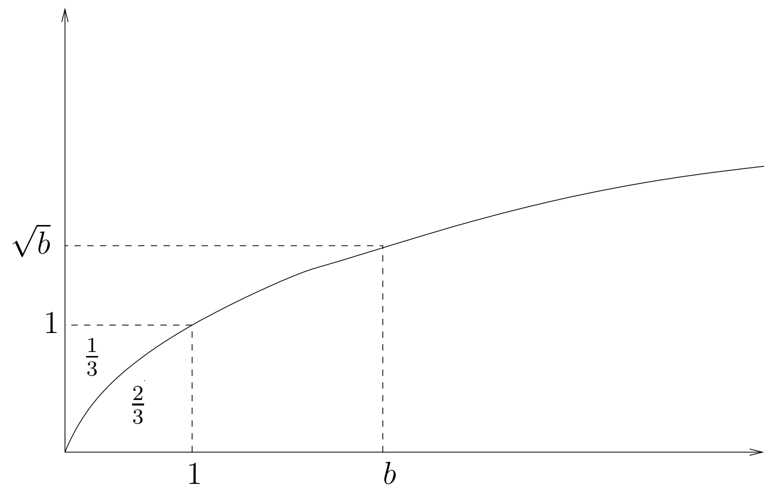

Figure 3: $y = \sqrt{x}$

PROBLEMS.

1. Set up and compute the sums to approximate the area under each given curve over the given interval. Sketch the curves and the rectangles. Is your approximation above or below the true area?

   a. $\ln x$ over $[1, 3]$ with $n = 6$ using right end points.

   b. $1/x^2$ over $[1, 2]$ with $\Delta x = 0.1$ using left end points. What would the exact value be it Conjecture 1 was true for $n = -2$?

   c. $\sin x$ over from $x = 0$ to $x = \pi$ with $n = 8$ using right end points. What do you think the exact value might be?

2. Redo the two problems above using **midpoints**, that this for each subinterval use the average of the left and right end points for the height of the rectangles. While a bit more cumbersome this method tends to have less error than using right or left end points. Why do you think this is?

3. Compute the following simple integrals using Theorem 4 and the examples done in this section.

   a. $\displaystyle\int_2^5 x^3 + 5x - 1\, dx$

b. $\displaystyle\int_{-2}^{1} 14x^2 - 8x + 3\,dx$

c. $\displaystyle\int_{7}^{8} x^3 dx$

d. $\displaystyle\int_{-3}^{-1} x^5 + 2x\,dx$

e. $\displaystyle\int_{1}^{7} |x^2 - 5|\,dx$

f. $\displaystyle\int_{-3}^{8} \Big||x+1| - |2x-5|\Big|\,dx$

4. Show that the signed area between the line $y = x$ and the x-axis from $x = a$ to $x = b$ is $b^2/2 - a^2/2$. Do this in two ways.
   a. Use the formula for the area of triangles.
   b. Set up a summation and take the limit.

5. Show that the formula for the area under the graph of $y = x^2$ gives the expected results when $a < b \leq 0$ and $a < 0 < b$. Hint: Draw pictures!

6. Show that the formula for the signed area under the graph of $y = x^3$ gives the expected results when $a < b \leq 0$ and $a < 0 < b$.

7. Let $f(x) = \begin{cases} 1 & \text{for} \quad x \in \mathbb{Q} \\ 0 & \text{for} \quad x \notin \mathbb{Q}. \end{cases}$ Show that

$$\int_0^1 f(x)\,dx = 1 \text{ while } \int_{\pi}^{\pi+1} f(x)\,dx = 0.$$

   This demonstrates a certain weakness in our theory.

8. Show that show that $\frac{3}{4}b^{\frac{4}{3}} - \frac{3}{4}a^{\frac{4}{3}}$ gives the signed area under the graph of $y = x^{1/3}$ from $x = a$ to $x = b$. What do you notice?

9.

2 Velocity & Distance

Example 8. If you travel at 5 mph for 2 hours, how far have you gone? Obviously 10 miles. But, if we graph $v(t) = 5$ mph over a 2 hour time interval we can interpret the area under the graph as the distance traveled.

Example 9. The graph in Figure 4 gives the velocity $v(t)$ in meters per seconds of an object as a function of time t for $0 \le t \le 8$ seconds.
a. What is the object's position when $t = 8$ seconds? (Assume the object's position starts at zero.)
b. How much distance did the object travel in 8 seconds?
c. Sketch a graph of the position $s(t)$ in meters as a function of time in seconds.

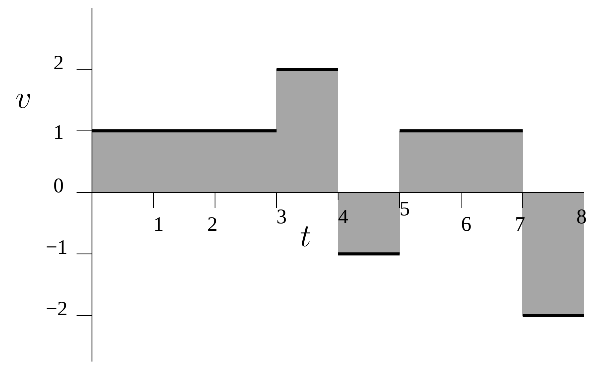

Figure 4: Velocity verses time for Example 9

Solution. a. In the first 3 seconds we are traveling 1 meter per second so the object travels 3 meters. In the next second the velocity is 2 meters per second so we have gone 2 more meters and thus 2+3=5 meters so far. During the fifth second the object is backing up 1 meter per second, so we are 4 meters from where we started. And so on. We can write

$$s(8) = \int_0^8 v(t)\,dt = 3 + 2 - 1 + 2 - 2 = 4 \text{ meters.}$$

b. In calculating the distance traveled going backwards counts positively just as would be the case when figuring the mileage on your car. Thus,

$$\text{distance traveled} = \int_0^8 |v(t)|\,dt = 3 + 4 + 2 + 2 + 3 = 14 \text{ meters.}$$

c. The graph of $s(t)$ is given in Figure 5. For the first three seconds the slope is 1. For the next second it is 2 and then -1 for the fifth second. And so on. □

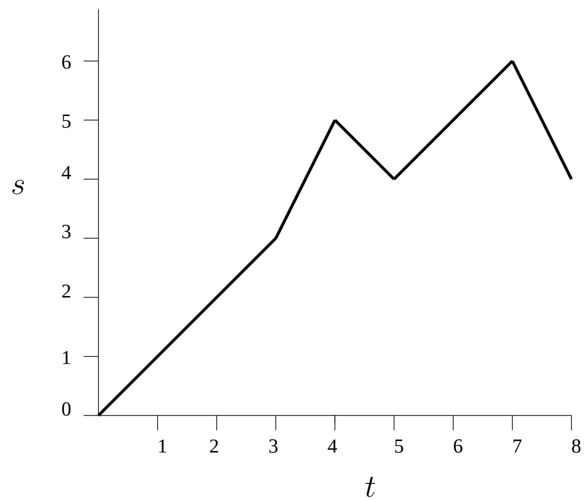

Figure 5: Position verses time for Example 9

Example 10. The graph in Figure 6 gives the velocity $v(t)$ in meters per second of an object as a function of time in seconds. Answer the same questions as in the last example.

Solution. a.

$$s(8) = \int_0^8 v(t)\,dt = 0.5 + 1.0 + 0.5 - 0.5 + 4.0 - 2.0 = 3.5 \text{ meters.}$$

b.

$$\text{distance traveled } = \int_0^8 |v(t)|\,dt = 0.5{+}1.0{+}0.5{+}0.5{+}4.0{+}2.0 = 7.5 \text{ meters.}$$

c. The graph of $s(t)$ is given in Figure 7. During the first second $s(t) = \frac{1}{2}t^2$. Notice the concavity is 1, which is the slope of the $v(t)$ graph and is equal to the acceleration. For the next second $s(t)$ is linear with slope 1. For the next two seconds the cavity of $s(t)$ is -1. And so on. □

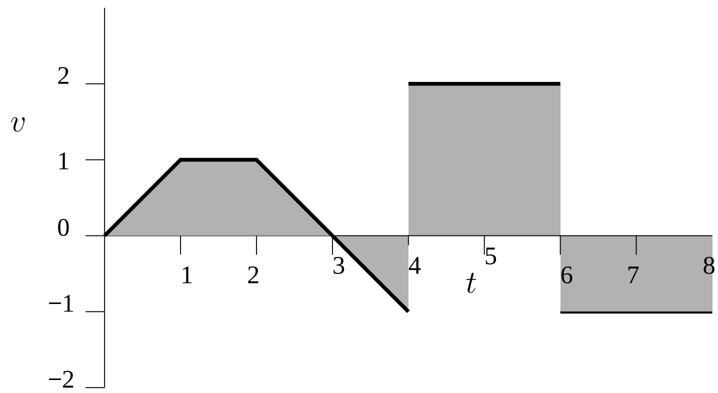

Figure 6: Velocity verses time for Example 10

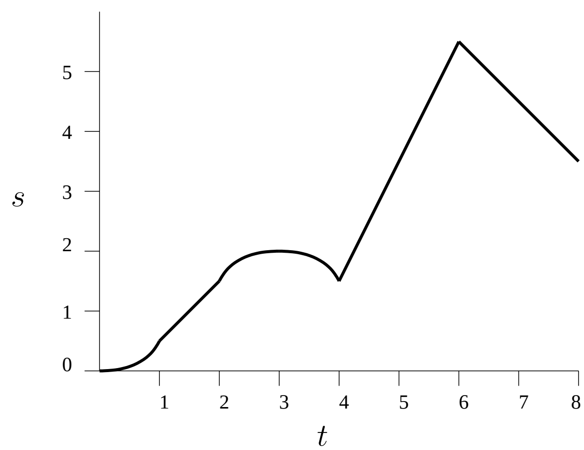

Figure 7: Position verses time for Example 10

Example 11. Suppose for an object we have $v(t) = -t^2 + 4$ in meters per second for $0 \leq t \leq 3$ seconds.
a. Assume the starting position is $s(0) = 0$. Find and graph $s(t)$, $v(t)$ and $a(t)$.
b. What are the positions when $t = 1$, 2 and 3 seconds?
c. What is the object's maximum distance away from the starting point? When does this occur?
d. What is the total distance traveled?

Solution. a&b. $a(t) = -2t$ and $s(t) = \int_0^t -t^2 + 4\,dt = -\frac{1}{3}t^3 + 4t$. Thus $s(1) = 3\frac{2}{3}$, $s(2) = 15\frac{1}{3}$ and $s(3) = 3$.

[insert graphs]

c. $v(t) = 0$ when $t = 2$ and we know $s(2) = 15\frac{1}{3}$; this is the maximum distance.

d. The distance traveled is $\displaystyle\int_0^3 |-t^2+4|\,dt = \int_0^2 -t^2+4\,dt + \int_2^3 t^2-4\,dt =$ $s(2) - s(0) + (-s(3) - (-s(2)) = 15\dfrac{1}{3} - 3 + 15\dfrac{1}{3} = 27\dfrac{2}{3}$. □

You may have noticed that in these examples when we integrated a function that had jump discontinuities the resulting function was continuous. This is indeed always the case. We record this as a theorem although we will prove it.

Theorem 5. *Suppose f is continuous over $[a, b]$ except at a finite number of jump discontinuities. For $x \in [a, b]$ define a new function by*

$$g(t) = \int_a^t f(x)\,dx.$$

Then $g(t)$ is continuous on $[a, b]$.

The examples of this section should reinforce the idea that there is a deep connection between the "area under a curve" problem and the "slope of a tangent line" problem. We again see that integration and differentiation are basically inverse operations of each over. We go from $v(t)$ to $s(t)$ via integration and from $s(t)$ to $v(t)$ by differentiation.

PROBLEMS.

1. Figure 8 shows the graph of the velocity $v(t)$ of an object in feet per second moving a straight line.

   a. Graph the position $s(t)$ assuming $s(0) = 0$.
   b. On the interval $(4, 6)$ is your graph is concave up or concave down?
   c. What is $s(6)$?
   d. What is $s'(2)$?
   e. What is $s'(5)$?
   f. What is $s''(5)$?

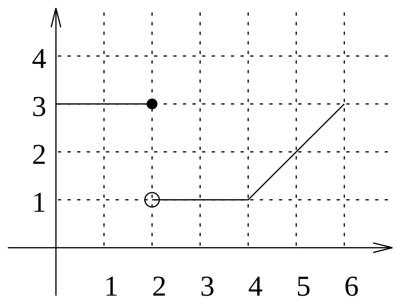

Figure 8: Graph of $v(t)$

2. Let $v(t) = \begin{cases} 2t+1 & \text{for } t \in [0,2] \\ -2 & \text{for } t \in (2,4] \\ \sqrt{t-1} & \text{for } t \in (4,8] \end{cases}$.
   a. Graph the acceleration $a(t) = v'(t)$.
   b. Graph the position $s(t) = \int_0^t v(t)\,dt$. Assume $s(0) = 0$ and that $s(t)$ is continuous.
   c. What is the final position at $t = 8$?
   d. What is the total distance traveled?

3. Suppose $v(t) = \sin t$. Sketch the graphs of $a(t)$ and $s(t)$ assuming $s(0) = 0$. Can you guess what function $s(t)$ is?

4. Suppose the acceleration of a object moving along a straight line is given by
$$a(t) = \begin{cases} 3 & \text{for } t \in [0,2] \\ -t & \text{for } t \in (2,4] \\ 36 - t^2 & \text{for } t \in (4,8] \end{cases}.$$
   Assume that at $t = 0$ the object is at rest at the origin.
   a. Graph $a(t)$
   b. Sketch the graphs of $v(t)$ and $s(t)$.
   c. Does the graph of $s(t)$ have an inflection point? If so where?
   d. When is the object moving forward? When is it moving backward?
   e. Estimate the final position and the total distance traveled.

3 The Riemann Integral

We define a new type of integral, called the **Riemann integral**, that is more flexible than the simple integral. When both exist they are equal. The motivation for this alternative definition is that for certain applications, for example finding the length of a curve, the extra flexibility is necessary. The definition however is rather cumbersome. Students should not get too bogged down if it seems overly abstract. It is covered in greater detail in more advanced courses.

Definition 6. Let $I = [a, b]$ be a closed bounded interval. Choose $n + 1$ points in I such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Let $I_i = [x_{i-1}, x_i]$ for $i = 1, ..., n$. Then the collection $\mathcal{P} = \{I_1, I_2, ..., I_n\}$ is a **partition** of I into n subintervals. Let $|\mathcal{P}| = n$, the number of partition elements and $\|\mathcal{P}\| = \max\{x_i - x_{i-1}\}_{i=1}^n$ be the length of the largest partition member.

Example 12. Let

$$\mathcal{P} = \{[0, 2.2], [2.2, 3.0], [3.0, 6.0], [6.0, 6.1], [6.1, 7.5], [7.5, 9.0], [9.0, 10.0]\}.$$

Then $\mathcal{P}$ is a partition of $[0, 10]$ with $|\mathcal{P}| = 7$ and $\|\mathcal{P}\| = 3$.

Definition 7. Let f be a function defined on a closed bounded interval $[a, b]$. Let $\mathcal{P}$ be a partition of $[a, b]$ with n members. For $i = 1, ..., n$ choose **any** $x_i^* \in I_i$ and let Δx_i be the length of I_i. Then the expression

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is called a **Riemann sum**.

The earlier examples we did with midpoints, and right and left end points were all examples of Riemann sums.

Example 13. Let $f(x) = \sin x$ and $I = [0, \pi]$. Let $x_0 = 0$, $x_1 = \pi/6$, $x_2 = \pi/2$, $x_3 = 3\pi/4$ and $x_4 = \pi$. Chose $x_1^* = \pi/7$, $x_2^* = \pi/4$, $x_3^* = 2\pi/3$ and $x_4^* = 4\pi/5$. Then $\Delta x_1 = \pi/6$. $\Delta x_2 = \pi/3$, $\Delta x_3 = \pi/4$ and $\Delta x_4 = \pi/4$. Thus the corresponding Riemann sum is

$$\sum_{i=1}^{4} \sin(x_i^*)\Delta x_i = \sin(\pi/7) \cdot \pi/6 + \sin(\pi/4) \cdot \pi/3 + \sin(2\pi/3) \cdot \pi/4+$$

$$\sin(4\pi/5) \cdot \pi/4 \approx 2.10948.$$

See Figure 9.

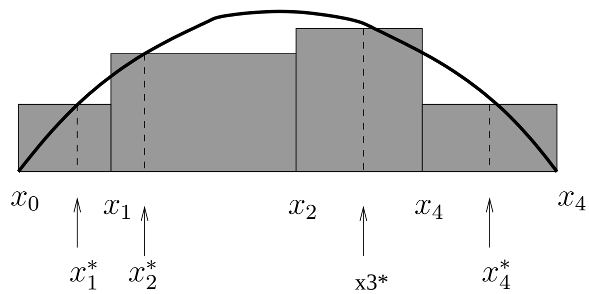

Figure 9: Rectangles for the Riemann sum in Example 13

Definition 8. We define the **limit of Riemann sums**. If this limit exists it is called the **Riemann integral**. Let f be a function defined on a closed bounded interval $[a, b]$. We say the limit of the Riemann sums of f over $[a, b]$ exists if there is a number A such that the following holds. For every $s > 0$ there exist numbers $N > 0$ and $\delta > 0$ such that for any partition $\mathcal{P}$ of $[a, b]$ with $|\mathcal{P}| > N$ and $\|\mathcal{P}\| < \delta$ we have

$$\left| A - \sum_{i=1}^{|\mathcal{P}|} f(x_i^*)\Delta x_i \right| < s$$

for every possible choice of the x_i^*. This is written

$$\lim_{\substack{|\mathcal{P}|\to\infty \\ \|\mathcal{P}\|\to 0}} \sum_{i=1}^{|\mathcal{P}|} f(x_i^*)\Delta x_i = A.$$

When the limit of Riemann sums of f over $[a, b]$ exists we call it the Riemann integral and write

$$\int_a^b f(x)\,dx = A.$$

It is convenient to define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ and $\int_a^a f(x)\,dx = 0.$

The next theorem states some basic facts about the Riemann integral. The formal definition of the Riemann integral is quite difficult to work with. We will not do the proofs. They are covered in more advanced courses.

Theorem 9. *1. For any continuous or piecewise continuous function with only finitely many jump discontinuities over $[a,b]$ the Riemann integrals exists.*

2. If the Riemann integral exists so does the simple integral and they are equal.

3. The properties of the simple integral in Theorem 4 hold for the Riemann integral.

4. Assume the Riemann integral of f over $[a,b]$ exists. Let $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, ...\}$ be an infinite family of partitions of $[a,b]$. Assume that $\lim_{n\to\infty} |\mathcal{P}_n| = \infty$ and $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. For each positive integer n let $x^*_{n,i}$ be a number in the i^{th} partition element of $\mathcal{P}_n$. For each positive integer n let $\Delta x_{n,i}$ be the length of the i^{th} partition element of $\mathcal{P}_n$. Then

$$\lim_{n\to\infty} \sum_{i=1}^{|\mathcal{P}_n|} f(x^*_{n,i})\Delta x_{n,i} = \int_a^b f(x)\,dx.$$

Example 14. Find the Riemann integral of $\sqrt{x}$ over $[0,1]$.

Solution. Since $\sqrt{x}$ is continuous over $[0,1]$ we know by Theorem 9(1) that the Riemann integral exists. We shall construct partitions and show that they satisfy the requirements of Theorem 9(4). We let $x_i = \left(\frac{i}{n}\right)^2$, for $i = 0, ..., n$ to determine the partition $\mathcal{P}_n$. For example, if $n = 5$ we would get

$$x_0 = 0,\ x_1 = \frac{1}{25},\ x_2 = \frac{4}{25},\ x_3 = \frac{9}{25},\ x_4 = \frac{16}{25},\ x_5 = \frac{25}{25} = 1$$

and $\|\mathcal{P}_5\| = 1 - 16/25 = 9/25$. In general $\|\mathcal{P}_n\| = 1 - \left(\frac{n-1}{n}\right)^2 = \frac{2n-1}{n^2}$. Thus $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$ as needed to apply Theorem 9(4). We will use right end points, so $x^*_i = x_i$. We can also compute that

$$\Delta x_i = x_i - x_{i_1} = \left(\frac{i}{n}\right)^2 - \left(\frac{i-1}{n}\right)^2 = \frac{2i-1}{n^2}.$$

The largest Δx_i is $\Delta x_n = \frac{2n-1}{n^2}$. This clearly goes to zero as n goes to infinity. Thus all the requirements to apply Theorem 9(4) are in place. The corresponding Riemann sum is,

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i^*)\Delta x_i &= \sum_{i=1}^{n} \sqrt{\left(\frac{i}{n}\right)^2} \cdot \frac{2i-1}{n^2} \\
&= \sum_{i=1}^{n} \frac{i(2i-1)}{n^3} \\
&= \frac{1}{n^3} \sum_{i=1}^{n} 2i^2 - i \\
&= \frac{1}{n^3} \left(2\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i \right) \\
&= \frac{1}{n^3} \left(2\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right) \\
&= \frac{4n^3 + 3n^2 - n}{6n^3}.
\end{aligned}$$

Next we take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3} = \frac{4}{6} = \frac{2}{3}.$$

Of course the is the same result we found in Example 7. □

PROBLEMS.

1. Let $f(x) = \begin{cases} 1 & \text{for} \quad x \in \mathbb{Q} \\ 0 & \text{for} \quad x \notin \mathbb{Q}. \end{cases}$

   Show that neither $\int_0^1 f(x)\,dx$ or $\int_\pi^{\pi+1} f(x)\,dx$ exists. Compare this to the results in Problem 7 in Section 1. Does this contradict Theorem 9(2)? Explain.

   Remark. There is another type of integral called the **Lebesgue integral**. With it functions like the one in the problem above can be defined unambiguously. The Lebesgue integral is studied in graduate level courses.

4 The Average Value of a Function

In this section we show a useful application of integration and in the next we use it to resolve a theoretical issue.

Definition 10. Let f be integrable over $[a, b]$. Then the **average value** of f over $[a, b]$ is

$$f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We give a justification for this definition. Let $y = f(x)$ be a function defined over $[a, b]$. To approximate the average value of $f(x)$ we shall divide $[a, b]$ into n equal length subintervals with end points $x_i = a + i\Delta x$ for $i = 0, ..., n$ where $\Delta x = (b-a)/n$. We shall evaluate f at the midpoints a take an average.

$$f\text{ave} \approx \frac{\sum_{i=1}^n f(x_i^*)}{n}$$

where $x_i^* = (x_i + x_{i-1})/2$ for $i = 1, ..., n$. For larger and larger values of n we expect this sum to approach the average value of f over $[a, b]$. However this sum is not in the form of a Riemann sum, so it is not clear how to compute this limit. The situation can be remedied as follows. Since $\Delta x = (b-a)/n$ we can write $n = (b-a)/\Delta x$. Thus we get

$$f\text{ave} \approx \frac{\sum_{i=1}^n f(x_i^*)\Delta x}{(b-a)}$$

and the numerator is a Riemann sum. Taking the limit as $n \to \infty$ gives

$$f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 15. Find the average value of x^2 over $[-1, 1]$.

Solution. $\frac{1}{2}\int_{-1}^1 x^2\,dx = \frac{1}{2}\frac{2}{3} = \frac{1}{3}.$ □

Theorem 11. *If $f(x) \le g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.*

The proof follows easily from the definition and properties of integration. The following observation will be used in the next section. Assume $f(x)$

is continuous over the interval $[a, b]$. By the Extreme Value Theorem there exist real numbers p and q in $[a, b]$ such that for all $x \in [a, b]$ we have

$$f(p) \leq f(x) \leq f(q).$$

By Theorem 11 we have

$$\int_a^b f(p)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b f(q)\,dx.$$

Of course $f(p)$ and $f(q)$ are constants. Thus,

$$f(p)(b-a) \leq \int_a^b f(x)\,dx \leq f(q)(b-a).$$

Whence,

$$f(p) \leq f_{\text{ave}} \leq f(q).$$

So, we arrive at the hardly startling conclusion that the average value of a function is in between its minimum and maximum values.

PROBLEMS.

1. Find the average value of $x^3 + x^2 - 3$ over $[1, 4]$. *Answer.* 223/9
2. Find the average value of $\sqrt{x}$ over $[0, 3]$.
3. Find the average value of $y = \sqrt{16 - x^2}$ over $[-4, 4]$. *Answer.* π

5 The Fundamental Theorem of Calculus

Theorem 12. *Let f be a continuous function over $[a, b]$ with anti-derivative F. Then*

I. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

II. If $\displaystyle g(x) = \int_a^x f(t)\,dt$, *then* $\displaystyle\frac{dg}{dx} = f(x)$.

Proof. We prove Part II first and then use it to prove Part I.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} =$$

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

We shall do the case where $h > 0$, that is $h \to 0^+$; the other case is similar. By the Extreme Value Theorem there exist real numbers p and q in $[x, x+h]$ such that $f(p) \le f(t) \le f(q)$ for all $t \in [x, x+h]$. The average value of $f(t)$ over $[x, x+h]$ will be between $f(p)$ and $f(q)$. Therefore,

$$f(p) \le \frac{1}{h} \int_x^{x+h} f(t)\,dt \le f(q).$$

Notice that the values of p and q will depend on h. As $h \to 0^+$ we see that $f(p) \to f(x)$ and $f(q) \to f(x)$ by continuity. By the squeeze theorem we get

$$\lim_{h\to 0^+} f(p) \le \lim_{h\to 0^+} \frac{1}{h} \int_x^{x+h} f(t)\,dt \le \lim_{h\to 0^+} f(q).$$

Thus

$$\lim_{h\to 0^+} \frac{1}{h} \int_x^{x+h} f(t) = f(x).$$

Since the argument for $h \to 0^-$ is similar the result follows.

Now Part I, the form we will use most often, follows easily. Let $c \in (a, b)$. Let $F(x) = \int_c^x f(t)\,dt$. By Part II we know $F'(x) = f(x)$. Then

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_c^b f(t)\,dt - \int_c^a f(t)\,dt = F(b) - F(a).$$

□

Remark. It is possible to prove Part I directly without assuming continuity - merely that the Riemann integral exists, and then use it to prove Part II.

Example 16. $\int_0^\pi \sin x\,dx = (-\cos x)|_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2.$

Example 17. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = (\arctan x)|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}.$

Example 18. $\displaystyle\int_1^2 \frac{1}{x}\,dx = (\ln x)|_1^2 = \ln 2 - \ln 1 = \ln 2.$

Example 19. Find $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

Wrong solution! The function $\dfrac{-1}{x}$ is an anti-derivative of $\dfrac{1}{x^2}$; just check that $\left(\dfrac{-1}{x}\right)' = \dfrac{1}{x^2}$. Therefore, by the FTC

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_{-1}^1 = \left(\frac{-1}{1}\right) - \left(\frac{-1}{-1}\right) = -1 - 1 = -2.$$

But that's impossible! The graph of $y = 1/x^2$ is never negative. What went wrong? The function $1/x^2$ is undefined at $x = 0$ and so in not continuous over the interval $[-1, 1]$. Thus the FTC does not apply. In fact it can be shown that the Riemann integral does not exist. □

Definition 13. The general anti-derivative of a function f is also called the **indefinite integral** and is denoted by

$$\int f(x)\,dx.$$

The Riemann integral over $[a, b]$ is also called the **definite integral**. These are the terms we will use from now on.

Example 20. Find $\displaystyle\int \sin x\,dx$. The answer is $-\cos x + C$.

Example 21. Find $\displaystyle\int \frac{1}{x}\,dx$. The answer is $\ln|x| + C$.

Example 22. Find $\displaystyle\int x^n\,dx$, $n \neq -1$. The answer is $\frac{x^{n+1}}{n+1} + C$.

All of the anti-derivative formulas can now be written as indefinite integrals.

$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C,\;\; r \neq 1$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int b^x\,dx = \frac{b^x}{\ln b} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \sinh x\,dx = \cos x + C$	$\int \cosh x\,dx = \sinh x + C$
$\int \operatorname{sech}^2 x\,dx = \tan x + C$	$\int \operatorname{csch}^2 x\,dx = -\coth x + C$
$\int \operatorname{sech} x \tanh x\,dx = -\sec x + C$	$\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$
$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$	$\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \csc x + C$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$	$\int \frac{-1}{1+x^2}\,dx = \cot^{-1} + C$

PROBLEMS.

Compute the following integrals when defined. If an integral does not exist explain why.

1. $\displaystyle\int_0^{\pi} \sec x \tan x\,dx$

2. $\int_0^{\pi/3} \sec x \tan x \, dx$ *Answer.* 1

3. $\int_1^e \frac{1}{x} \, dx$. *Answer.* 1

4. $\int_4^9 \sqrt{x} \, dx$. *Answer.* 38/3

5. Find the average value of $\sin x$ over $[0, \pi]$. *Answer.* $2/\pi$

6. Find the average value of $\sec^2 x$ over $[-\pi/4, \pi/4]$. *Answer.* $4/\pi$

7. Find the average value of $\sqrt{4 - x^2}$ over $[-2, 2]$. *Answer.* $\pi/2$

8. The energy of a wave is computed by finding the average value of the square of the wave function over one period. Find the average value of $\sin^2 x$ over $[0, 2\pi]$. Do this by using the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

You will need to figure out an anti-derivative for $\cos 2x$.

6 Geometric Tricks

You can easily check that $\int_{-\pi}^{\pi} \sin x \, dx = 0$. But it is clear from the graph of $y = \sin x$ that the area from $-\pi$ to 0 will be the negative of the area from 0 to π. Thus they cancel out. This would be true of any odd function being integrated from $-a$ to a. Formally, since the anti-derivative of an odd function is an even function we get

$$\int_{-a}^{a} f(x) \, dx = F(-a) - F(a) = F(a) - F(a) = 0,$$

where f is odd and F is any anti-derivative.

Example 23. Show that $\int_{-\pi}^{\pi} x^4 \sin(x^3) \, dx = 0$.

Solution. We don't know how to find an anti-derivation, but we can check that the function is odd.

$$(-x)^4 \sin((-x)^3) = x^4 \sin(-x^3) = -x^4 \sin(x^3).$$

Since we are integrating from $-\pi$ to π the answer must be zero. □

Example 24. Show that $\displaystyle\int_0^{\pi} \cos^3 x\, dx = 0$.

Solution. The interval is not symmetric about the origin but a similar idea works. From the graph, see Figure 10, it is clear that

$$\int_0^{\pi/2} \cos^3 x\, dx = -\int_{\pi/2}^{\pi} \cos^3 x\, dx$$

Therefore

$$\int_0^{\pi} \cos^3 x\, dx = \int_0^{\pi/2} \cos^3 x\, dx + \int_{\pi/2}^{\pi} \cos^3 x = -\int_{\pi/2}^{\pi} \cos^3 x + \int_{\pi/2}^{\pi} \cos^3 x = 0.$$

In fact, if k is an odd positive integer then for any integer n we have $\int_0^{n\pi} \cos^k x\, dx = 0$. □

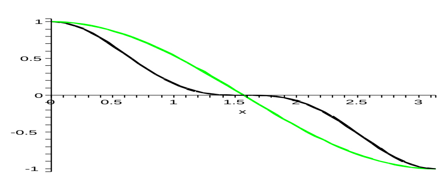

Figure 10: Graphs of $y = \cos x$ in gray/green and $y = \cos^3 x$ in black.

Example 25. Find $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\, dx$.

Solution. The function is even and is never negative, so the answer won't be zero. Here a different trick will be used. The graph of $y = \sqrt{4 - x^2}$ is the upper half of a circle of radius 2 centered at the origin. The area inside the whole circle is $\pi r^2 = 4\pi$. Half of this is 2π so the answer is 2π. □

Example 26. Show that $\displaystyle\int_0^1 \arcsin x\,dx = \frac{\pi}{2} - 1.$

Solution. We don't yet know how to find an anti-derivative of $\arcsin x$. But we can use the same method an in Example 7. In Figure 11 we want to find the area between the arcsine curve and the x-axis. But the area between the curve and the y-axis is the same as the $\int_0^{\pi/2} \sin x\,dx = 1$. The area of the box is $1 \times \pi/2 = \pi/2$. Therefore the desired area is $\frac{\pi}{2} - 1$. □

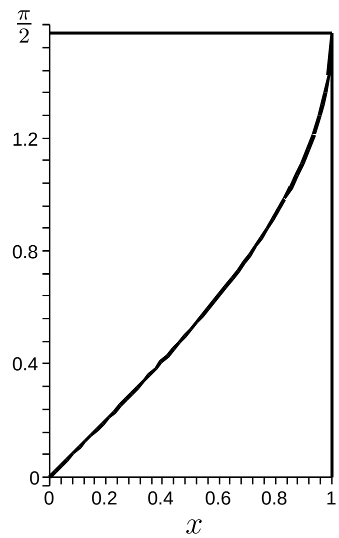

Figure 11: $y = \arcsin x$

Example 27. Show that $\displaystyle\frac{\pi}{4} \le \int_0^1 \sqrt{1 - x^3}\,dx \le 1.$

Solution. For $x \in [0, 1]$ we have $0 \le x^3 \le x^2$. Therefore, we have

$$\sqrt{1 - x^2} \le \sqrt{1 - x^3} \le 1$$

for $x \in [0, 1]$. By Theorem 11 we have

$$\int_0^1 \sqrt{1 - x^2}\,dx \le \int_0^1 \sqrt{1 - x^3}\,dx \le \int_0^1 1\,dx.$$

The first integral is the area of a quarter of the unit circle and the last is the area of a 1×1 square. Thus, $\dfrac{\pi}{4} \le \displaystyle\int_0^1 \sqrt{1-x^3}\,dx \le 1$. □

PROBLEMS.

1. Compute $\displaystyle\int_2^5 |x-3|\,dx$. Hint: Graph the function first.
2. Compute $\displaystyle\int_1^6 \Big||x-3| - |x-5|\Big|\,dx$. Hint: Graph the function first.
3. Compute $\displaystyle\int_{-2}^2 x\,|x|\,dx$. Explain!
4. Compute $\displaystyle\int_{-7\pi}^{7\pi} x^2 \sin x + \cos x\,dx$. Explain!
5. Compute $\displaystyle\int_{-2/\pi}^{2/\pi} x \cos^2 x\,dx$. Explain!
6. Compute $\displaystyle\int_{-3}^3 (x+2)\sqrt{9-x^2}\,dx$. Explain!
7. Verify that $\ln \sec x$ is an anti-derivative of $\tan x$. Compute the following.
   a. $\displaystyle\int_0^{\pi/4} \tan x\,dx$
   b. $\displaystyle\int_0^1 \arctan x\,dx$.
8. Compute $\displaystyle\int_1^3 \ln x\,dx$. Hint: $\ln x$ is the inverse of e^x.
9. Let $f(x) = x^3 + x$. Verify that f is increasing everywhere and is therefore one-to-one and has a well defined inverse. Compute $\displaystyle\int_0^2 f^{-1}(x)\,dx$.
   Answer: 1.25

10. Find a formula for the area under the curve $y = \sin^{-1} x$ from $x = 0$ to t. Take the derivative of this formula with respect to t and show that you get $\sin^{-1} t$. What can you conclude about the general anti-derivative of $\sin^{-1} x$?

11. Suppose you know that $\displaystyle\int_1^2 f(x)\,dx = 7$. What can you say about $\displaystyle\int_4^5 f(x-3)\,dx$?

12. Give an example showing that even if $f(x)$ is never negative then $\displaystyle\int_a^b (f(x))^2\,dx$ is not in general equal to $\displaystyle\left(\int_a^b f(x)\,dx\right)^2$.

7 Another Implication of the FTC

Let $g(x) = \int_0^x f(t)\,dt$. The the FTC tells us that $\dfrac{dg}{dx} = f(x)$. But now suppose that $h(x) = g(x^6)$. Then $h'(x) = g'(x^6)(x^6)' = f(x^6)6x^5$.

Here's another way to look at it. Let F be an anti-derivative of f. Then $h(x) = \displaystyle\int_0^{x^6} f(t)\,dt = F(x^6) - F(0)$. Now we can write

$$h'(x) = (F(x^6))' - (F(0))' = F'(x^6)(x^6)' - 0 = f(x^6)6x^5.$$

Example 28. Let $q(x) = \displaystyle\int_0^{x^2} \sqrt{t^3+1}\,dt$. Find $q'(x)$.

Solution. We do not know how to find an anti-derivative of $\sqrt{t^3+1}$. Undeterred we use our imagination and suppose that F is a function such that $F'(t) = \sqrt{t^3+1}$. Then $q(x) = F(x^2) - F(0)$. Hence,

$$q'(x) = F'(x^2)(x^2)' - 0 = \left(\sqrt{(x^2)^3+1}\right) 2x = 2x\sqrt{x^6+1}.$$

□

Example 29. Let $f(x) = \displaystyle\int_3^{e^x} \sin^3(t^2)\,dt$. Find $f'(x)$.

Solution. Suppose $F'(t) = \sin^3(t^2)$. Then

$$f'(x) = F'(e^x)(e^x)' - (F(3))' = \sin^3((e^x)^2) \cdot e^x - 0 = e^x \sin^3(e^{2x}).$$

$\square$

Example 30. Let $h(x) = \int_{x^3}^{\sin x} \cot^5(t^2)\,dt$. Find $h'(x)$.

Solution. Suppose $F'(t) = \cot^5(t^2)$. Then

$$h'(x) = (F(x^3))' - (F(\sin x))' = F'(x^3)(x^3)' - F'(\sin x)(\sin x)' =$$

$$\cot^5((x^3)^2) \cdot 3x^2 - \cot^5((\sin x)^2)\cos x = 3x^2 \cot^5 x^6 - \cot^5(\sin^2 x)\cos x.$$

$\square$

Example 31. Suppose f is a continuous function such that $\int_0^{x^3} f(t)\,dt = x^4 + 2x^2$. What is $f(t)$?

Solution. We take the derivative of both sides with respect to x. This gives,

$$f(x^3)3x^2 = 4x^3 + 4x.$$

Therefore, $f(x^3) = \dfrac{4x^3 + 4x}{3x^2}$. If we let $t = x^3$, then $x = t^{\frac{1}{3}}$. Hence,

$$f(t) = \frac{4t + 4t^{\frac{1}{3}}}{3t^{\frac{2}{3}}} = \frac{4}{3}(t^{\frac{1}{3}} + t^{-\frac{1}{3}}).$$

We note that $f(t)$ is not defined at $t = 0$. $\square$

Problems.

1. Let $f(x) = \int_8^{x^7} e^{t^3}\,dt$. Find $f'(x)$.
2. Let $r(x) = \int_x^{\tan x} \csc^3 t\,dt$. Find $r'(x)$.
3. Let $E(p) = \int_0^{\sinh p} x \tan x\,dx$. Find $E'(p)$.
4. Let $t(x) = \int_{\cos x}^{x^2} 13\,\mathrm{sech}\, s\,ds$. Find $t'(x)$.