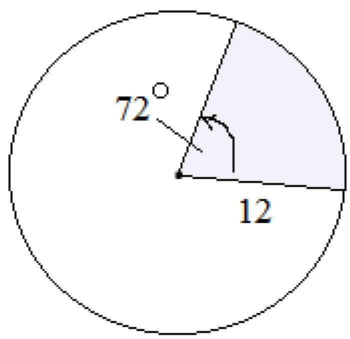
Trigonometry: Arc Length and Sector Area



Includes formulas, examples, illustrations, and quick quiz (w/solutions)

Mathplane.com

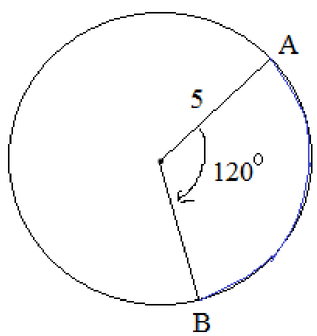
Arc Length (using degrees)

$$\text{Arc Length} = (2\pi r)\frac{(\text{measure of central angle})}{360^{\circ}}$$

$(2\pi r)$ — circumference of *entire circle*

$\frac{(\text{measure of central angle})}{360^{\circ}}$ — percentage (portion) of the entire circle

Example: Find the arc length $\overset{\frown}{AB}$



Since the radius is 5, the circumference of the whole circle is

$$2\pi(5) = 10\pi$$

Arc AB is $\frac{120^{\circ}}{360^{\circ}} = \frac{1}{3}$ of the circumference

Therefore, $\overset{\frown}{AB} = \frac{1}{3}$ of $10\pi \longrightarrow \frac{10\pi}{3}$

10.47 units

(converting radians/degrees)

2π (radians) $= 360^{\circ}$ so, if we use substitution in the above formula:

$$\text{Arc Length} = (2\pi r)\frac{(\text{measure of central angle})}{360^{\circ}}$$

and cancel the 2π's $= (\cancel{2\pi} r)\frac{(\text{measure of central angle})}{\cancel{2\pi}\text{ (radians)}} = r(\text{central angle})$

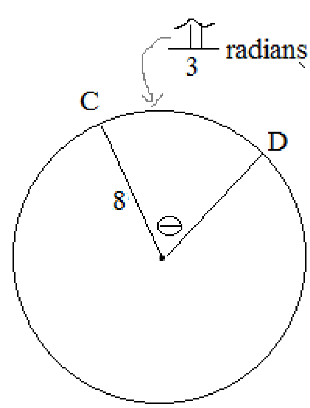
Arc Length (using radian measure)

$$\text{Arc Length} = r\theta$$

r — radius of the circle

θ — *radian* measure of the arc

Example: Find the arc length of the $\overset{\frown}{CD}$



Since the radius (r) = 8 and

$$\theta = \frac{\pi}{3}$$

the arc length of CD is $\frac{8\pi}{3}$

8.38 units

Quick check: $\frac{\pi}{3} = 60$ degrees

60 degrees is 1/6 of the entire circle

circumference of circle $= 16\pi$

therefore, arc length is (1/6) of $16\pi \longrightarrow \frac{8\pi}{3}$

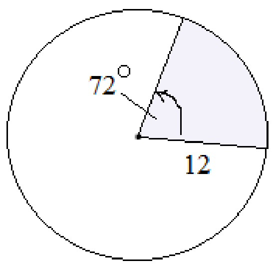
Sector Area (using degrees)

$$\text{Sector Area} = \pi r^2 \; \frac{\text{(measure of central angle)}}{360^\circ}$$

πr^2 — area of entire circle

$\frac{\text{(measure of central angle)}}{360^\circ}$ — percentage (portion) of the entire circle

Example: Find the sector area (shaded region)



Since the radius is 12 units, the area of the <u>entire</u> circle is

$$\pi (12 \text{ units})^2 = 144\pi \text{ sq. units}$$

Then, $\frac{72^\circ}{360^\circ} = \frac{1}{5}$ so, the "piece is 1/5 of the pie"

$$\frac{1}{5} \cdot 144\pi \text{ sq. units} = \frac{144\pi}{5} \text{ sq. units}$$

90.48 sq. units

(converting radians/degrees)

2π (radians) = 360° so, if we use substitution in the above formula:

$$\text{Sector Area} = \pi r^2 \; \frac{\text{(measure of central angle)}}{360^\circ}$$

and cancel the π's

$$= \pi r^2 \; \frac{\text{(measure of central angle)}}{2\pi \text{ (radians)}} = \frac{r^2 \text{(central angle)}}{2}$$

Sector Area (using radian measure)

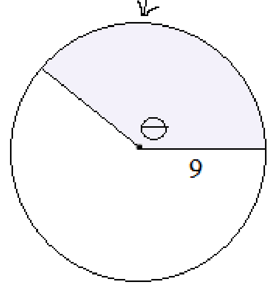
$$\text{Sector Area} = \frac{r^2 \theta}{2}$$

r — radius of circle

θ — radian measure of the arc

Example: Find the sector area of the shaded region.

$\frac{5\pi}{6}$ radians



Using the formula:

radius (r) = 9 units

$\theta = \frac{5\pi}{6}$

$$\frac{(9)^2 \cdot \frac{5\pi}{6}}{2} = \frac{405\pi}{12} = \frac{135\pi}{4}$$

106.03 square units

Quick Check: $\frac{5\pi}{6}$ radians = 150°

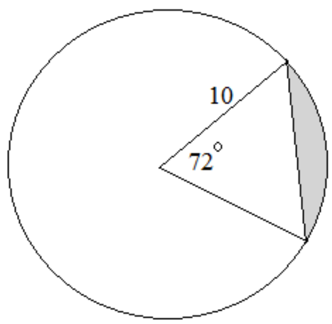
$\frac{150^\circ}{360^\circ} = \frac{5}{12}$ (shaded portion of the circle)

area of circle: 81π

therefore, sector is 5/12 of 81π → $\frac{405\pi}{12}$

Sector Area Trigonometry

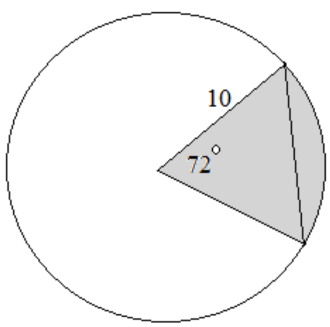
Example Find the shaded area. Then, find the perimeter of the shaded boundary.



sector area of circle: $\frac{\theta}{360^\circ}\pi r^2$

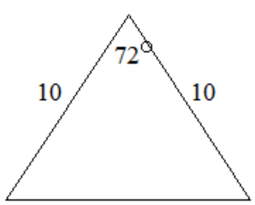
arc length in a circle: $\frac{\theta}{360^\circ}(2\pi r)$

area of triangle: 1/2(base)(height)

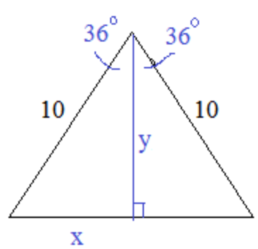


sector area of circle: $\frac{72^\circ}{360^\circ}\pi(10)^2 = 20\pi$

$= 62.8$ (approx.)



(all radii congruent and property of isosceles triangles)



Use trig functions to identify base and height

$\sin(36^\circ) = \frac{x}{10}$ $\cos(36^\circ) = \frac{y}{10}$

$x = 10 \cdot (.588) = 5.88$ $y = 10 \cdot (.809) = 8.09$

area of triangle $= \frac{1}{2}(11.76)(8.09) = 47.6$

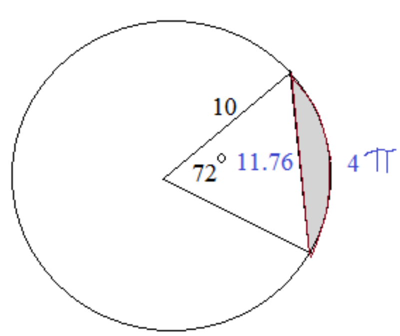
shaded area = sector area - triangle area

$= 62.8 - 47.6 = 15.2$ square units

arc length in circle: $\frac{72^\circ}{360^\circ}(2\pi(10)) = 4\pi$

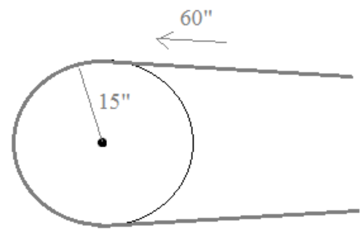
The border of the shaded area is

$11.76 + 12.57 = 24.3$ units



Angular and Linear Speed

Example: A pulley with a 15" radius pulls 60" of rope every 20 seconds.
What is the angular speed in radians/seconds?



$r\theta$ = linear distance

$(15")\ \theta = \cdot\ 60"$

θ = 4 radians

so, the angular speed is 4 radians/20 seconds or .2 radians/second

Example: A skateboard cruises down a hill at 15 miles per hour.
If the diameter of each wheel is 2.3", what is the angular speed in radians/second?

$r\theta$ = linear distance

$1.15"\ \theta = \frac{15 \text{ miles}}{1 \text{ hour}}$

Convert the units:

$$\frac{15 \text{ miles}}{1 \text{ hour}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = 264 \frac{\text{inches}}{\text{second}}$$

$1.15"\ \theta = 264 \frac{\text{inches}}{\text{second}}$

229.6 radians/second

Example: A bicycle has wheels with diameter 27.6 inches.
The diagram shows the dimensions of the chain mechanism.
If the pedal turns 180 degrees, how far does the bicycle travel?

First, find the linear distance of the big gear. (i.e. the arc length)

$r\theta$ = linear distance

$4.7"(\pi) = 14.765"$

180 degrees

So, the chain moves 14.765 inches...

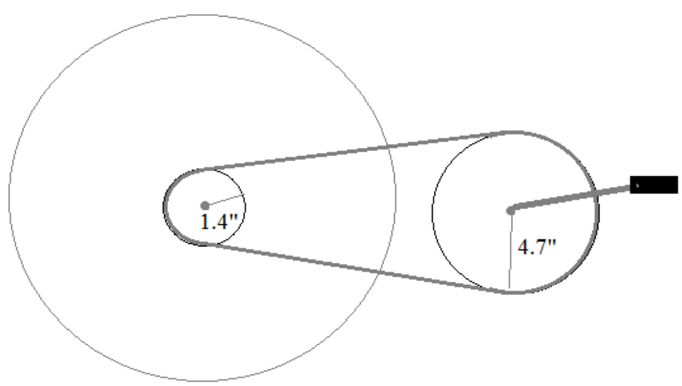
Then, find the angular distance of the small gear.

If the chain moves 14.765 inches,

$1.4"(\ \theta\ ) = 14.765"$

θ = 10.55 radians..

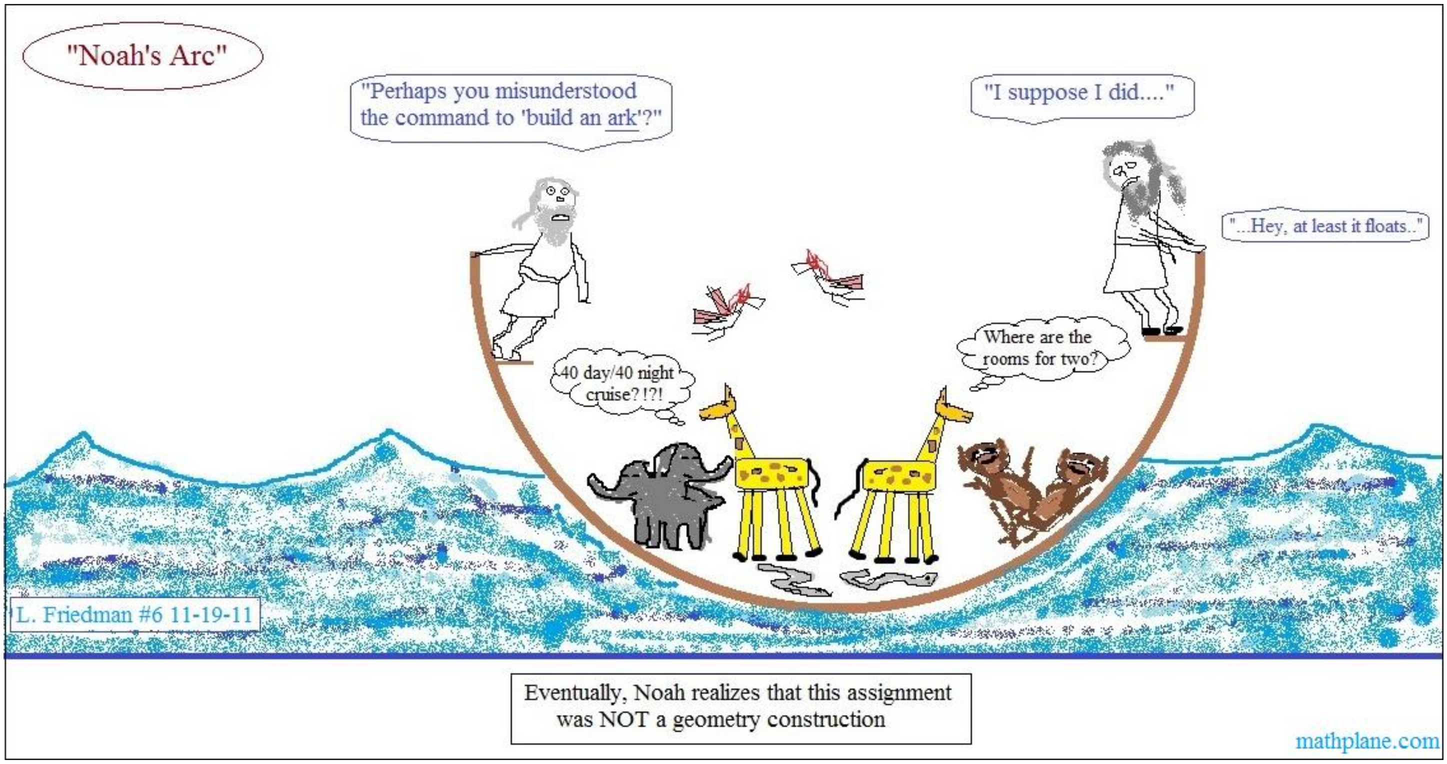
Therefore, the bicycle wheel will turn 10.55 radians..



Finally, determine the distance the bike travels.

Since the diameter is 27.6",

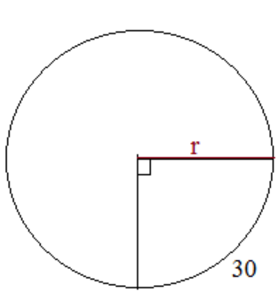
13.8 inches (10.55 radians) = 145.5 inches

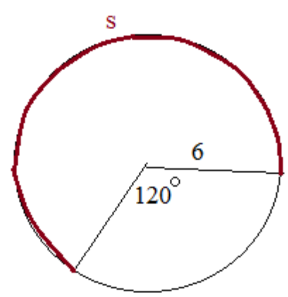


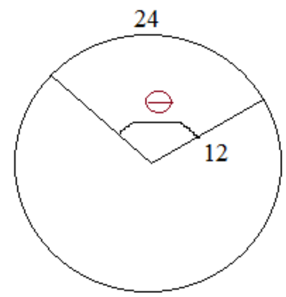
Practice Quiz and Solutions -→

Arc Length and Sector Area

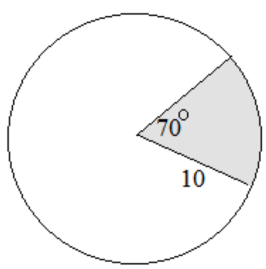
I. Arc Length -- Evaluate the unknown variable:

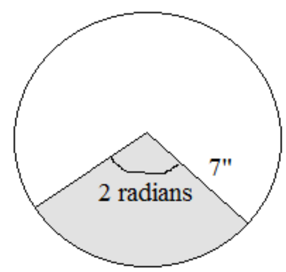






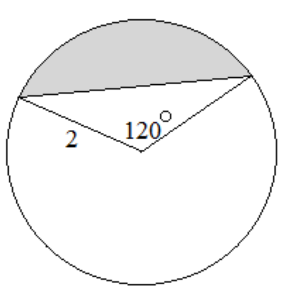
II. Sector Area -- Find the shaded areas:



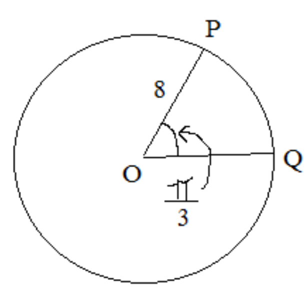


III. Miscellaneous Questions

a) Find the shaded area:

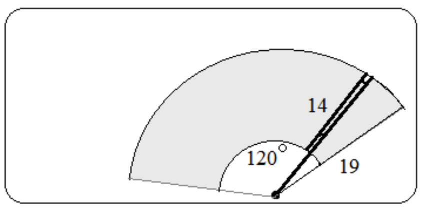


b) Find the perimeter of OPQ

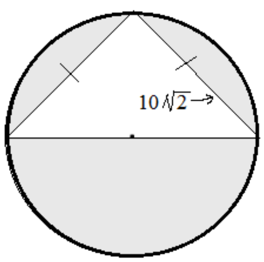


c) A sprinkler rotates 150 degrees back and forth and sprays water up to 20 feet.
How much of the lawn space can the sprinker cover with water?

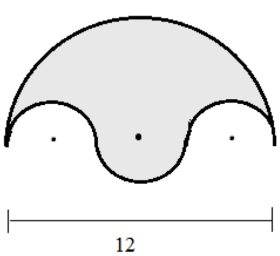
d) A windshield wiper extents 130 degrees. (See diagram)
What area of glass is cleared by the wiper?



e) Find the shaded area:

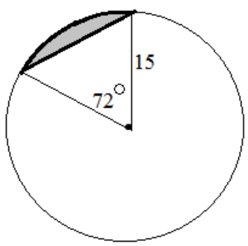


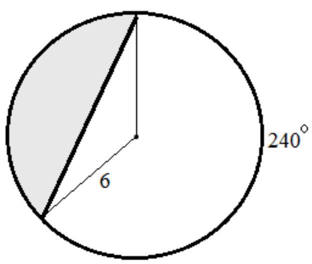
f) Find the shaded area and perimeter of the ("circular") figure:



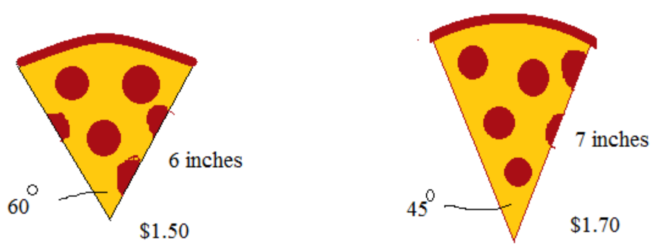
g) The area of a circle is 58 square feet.
What is the circle's diameter?

h) Find the area and perimeter of shaded segments





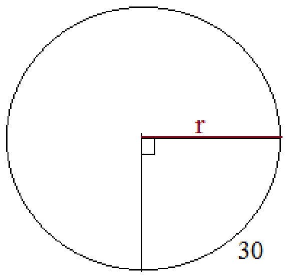
i) Which pizza slice is a better deal?



Arc Length and Sector Area

SOLUTIONS

I. Arc Length -- Evaluate the unknown variable:

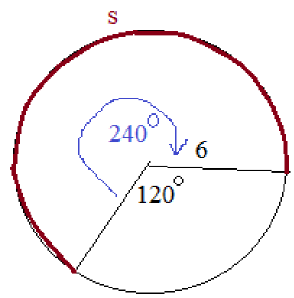


formula for arc length: $s = r\theta$

$30 = r\frac{\pi}{2}$

$30 = 1.57r$

$r \approx 19.1$

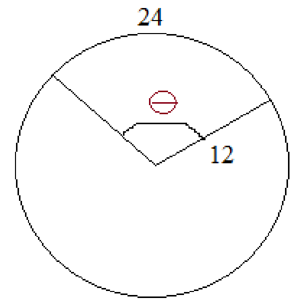


formula for arc length: $\frac{\theta}{360°} \cdot 2\pi r$

$\frac{240}{360} \cdot 2\pi(6) = s$

$s = 8\pi$

$s \approx 25.2$



$s = r\theta$

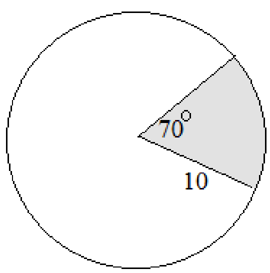
$24 = 12\theta$

$\theta = 2$ radians

(not 2 degrees!)

2 radians is approximately 114 degrees

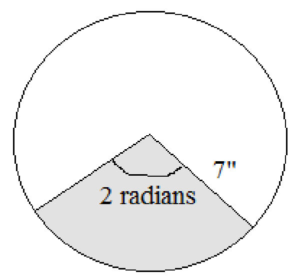
II. Sector Area -- Find the shaded areas:



Since the measure of the central angle is given in degrees, we'll use the following formula:

$\frac{\theta}{360°} \cdot \pi r^2$

$\frac{70°}{360°} \cdot \pi(10)^2 \approx 61.1$ sq. units



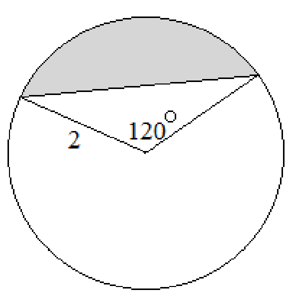
Since the measure is expressed in radians, we'll use the following formula:

$\frac{r^2\theta}{2}$

$\frac{(7")^2 \cdot 2}{2} = 49$ square inches

III. Miscellaneous Questions

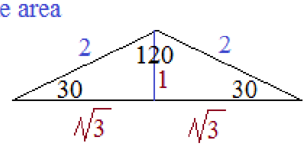
a) Find the shaded area:



sector area ("piece of the pie")

$\frac{120}{360}\pi r^2 = \frac{1}{3} \cdot 4\pi \approx 4.2$

triangle area

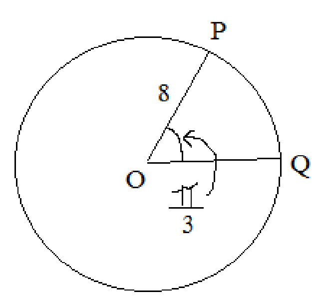


(30-60-90 triangles)

area = 1/2(base)(height) = $1/2(2\sqrt{3})(1) \approx 1.73$

shaded area = sector area − triangle area ≈ 2.47

b) Find the perimeter of OPQ



arc length of $\overset{\frown}{PQ}$

$r\theta = 8 \cdot \frac{\pi}{3}$

≈ 8.4

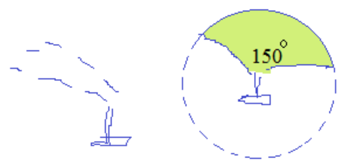
then, since PO = 8 and QO = 8,

the perimeter is

approximately 24.4

SOLUTIONS

c) A sprinkler rotates 150 degrees back and forth and sprays water up to 20 feet.
How much of the lawn space can the sprinkler cover with water?

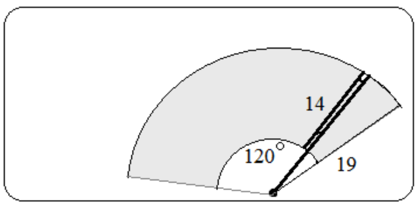


$$\text{Sector Area} = \frac{\theta}{360}\pi(\text{radius})^2$$

$$= \frac{150}{360}\pi(20')^2$$

$$= \boxed{166\tfrac{2}{3}\pi \text{ square feet}}$$

d) A windshield wiper extents 130 degrees. (See diagram)
What area of glass is cleared by the wiper?



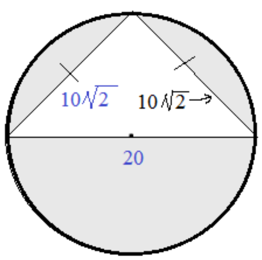
Area of wiper blade = sector area of "outer circle" − sector area of "inner circle"

"outer circle" radius = 19, sector: 120 degrees. Area $= \frac{120}{360}\pi(19)^2 = \frac{361}{3}\pi$

"inner circle" radius = 19 - 14 = 5, sector: 120 degrees. Area $= \frac{120}{360}\pi(5)^2 = \frac{25}{3}\pi$

Area covered by wiper blade $= \boxed{112\pi}$

e) Find the shaded area:



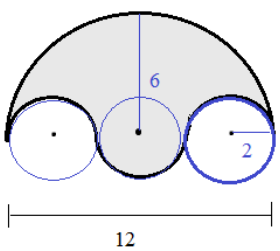
Since a triangle inscribed in a semicircle is a right triangle, we know the diameter is 20.. (radius is 10)

Area of entire circle: 100π

Area of triangle: $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\cdot 10\sqrt{2}\cdot 10\sqrt{2} = 100$

$\boxed{\text{Shaded area} = 100\pi - 100}$ (approximately 214 square units)

f) Find the shaded area and perimeter of the ("circular") figure:



The area of the 'big semicircle' would be

$$\frac{1}{2}\pi(6)^2 = 18\pi$$

Then, we have to cut out 2 'small semicircles' and add 1 'small semicircle'!!

area of each 'small semicircle' is $\frac{1}{2}\pi(2)^2 = 2\pi$

therefore, the shaded area is $18\pi - (2)\,2\pi + (1)\,2\pi = \boxed{16\pi \text{ square units}}$

The perimeter of the 'big semicircle' is $\frac{1}{2}\cdot 2\pi(6) = 6\pi$

then, the perimeter of each 'small semicircle' is $\frac{1}{2}\cdot 2\pi(2) = 2\pi$

Therefore, the perimeter is 1 'big semicircle' and 3 'small semicircles'

$\boxed{12\pi \text{ units}}$

SOLUTIONS

g) The area of a circle is 58 square feet. What is the circle's diameter?

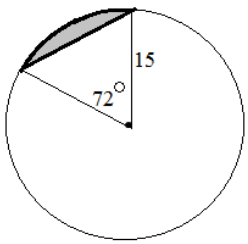
$\text{Area} = \pi(\text{radius})^2$

$58 = \pi(\text{radius})^2$

radius = 4.3 (approximately)

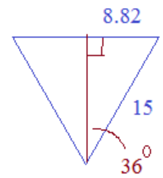
Therefore, the diameter is **8.6 feet (approximately)**

h) Find the area and perimeter of shaded segments



Arc length: $\frac{72}{360}(2\pi \cdot 15) = 6\pi$

Use trig to find segment:



Perimeter = $6\pi + 17.64$

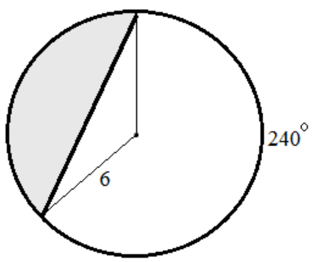
36.5 units

Sector Area: $\frac{72}{360}\pi(15)^2 = 45\pi$

Then, subtract the triangle.

$\frac{1}{2}(17.64)(12.14) = 107$

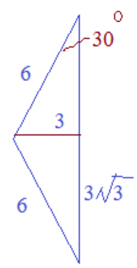
$45\pi - 107 =$ **34.4 sq units**



Arc length: $\frac{120}{360}(2\pi \cdot 6) = 4\pi$

Perimeter = $4\pi + 6\sqrt{3}$

approx 23 units



Sector Area: $\frac{120}{360}\pi(6)^2 = 12\pi$

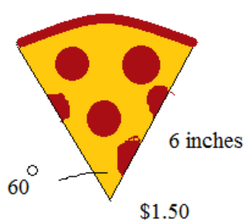
Then, subtract the triangle.

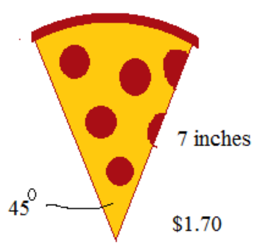
$\text{area} = \frac{1}{2}ab\sin C$

$\frac{1}{2}(6)(6)\sin(120^\circ) = \frac{18\sqrt{3}}{2}$

$12\pi - 9\sqrt{3} =$ **22.1 sq units**

i) Which pizza slice is a better deal?





We're seeking the lower price/area or higher area/price

sector area of 6-inch slice:

$\frac{60}{360}\pi(6\text{ inches})^2 = 6\pi$

$\frac{1.50}{6\pi} = .0796$ / sq inch

12.566 sq inches/dollar

sector area of 7-inch slice:

$\frac{45}{360}\pi(7\text{ inches})^2 = \frac{49}{8}\pi$

$\frac{1.70}{6.125\pi} = .0883$ / sq inch

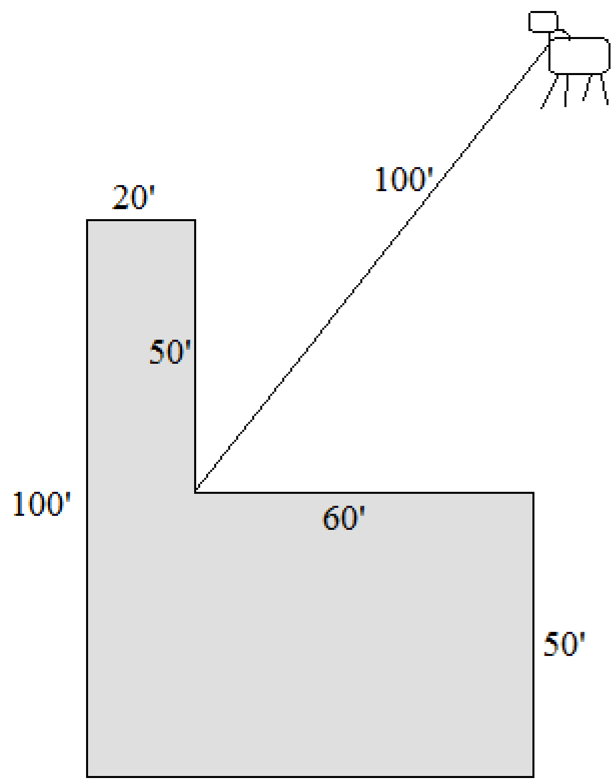
11.319 sq inches/dollar

The 6-inch slice is the better deal!

One more Question:

A cow is tethered to a 100-foot rope, attached to the inside corner of an L-shaped building (as shown in the diagram).

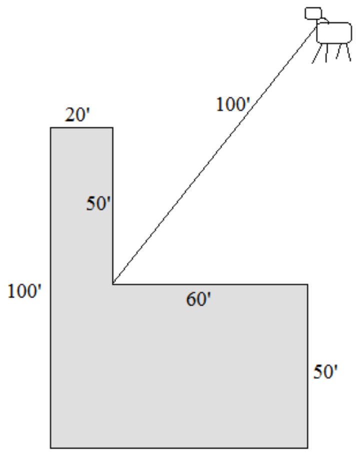
Find the grazing area of the cow.



Solution on next page --→

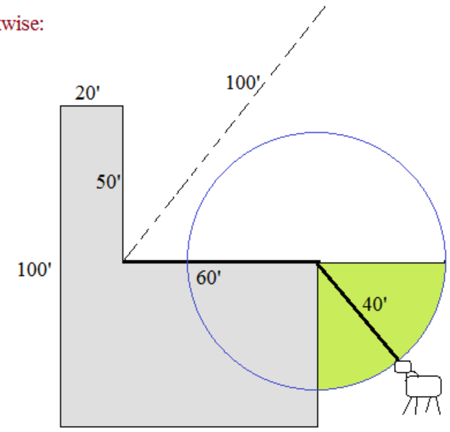
A cow is tethered to a 100-foot rope, attached to the inside corner of an L-shaped building (as shown in the diagram).

Find the grazing area of the cow.



Sector Area = $\frac{\text{angle measure}}{360 \text{ degrees}} \pi r^2$

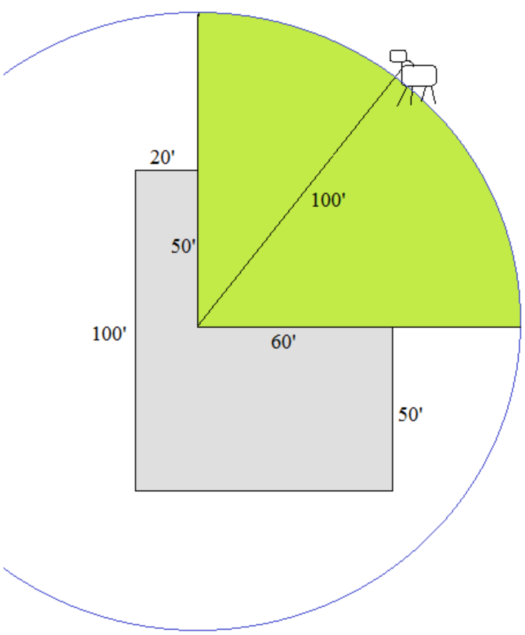
Going clockwise:

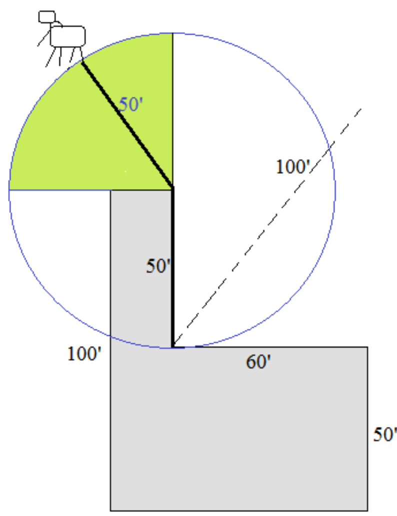


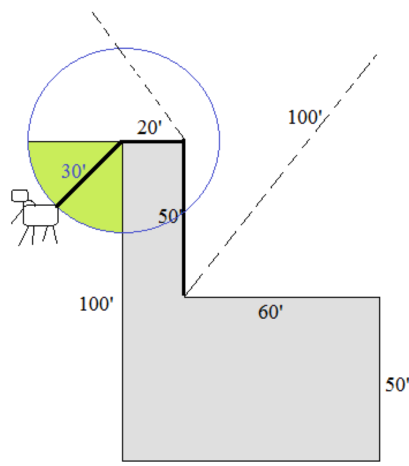
sector area: radius 40'
θ = 90 degrees

$\frac{90 \text{ degrees}}{360 \text{ degrees}} \pi (40')^2 = 400\pi$ square feet

Going Counterclockwise:







Second sector area: radius 50'
θ = 90 degrees

$\frac{90 \text{ degrees}}{360 \text{ degrees}} \pi (50')^2 = 625\pi$ square feet

Third sector area: radius 30'
θ = 90 degrees

$\frac{90 \text{ degrees}}{360 \text{ degrees}} \pi (30')^2 = 225\pi$ square feet

First sector area: radius 100'
θ = 90 degrees

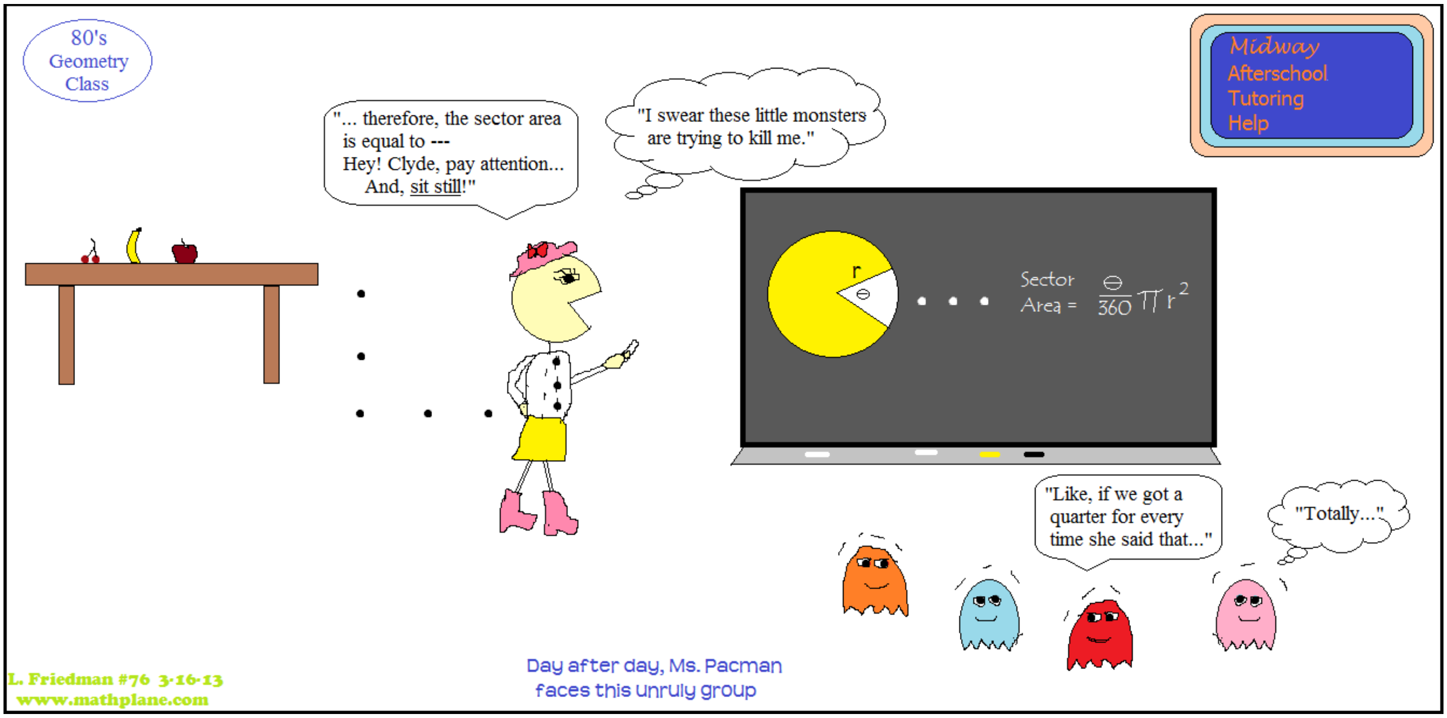
$\frac{90 \text{ degrees}}{360 \text{ degrees}} \pi (100')^2 = 2500\pi$ square feet

Total Grazing area: 3750π square feet

(approximately 11,781 sq. feet)

Thanks for visiting! (Hope it helped!)

If you have questions, suggestions, or requests, let us know.



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