Bewley-Huggett-Aiyagari with Aggregate Shocks

Per Krusell and Tony Smith

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Non-representative-agent macro

- The distribution of wealth and income matters non-trivially for aggregate outcomes: there is no aggregation theorem.
- There is no simple way to summarize aggregate price processes—the ones that matter for decision makers—without understanding how the distribution moves over time.
- This is potentially an impossible problem to solve even numerically. Yet it is a very important (if nothing else, robustness) question in macro.
- There has been a lot of work on this over the last 20 years and is a very active area today.
- Some key insights:
  - Some key decisions "almost" aggregate, making computation feasible, yet delivering interesting results.
  - Some brute-force methods are also available now.
- Not much work has been done, though, with a focus on distributional dynamics (i.e., non-macro work).

The Huggett model with aggregate shocks

- Shock process: labor income is $\epsilon \in \{\epsilon_h, \epsilon_l\}$, but now the transition probabilities depend on an aggregate event $z \in \{g, b\}$. Let $\Pr[Z' = z'|Z = z] = \phi_{zz'}$ and $\Pr[\epsilon = \epsilon_{s'}|\epsilon = \epsilon_s, Z = z, Z' = z'] = \pi_{ss'|zz'}$.
  It is possible to select probabilities so that the fraction of agents with ϵ_h depends only on z: the employment rate jumps up and down between two values (as do aggregate resources).
- Asset markets: contingent claims, one for each aggregate state. Borrowing constraints on each: $a_{g,t+1} \geq \underline{a}_g$ and $a_{b,t+1} \geq \underline{a}_b$ for all t. The contingent claims are in zero net supply: in equilibrium, $\int_{\underline{a}_z} a_{i,z,t+1} di = 0$ for all z, t.
- Aggregate state: includes the distribution of wealth, so it is (Γ, z).
- Individual state: aside from the aggregate state: (ω, ϵ), where ω is all the asset income.

Recursive competitive equilibrium (RCE)

V, h_g, h_b, H, Q_g, and Q_b such that

1. $V(\omega, \epsilon, \Gamma, z)$ solves

$$V(\omega, \epsilon, \Gamma, z) = \max_{a'_g \geq \underline{a}_g, a'_b \geq \underline{a}_b} u(\omega + \epsilon - Q_g(\Gamma, z)a'_g - Q_b(\Gamma, z)a'_b) +$$

$$E[V(a'_{z'}, \epsilon', H(\Gamma, z, z'), z')|z, \epsilon]$$

   for all $(\omega, \epsilon, \Gamma, z)$.

2. $h_g(\omega, \epsilon, \Gamma, z)$ and $h_b(\omega, \epsilon, \Gamma, z)$ attain the maximum, for all $(\omega, \epsilon, \Gamma, z)$, in the above maximization problem.
3. Markets clear: for all (Γ, z) and $j = g, b$:

$$\sum_{\epsilon} \int_{\omega} h_j(\omega, \epsilon, \Gamma, z)\Gamma(d\omega, \epsilon) = 0.$$

RCE (continued)

4. Consistency: for all (Γ, z) and (when applicable) (B, ϵ) and z',

$$H(\Gamma, z, z')(B, \epsilon) = \sum_{\tilde{\epsilon}} \pi_{z', \epsilon | z, \tilde{\epsilon}} \int_{\omega : h_{z'}(\omega, \tilde{\epsilon}, \Gamma, z) \in B} \Gamma(d\omega, \tilde{\epsilon}).$$

Asset pricing with extreme borrowing constraints

- Let $\underline{a}_g = \underline{a}_b = 0$. (Different "trick" than in Constantinides and Duffie (1996), who use permanent shocks, and hence no self-insurance can occur in equilibrium.)
- Then it is straightforward again to show that the equilibrium is autarky; and that the rich agent prices both assets.
- We obtain, for all (z, z'),

$$Q_{zz'} = \beta\phi_{zz'}\left[\pi_{hh|zz'} + (1 - \pi_{hh|zz'})\frac{u'(\epsilon_l)}{u'(\epsilon_h)}\right].$$

  These prices can then be used to price any asset contingent on aggregate events: a bond; "stock"; options; long-term bonds.
- Less extreme borrowing constraints: discussed in the context of the Aiyagari setup, model has to be solved numerically.

The Aiyagari model

- Stochastic structure: as in Huggett case, i.e., z_g (good times) and $z = z_b$ (bad times) for aggregate shock, with

$$\begin{pmatrix} \pi_{g|g} & \pi_{g|b} \\ \pi_{b|g} & \pi_{b|b} \end{pmatrix}.$$

  Also, if $z = z_g$ ($z = z_b$), then the number of unemployed always equals u_g (u_b), with $u_g < u_b$. The joint Markov structure on (z, ϵ) is

$$\Pi' = \begin{pmatrix} \pi_{g1|g1} & \pi_{g1|b1} & \pi_{g1|g0} & \pi_{g1|b0} \\ \pi_{b1|g1} & \pi_{b1|b1} & \pi_{b1|g0} & \pi_{b1|b0} \\ \pi_{g0|g1} & \pi_{g0|b1} & \pi_{g0|g0} & \pi_{g0|b0} \\ \pi_{b0|g1} & \pi_{b0|b1} & \pi_{b0|g0} & \pi_{b0|b0} \end{pmatrix}.$$

- Assets: now a bond, in addition to capital; borrowing constraint on each. One can use contingent claims as well, as earlier.

Recursive competitive equilibrium (RCE)

- State variable: $(\omega, \epsilon; \Gamma, z)$. Key law of motion:

$$\Gamma' = H(\Gamma, z, z').$$

  Note that z' appears!
- Objects: functions V, h^k, h^n, h^b, H, H^n, r, w, and q.
- Consumer maximization: $V(\omega, \epsilon, \Gamma, z)$ solves

$$V(\omega, \epsilon, \Gamma, z) = \max_{k', n, b'} u(\omega + a + n\epsilon w(\Gamma, z) - k' - q(\Gamma, z)b', 1 - n) +$$

$$E[V(b' + k'(1 - \delta + r(H(\Gamma, z, z'), z')), \epsilon', H(\Gamma, z, z'), z')|z]$$

  for all $(\omega, \epsilon, \Gamma, z)$. Also, $h^k(\omega, \epsilon, \Gamma, z)$, $h^n(\omega, \epsilon, \Gamma, z)$, and $h^b(\omega, \epsilon, \Gamma, z))$ attain the maximum, for all $(\omega, \epsilon, \Gamma, z)$, in the above maximization problem.

RCE (continued)

- Input pricing: these functions satisfy, for all (Γ, z),

$$r(\Gamma, z) = F_k(z, \bar{k}, \bar{n}) \text{ and } w(\Gamma, z) = F_n(z, \bar{k}, \bar{n}),$$

  where now $\bar{k} = (\sum_\epsilon \int \omega \Gamma(d\omega))/(1 - \delta + r(\Gamma, z))$ and $\bar{n} = H^n(\Gamma, z)$.

- Market clearing in bonds: $\forall(\Gamma, z)$

$$0 = \sum_\epsilon \int_\omega h^b(\omega, \epsilon, \Gamma, z)\Gamma(d\omega, \epsilon).$$

- Market clearing in labor: $\forall(\Gamma, z)$

$$H^n(\Gamma, z) = \int_\omega h^n(\omega, 1, \Gamma, z)\Gamma(d\omega, 1).$$

RCE (continued)

- Consistency: for all (Γ, z) and (when applicable) (B, ϵ) and z',

$$H(\Gamma, z, z')(B, \epsilon) =$$

$$= \sum_{\tilde{\epsilon}} \pi_{z', \epsilon | z, \tilde{\epsilon}} \int_{\omega : h^k(\omega, \tilde{\epsilon}, \Gamma, z)(1-\delta+r(H(\Gamma,z,z'),z'))+h^b(\omega,\tilde{\epsilon},\Gamma,z) \in B} \Gamma(d\omega, \tilde{\epsilon}).$$

Interlude: why does the distribution matter?

An aggregation theorem (in wealth/assets) holds if

- all agents have the same utility function;
- this utility function is time-separable and of the "right form" (the right form includes $u(c)$ a quadratic, exponential, or power function of an affine function of c, and leisure can be valued too if the aggregator of c and l is homothetic);
- markets are complete.

Here, in the baseline case, these assumptions are met, except for the last one. Markets for insurance against idiosyncratic risk are missing.

So here, the following features are present:

- people of different wealth levels (and with different employment status) have different propensities to save;
- they have different propensities to work;
- they make different portfolio choices (they have different relative demands for contingent claims);
- and the distribution of wealth is nondegenerate;
- so that a redistribution of wealth would influence aggregates.

What is the (quantitative) effect of these deviations from aggregation?

We will also note later that the incomplete-markets friction will interact with other potential frictions not considered in the baseline model.

Computation: overall challenges

Main problem: huge (infinite-dimensional) state variable (Γ), so hard (impossible) to solve the dynamic programming problem for given equilibrium functions.

In addition, even if the dynamic-programming problem can be solved, the nontrivial equilibrium functions (H and H^n) need to be determined—iterated on—and their arguments are infinite-dimensional too.

Different approaches

Approach 1: Krusell and Smith (1998), uses "solution by simulation". Two key steps:

1. solve dynamic-programming problem with few state variables capturing Γ and its evolution over time; interpretation of boundedly rational perceptions of how prices evolve
2. resulting decision rules simulated for many agents, implied behavior compared to boundedly rational perceptions; if close enough, approximation deemed accurate.

It turns out: in (almost) all quantitative applications, has appeared to work well!

Key finding: in many (most) models, what matters for outcomes is only the mean of aggregate wealth. KS label this *approximate aggregation*.

The method itself emphasizes this finding.

Approach 2: Krueger-Kubler—Smolyak collocation approach.

1. Brute force: literally uses a finite number of agents but "efficient" way of dealing with many states. Key: polynomial approximation terms AND grid points constructed that as states are added, not both of these grow exponentially.
2. In "difficult" applications been shown to work well.

Approach 3: Reiter—linearizes the aggregate state and formulates a linear law of motion for it.

1. Non-iterative, building on linearization.
2. Solves individual's problem in steady state fully nonlinearly (linearizes at different points in the state space).
3. Parameterizes the decision rules. All these are then allowed to depend on an aggregate state vector which can be large, e.g., a "histogram vector".
4. Movements in aggregate state, and the dependence on individuals' parameters on aggregate state, all described in linear system.

Comments

- Rare with departures from approximate aggregation. Prize money for finding reasonably calibrated model where approximate aggregation does not hold (two examples notwithstanding).
- Short of such findings, not clear what the "need for computational methods" is.
- Other forms of heterogeneity studied: firms and lumpy investment, price setting, ...
- Many remaining problems in the literature are still perceived as (and may be) difficult to solve.

In the following, KS approach and results are described. Purpose:

- explain computational method
- explain properties of a class of models

Krusell and Smith (1998): “solution by simulation”

Here: explain solution by simulation, i.e., how it works and why it works. This also gives insights into the value of different other approaches.

Solution by simulation, based on boundedly rational forecasts:

- People have a perceived law of motion for the aggregate state (H), allowing enough simplification that their dynamic problems can be solved.
- The decision rules are then used to simulate a large cross-section of consumers, thus obtaining a time series for the aggregate state.
- The obtained time series is compared to the perceptions, and an update is made. Continue until “convergence”.

Argument to be made: this method works. Reason: not that the method is great per se, but that this class of economies allows it to work well!

Algorithm I: trivial price determination within the period

No bonds, no leisure choice. This implies that the prices, w and r, are directly given by $\bar{k}$.

- We assume that agents have incorrect perceptions regarding how the economy works: agents only think prices depend on a finite set of moments of Γ: $m \equiv (m_1, m_2, \ldots, m_I)$.
- The function H is then represented by the function H_I: $m' = H_I(m, z, z')$.
- Given that agents behave based on these perceptions, derive the implied aggregate behavior and check the extent to which the agents' perceptions differ from how the economy behaves.

Iterative procedure

1. Select I.
2. Guess on H_I in the form of some given parameterized functional form, and guess on parameter values.
3. Solve the consumer's problem given H_I, and obtain the implied decision rules f_I.
4. Use these decision rules to simulate the behavior of N agents (with N a large number).
5. Use the "stationary region" of the simulated data to estimate a new set of parameters for H_I.
6. Update the parameters/iterate until a fixed point in these parameters is found. At this stage, we obtain a goodness-of-fit.
7. If the goodness-of-fit is satisfactory, stop. If it is not satisfactory, increase I, or, as a less ambitious step, try a different functional form for H_I.

Example that will work surprisingly well

$I = 1$ and H_I is linear:

$$\begin{aligned} z = z_g: \quad \bar{k}' &= a_0 + a_1 \bar{k} \\ z = z_b: \quad \bar{k}' &= b_0 + b_1 \bar{k} \end{aligned}$$

The agent solves the following problem:

$$v(k, \epsilon; \bar{k}, z) = \max_{c,k'} \{u(c) + \beta E[v(k', \epsilon'; \bar{k}', z')|z, \epsilon]\} \quad \text{s.t.}$$
$$c + k' = F_k(\bar{k}, \bar{l}, z)k + F_l(\bar{k}, \bar{l}, z)\tilde{l}\,\epsilon + (1 - \delta)k$$
$$\bar{k}' = a_0 + a_1 \bar{k} \text{ if } z = z_g$$
$$\bar{k}' = b_0 + b_1 \bar{k} \text{ if } z = z_b$$

Markov law of motion for (z, ϵ), and $k' \geq 0$

$\Rightarrow$ optimal decision rule: $k' = f(k, \epsilon; \bar{k}, z)$.

<u>Chief computational task</u>: Find the fixed point $(a_0^\star, a_1^\star, b_0^\star, b_1^\star)$.

Detail on solving the consumer's problem

General idea: Approximate each of the functions $v(k, 1; \bar{k}, z_g)$, $v(k, 1; \bar{k}, z_b)$, $v(k, 0; \bar{k}, z_g)$, and $v(k, 0; \bar{k}, z_b)$ on a grid of points in the $(k, \bar{k})$ plane. Use cubic spline and polynomial interpolation to compute the value function at points not on the grid. Choices for capital are not restricted to the grid.

1. Choose a grid. Choose an initial set of values for each of the above functions above at each of the grid points.
2. Compute cubic splines using the initial set of values.
3. For each of the four (z, ϵ) pairs, maximize the right-hand side of the Bellman equation at each point in the grid. Record the new optimal value at this grid point. (Use Newton-Raphson optimization routine at grid points for which the borrowing constraint does not bind.)
4. Replace the initial values with the new optimal values. Repeat steps 2 and 3 until the new and old values are close.

Parameter values

A. Model Parameters

- Quarterly model: $\beta = 0.99$, $\delta = 0.025$.
- σ (CRRA) = 1,2,3,4,5
- α (capital share) = 0.36
- average duration of good times = 8 periods
- average duration of bad times = 8 periods
- value of technology shock in good times = 1.01
- value of technology shock in bad times = 0.99
- unemployment rate in good times = 0.04
- unemployment rate in bad times = 0.10
- average duration of an unemployment spell in good times = 1.5
- average duration of an unemployment spell in bad times = 2.5

This implies:

$$\Pi' = \begin{pmatrix} 0.851 & 0.123 & 0.583 & 0.094 \\ 0.116 & 0.836 & 0.031 & 0.350 \\ 0.024 & 0.002 & 0.292 & 0.031 \\ 0.009 & 0.039 & 0.094 & 0.525 \end{pmatrix}.$$

B. Solution algorithm/simulation parameters

- Solving the agent's problem: We use one of many available methods (use cubic splines to build an approximation to the value function).
- Simulations: We simulate the behavior of 5,000 agents over 11,000 periods (first 1,000 periods are dropped). Initial condition: all agents have the same wealth (results are not sensitive to this initial condition).

Findings: approximate aggregation

Good times:

$$\log \bar{k}' = 0.095 + 0.962 \log \bar{k}$$

$$R^2 = 0.999998 \quad \hat{\sigma} = 0.0028\%$$

Bad times:

$$\log \bar{k}' = 0.085 + 0.965 \log \bar{k}$$

$$R^2 = 0.999998 \quad \hat{\sigma} = 0.0036\%$$

Very good fit with only one moment. Evidence:

1. The law of motion for aggregate capital track the data extremely well.
2. The agents' ability to forecast prices is extremely high.
3. Would more moments help in these forecasts? No.

Good fit:

Forecasting accuracy				
	1 quarter ahead		*25 years ahead*	
Variable	*corr*$(x, \hat{x})$	*max % error*	*corr*$(x, \hat{x})$	*max % error*
Capital	0.999999	0.0143	0.999614	0.2373
Rental rate	1.000000	0.0091	0.999888	0.152
Wage rate	0.999999	0.0051	0.999583	0.0855

Can the forecasts be improved in equilibrium?

Include standard deviation, skewness, and kurtosis

Good times:

$$\log \bar{k}' = 0.092 + 0.963 \log \bar{k} + \underset{(69.9)}{0.00087 s_2} - \underset{(-18.0)}{0.00018 s_3} + \underset{(11.9)}{0.00011 s_4}$$

$$\begin{aligned} R^2 &= 0.999999 \quad \text{(before: 0.999998)} \\ \hat{\sigma} &= 0.0018\% \quad \text{(before: 0.0028\%)} \end{aligned}$$

Bad times:

$$\log \bar{k}' = 0.081 + 0.965 \log \bar{k} + \underset{(57.5)}{0.0012 s_2} - \underset{(-23.3)}{0.00029 s_3} + \underset{(15.3)}{0.00019 s_4}$$

$$\begin{aligned} R^2 &= 0.999999 \quad \text{(before: 0.999998)} \\ \hat{\sigma} &= 0.0024\% \quad \text{(before: 0.0036\%)} \end{aligned}$$

Solve the model with more moments: $I = 2$

How do the results change if we let agents perceive that prices depend on an additional moment, i.e. use $I = 2$? They (almost) do not change:

$$\log \bar{k}' = 0.094 + 0.963 \log \bar{k} + 0.00016 \log s_2$$

$$R^2 = 0.999999 \quad \hat{\sigma} = 0.0019\%,$$

$$\log s_2' = 0.048 - 0.019 \log \bar{k} + 0.999 \log s_2$$

$$R^2 = 0.9998 \quad \hat{\sigma} = 0.043\% \qquad \text{in good times and}$$

$$\log \bar{k}' = 0.084 + 0.965 \log \bar{k} + 0.00030 \log s_2$$

$$R^2 = 0.999999 \quad \hat{\sigma} = 0.0026\%,$$

$$\log s_2' = 0.057 - 0.019 \log \bar{k} + 0.994 \log s_2$$

$$R^2 = 0.9995 \quad \hat{\sigma} = 0.073\% \qquad \text{in bad times.}$$

Why doesn't the distribution matter?

- Are the marginal propensities to consume the same across all individuals?
  1. The decision rules for capital are close to linear $\Rightarrow$ most agents with the same employment status have the same savings propensities.
  2. Across agents with different employment status, the marginal propensities to save differ, but by very little.
- Although the propensities to consume do not differ by much, they do differ: it is possible to redistribute capital to get heterogeneity to matter much more. Moreover, the distribution of capital does move significantly over time. (Recall Huggett model with extreme borrowing constraints!)
- However: propensities are significantly different only for the very poorest people, and their effect on $\bar{k}'$ is (almost) nil.
- A multiple-equilibrium phenomenon nevertheless? No. Nothing in the model suggests it. Also, see 2-period model later.

Why are the marginal propensities so similar?

This is not super-well understood. It has to do with the utility costs of variations in consumption:

- Lucas (1987), Cochrane (1989), and others: in representative-agent models, they are very small.
- Asset pricing literature (Telmer (1993), Heaton and Lucas (1996)): with idiosyncratic risks, one asset does very well in terms of providing insurance (in utility terms; here, consumption is much more variable for individuals than for aggregate).
- In a 2-period model, can prove that decisions become linear as wealth increases (reminiscent of Bewley's (1977) foundation for the permanent-income hypothesis).

What model environments lead to more heterogeneity in marginal propensities?

Very good, and important, question!

Algorithm II: nontrivial within-period price determination

Back to the infinite-horizon model, now with bonds and endogenous leisure. Can we use the same method?

This would mean that market clearing will not hold in the simulation (actual choices of bonds and labor will not sum up to zero and H^b, respectively, which is what the perceptions are). Two problems:

- this makes the approximation of a different nature than before (it is not just perceptions that are a little off but other equilibrium conditions as well);
- and the deviation from bond-market clearing explodes over time.

Thus, alternative method needed.

A two-stage procedure using solution by simulation

The first of the steps below is as before; the second is new.

1. Consumers view future prices as being given by the aggregate laws Q and H^n (and, of course, H^k, as before, so that w and r can be computed), but
2. but they observe directly all current prices while making decisions: current prices are parameters in a consumer's problem, and all decision rules depend explicitly on them. In the simulation, prices are then varied to ensure market clearing period by period. At the end, comparison with perceived price functions and update.

The two steps are thus that we need consumers both to solve the kind of problem used in Algorithm I, and an additional one that can be viewed as a one-period deviation from the first kind, where prices in the current period are not exactly given by the perceived functions.

The two-state procedure: more detail

1. Solve dynamic problem, as before, to obtain value functions V, given

$$
\begin{array}{rcrcl}
\bar{k}' & = & H^k(\bar{k}, z) & = & a_{0z} + a_{1z}\bar{k} \\
q & = & Q(\bar{k}, z) & = & b_{0z} + b_{1z}\bar{k} \\
\bar{n} & = & H^n(\bar{k}, z) & = & c_{0z} + c_{1z}\bar{k}.
\end{array}
$$

2. In the actual simulation, the agent solves a "static" problem given the V: he maximizes, by choosing $k' \geq \underline{k}$, $n \in [0, 1]$, and $b' \geq \underline{b}$,

$$u(\omega + F_2(\bar{k}, \bar{n}, z)n\epsilon + a - k' - qb', 1 - n) + \beta\cdot$$

$$E[V(k'(1-\delta+F_1(H^k(\bar{k}, z), H^n(H^k(\bar{k}, z), z'), z')), \epsilon', H^k(\bar{k}, z), z')|z, \epsilon$$

3. Given the implied $h^k(\omega, \epsilon, \bar{k}, z; q, \bar{n})$, $h^n(\omega, \epsilon, \bar{k}, z; q, \bar{n})$, and $h^b(\omega, \epsilon, \bar{k}, z; q, \bar{n})$, vary q and $\bar{n}$ to clear markets in every period of the simulation.

Results

Approximate aggregation works again (really needs detailed explanation. . .).

The aggregate time series				
Model	mean(k_t)	corr(c_t, y_t)	std.dev.(i_t)	corr(y_t, y_{t-4})
Complete markets	11.54	0.691	0.031	0.486
Benchmark	11.61	0.701	0.030	0.481
$\sigma = 5$	12.32	0.741	0.033	0.524
RBC	11.58	0.669	0.027	0.339
Stochastic β	11.78	0.825	0.027	0.459

“Hand-to-mouth consumption” as in Campbell and Mankiw (1989).

In general, looks like a “2-agent model”.

Approximate aggregation $\Rightarrow$ boring model?

No. We have just learned that it does not. The benchmark model is boring, but not the extensions to deal with

- the price of risk
- wealth inequality
- consumption-income fluctuations

and the interpretations are interesting too!

In other work: indivisible labor supply with heterogeneous costs/benefits of working, welfare costs of fluctuations, ...