

Liquidity and Interest Rates*

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Received September 13, 1988; revised April 5, 1989

This paper analyzes a series of models in which money is required for asset transactions as well as for transactions in goods. In these models, government open-market operations induce liquidity effects that lead to interest rate behavior quite different from the behavior one would predict on the basis of Fisherian fundamentals. The paper characterizes these effects under various assumptions about the nature of securities traded and the behavior of shocks. *Journal of Economic Literature* Classification Numbers: 023, 311, 313. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper analyzes a series of models in which money is required for asset transactions as well as for transactions on goods. This modification to more familiar cash-in-advance models of monetary economies is a step toward realism: According to the *Federal Reserve Bulletin*, about 11% of all demand deposits in the United States are held by financial businesses, and financial businesses hold about twice as many deposits per employee as do other businesses. One can imagine societies in which at least the most sophisticated financial markets clear, Arrow–Debreu style, without the use of non-interest-bearing reserves, but this is not the way U.S. financial markets operate today, nor do they show any trend toward operating in such a way.

If cash is required for trading in securities, then the quantity of cash—of “liquidity”—available for this purpose at any time will in general influence the prices of securities traded at that time. That is, the price of a security will in general depend not only on the properties of the income stream to

* Lars Peter Hansen, Robert Hodrick, Nancy Stokey, and Neil Wallace provided very helpful criticism. Chi-Wa Yuen carried out all the calculations. I am also grateful for the comments of seminar participants at U.C.L.A., Yale, the Money and Banking Workshop at Chicago, and the 1988 Summer Research Workshop at Northwestern, and for research support provided by NSF Grant SES 8420420.

which it is a claim—its “fundamentals”—but also on the liquidity in the market at the time is traded. In view of the mounting evidence that theories of asset pricing based solely on fundamentals cannot adequately account for observed movements in securities prices, there should be no difficulty in motivating a theoretical study of a non-fundamental influence on asset prices.¹ Since the paper is frankly exploratory, however, I will defer discussion of the particular observations the theory may help to explain until the concluding section of the paper.

The term “liquidity” is used here in exactly the sense of recent papers by Grossman and Weiss [2] and Rotemberg [13].² These papers worked out the effects of open market operations on interest rate behavior in settings in which the agents on the opposite side of a government sale (or purchase) of bonds hold only a fraction of the economy’s money supply, and have no ability to obtain more money—more liquidity—in time to affect their ability to purchase bonds. In this situation, these authors show, a bond issue will raise interest rates, for reasons having nothing to do with changes in expected inflation or in marginal rates of substitution—the Fisherian fundamentals of interest rate determination. These papers were the first to isolate such an effect, long thought by monetary and financial theorists to be present in reality, in a model of economic equilibrium. This paper is thus a sequel to their analyses.

In [2] and [13], an open market operation that induces a liquidity effect will also alter the distribution of wealth, since agents who participate in the trade will have different post-trade portfolios than those who were absent. These distributional effects linger on indefinitely (as they no doubt do in reality), a fact that vastly complicates the analysis, effectively limiting both papers to the study of a one-time, unanticipated bond issue in an otherwise deterministic setting. This paper studies this same liquidity effect using a simple device that abstracts from these distributional effects. The

¹ A long line of econometric research from Sargent [14] through Hansen and Singleton [5] has failed to confirm a relationship between short-term interest rates and their Fisherian fundamentals, real interest rate movements and expected inflation. An equally long line of work stimulated by the research of LeRoy and Porter [8] and Shiller [15] identifies movements in stock prices that cannot be accounted for by their fundamentals. Though this work has been forcefully challenged, for example by Kleidon [7] and Marsh and Merton [11], I interpret West [16] as confirming these authors’ original conclusions. There is, of course, a vast literature bearing on this issue in addition to these few papers.

² Helpman and Razin [6] also apply a cash-in-advance constraint to securities purchases, with different analytical objectives in mind.

The term “liquidity” is also used in an entirely different sense, to refer to a quality of “moneyness” that different, non-money securities are supposed to possess in differing degrees. In this paper, as in [2, 6, 13], this second sense of liquidity is entirely absent. There are assumed payment functions that can be served by money and for these purposes all other securities are assumed to be equally useless or “illiquid.”

idea is to view agents trading in securities and agents engaging in other activities as members of a single "family" that meets periodically to pool resources and information. This device serves the purpose of permitting us to study situations in which different people face different trading opportunities while still retaining the convenience of the representative household fiction.³ As we will see, it permits us to analyze in a stochastic setting the effects of a very wide variety of monetary policies.

The paper consists of a series of examples, with an emphasis on special cases that can be solved by pencil-and-paper methods. When a new effect is introduced it is useful to experiment with many variations before investing much in any one of them. In the next section a benchmark example, taken from Lucas and Stokey [9], will be used to introduce the liquidity effect in its simplest form. In this example, inflation and liquidity effects determine the interest rate on one period bonds. Section 3 introduces a more general formulation that can accommodate a wide variety of bonds and other securities. Section 4 then specializes to the case in which shocks to the system are serially independent and liquidity effects are transient. Section 5 studies a logically inconsistent case in which goods prices are held fixed, an analysis that will, I hope, provide a useful introduction to the full treatment of the case of Markovian shocks (with a finite number of states) in Section 6. Section 7 describes some numerical simulations designed to give a sharper picture of the effects of serially correlated shocks. Section 8 contains concluding remarks.

2. A BENCHMARK EXAMPLE

Throughout the paper, I will consider a representative agent economy, in which the typical household has preferences

$$E \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\},$$

where $\{c_t\}$ is a stochastic stream of consumption of a single good, β is a discount factor between zero and one, and U is a bounded, twice-differentiable function with $U'(c) > 0$, $U'(0) = \infty$, and $U''(c) < 0$. The household has a constant, non-storable goods endowment y , and in equilibrium, $c_t = y$ for all t and all realizations of shocks that I will specify in a moment. Hence the sole concern of the analysis will be the determination of goods and securities prices under a particular set of trading conventions.

³ Grossman [3] shows that distributional effects are not *necessary* for the occurrence of liquidity effects, using a perfect insurance argument that serves the same function as this "family" construct.

Trading is assumed to proceed in the following way. Think of the typical household as consisting of three members, each of whom goes his own way during a period, the three regrouping at the end of a day to pool goods, assets, and information. One member of the household collects the endowment y , which he must sell to other households on a cash-in-advance basis. A household *cannot* consume any of its own endowment. Cash receipts from the sale of date- t endowment cannot be used for any purpose during period t . A second member of the household takes an amount $N_t - Z_t \geq 0$ of the household's initial cash balances N_t and uses it to purchase goods from other households. If the dollar price of goods is P_t , and if this member spends all of his balances, his household thus consumes the amount $c_t = (N_t - Z_t)/P_t$. A third member of the household takes the remaining cash balances, $Z_t \geq 0$, and engages in securities trading.

This construction of a multiple-member household that pools its resources at the end of each day is the device the permits us to study situations in which different individuals have different trading opportunities during a period, while retaining the simplicity of the representative household. It will be retained in all the examples I consider. It is very similar in its effect to the perfect insurance assumptions used by Rogerson [12] and Hansen [4] to achieve the same analytical end in different contexts.

These features are common to all the examples considered in the paper. The examples differ with respect to the securities that are assumed to be traded and the nature of the policies affecting the supplies of these securities. In the initial example considered in this section, the *only* security we consider is a one period, dollar denominated government bond that entitles its purchaser to one dollar at the beginning of the following period, prior to any trading. These bonds are auctioned off in the securities market at a price q_t . Thus a household beginning with N_t dollars that chooses the division Z_t of these balances can acquire $B_t \leq Z_t/q_t$ bonds. This household will begin the following period with cash balances given by⁴

$$N_{t+1} = P_t y + Z_t + (1 - q_t) B_t. \quad (2.1)$$

⁴ Neil Wallace suggested the following alternative conventions about timing. Require the household to attend securities trading at date $t + 1$ to obtain—in cash—the face value B_t of the bonds it purchased at date t (as opposed to getting B_t dollars in cash mailed to the house, as I am assuming). Under this assumption, a dollar unspent in securities trading is a different security from a zero-interest bond, since the dollar can be spent on goods next period while the bond cannot. This assumption gives a kind of “liquidity” (in a different sense from the way I am using the term in this paper) advantage to cash over bonds that is, I think, a step toward realism from the present model. On the other side, it complicates the analysis by necessitating the use of *two* variables to describe the state of a household, one to describe its cash holdings and another to describe where these holdings are located. I have analyzed this interesting variation in [10].

The size of the government bond issue, expressed relative to the economy's beginning-of-period money stock, will be taken to be an i.i.d. random variable x_t , with a probability distribution λ on a compact set $X \subset (0, \infty)$. That is, if there are M_t dollars outstanding, the government auctions off claims to $x_t M_t$ dollars payable one period hence. A bond issue of size x_t thus withdraws $q_t x_t M_t$ dollars from private circulation today and returns $x_t M_t$ tomorrow. The ratio M_{t+1}/M_t is then the random variable $1 + (1 - q_t) x_t$. Aside from these stochastic open market operations there are no other shocks to this economy.

For this model, a critical issue will be whether the open market shock x_t is taken to be realized before agents commit themselves to a cash division Z_t or after this decision is made. Throughout the paper, attention is restricted to the case where x_t is announced *after* households have made their decisions on the allocation of cash between its two uses. I assume as well that only agents present in the securities market observe the current shock; agents in goods markets do not. With i.i.d. shocks, this will mean that neither the division of cash balances nor the price of goods will depend on the open market shock x_t . The price of one period bonds will be the only variable responding to these shocks.

It will be convenient to use a normalization employed in [9]. Let $m_t = N_t/M_t$ denote a household's money holdings *relative to* the economy-wide average beginning-of-period money holdings (so that $m_t = 1$ in equilibrium). Similarly, let $z_t = Z_t/M_t$, $b_t = B_t/M_t$, and $p_t = P_t/M_t$. In terms of these normalized variables, (2.1) becomes

$$m_{t+1} = [1 + (1 - q_t) x_t]^{-1} [p_t y + z_t + (1 - q_t) b_t]. \quad (2.2)$$

I will define a stationary equilibrium consisting of a constant (normalized) price level $p > 0$, a constant division of money balances $0 \leq z < 1$, and a bond price $q(x)$ consistent with utility maximizing behavior and market clearing. Let $v(m)$ be the maximized objective function for a household beginning a period with (normalized) balances m . The above description of the household's decision problem motivates the Bellman equation,

$$v(m) = \max_{0 \leq z \leq m} \left\{ U[(m - z)/p] + \beta \int_X \max_{0 \leq q(x) b \leq z} [v(m')] \lambda(dx) \right\}, \quad (2.3)$$

where m' is defined by

$$m' = [1 + (1 - q(x))x]^{-1} [py + z + (1 - q(x))b]. \quad (2.4)$$

Then an equilibrium is defined as a value function $v: \mathbf{R}_+ \rightarrow \mathbf{R}$, a number $z \in [0, 1]$, a number $p > 0$, a bond purchase function $b: X \rightarrow \mathbf{R}$, and a bond

price function $q: X \rightarrow (0, 1]$ such that (i) given p and $q(x)$, $v(m)$ satisfies (2.3); (ii) z and b attain the right side of (2.3) at $m = 1$; (iii) $1 - z = py$; and (iv) $b(x) = x$ for all $x \in X$. Conditions (i) and (ii) describe utility maximization at equilibrium prices. Conditions (iii) and (iv) require cleared goods and bonds markets.⁵

Here and in later sections, I will proceed to use first order and envelope conditions for the problem (2.3) to characterize equilibrium behavior, assuming that value functions exist and are increasing, differentiable, and concave. For the inner maximization in (2.3), the possibilities are $q(x) = 1$ and any feasible value of b , $q(x) < 1$ and $b = z/q(x)$, or $q(x) > 1$ and $b = 0$. This last is not an equilibrium possibility, since $x > 0$ and (iv) must hold.

For any m , the unique maximizing value of z in the outer maximization in (2.3) satisfies the first-order condition

$$\frac{1}{p} U' \left(\frac{m-z}{p} \right) = \beta \int_X v'(m') \frac{1}{q(x)[1-xq(x)+x]} \lambda(dx),$$

where m' is given in (2.4). The envelope condition for m is

$$v'(m) = \frac{1}{p} U' \left(\frac{m-z}{p} \right).$$

In equilibrium, $m = m' = 1$ and $(1-z)/p = y$. Thus we can cancel the term $(1/p) U'(y)$ and obtain the equilibrium condition

$$1 = \beta \int_X \frac{1}{q(x)[1-xq(x)+x]} \lambda(dx). \quad (2.5)$$

Since $q(x) = \min[1, z/x]$, we can eliminate $q(x)$ from (2.5) to obtain

$$z = \beta \int_X \max \left\{ z, \frac{x}{1+x-z} \right\} \lambda(dx). \quad (2.6)$$

The right side of (2.6) is positive at $z = 0$ and equal to β at $z = 1$; it is a continuous and strictly increasing function of z , with a slope strictly less than one if $z < 1$. Hence (2.6) has a unique solution $z^* \in (0, 1)$. Then $p = (1 - z^*)/y$ is the equilibrium price level, and $q(x) = \min[1, z^*/x]$ is the equilibrium bond price function.

⁵ Of course, this narrow definition of equilibrium rules out many possibilities characterized by optimal consumer behavior and cleared markets that one might well want to call equilibria. For example, I have no reason to believe that the assumptions I am using preclude the existence of nonstationary equilibria or sunspot equilibria. On the contrary, on the basis of Woodford's analysis [17] of a version of the model in [9], I would conjecture that the present model does have sunspot equilibria, in addition to the stationary equilibrium I will characterize.

This example can be used to illustrate the potential force of the liquidity effect on the stochastic behavior of interest rates. Consider the case when $q(x) < 1$ for all $x \in X$ so that $q(x) = z^*/x$.⁶ Then the interest rate is the i.i.d. random variable

$$r_t \cong -\ln[q(x_t)] = -\ln(z^*) + \ln(x_t).$$

The Fisherian fundamentals for the interest rate are the real rate, constant in this example at $-\ln(\beta)$, plus the expected inflation rate, $\ln(P_{t+1}/P_t) = \ln(M_{t+1}/M_t) \cong -x_t q(x_t) + x_t = -z^* + x_t$. (Here I am assuming that the expectation of inflation is formed after x_t is realized.) The variance of the interest rate is thus $\text{Var}[\ln(x_t)]$ while the variance one would predict on the basis of the variability of fundamentals is $\text{Var}(x_t)$. Since x_t is a small fraction (the short term bond issue in period t divided by the total money supply), the interest rate in this example is *much* more variable than one would predict on the usual Fisherian grounds.

In the rest of the paper, I will work through a number of variations on this example, in an attempt to get a better idea of which aspects of this liquidity effect are due to the peculiarities of the example and which obtain more generally. One possibility would be to examine the liquidity effect in a context, such as that used in [9], in which real and monetary fundamentals follow a much more generally specified stochastic process. It is clear from the example just discussed that liquidity effects and fundamentals can interact, and introducing a more general process for the fundamentals would enable us to study these interactions more fully. But trying to study too many complicated things at once carries the risk of misunderstanding any one of them, so instead I will turn in the opposite direction and analyze examples of what might be called a "pure" liquidity effect. I will use the present example to explain what I mean by this.

In the example above endowments and real consumption are constant, so marginal rates of substitution and hence real interest rates are constant. Changes in interest rates result from a mix of expected inflation effects and liquidity effects, both driven by the same random variable x_t . Now suppose we introduce into the model a lump sum tax of the magnitude $\pi(x_t) M_t$, payable at the beginning of period $t+1$, prior to any trading. With this modification, the ratio M_{t+1}/M_t in the model becomes $1 + (1 - q(x_t))x_t - \pi(x_t)$. Given any equilibrium bond price function q it is clearly possible to choose this transfer function π so as to maintain money growth at zero, $M_{t+1}/M_t = 1$, for all realizations x_t . Under the assumption that π is so

⁶ This is a logical possibility. For example, let $\beta = \frac{1}{2}$, let $x = \frac{1}{4}$ with probability $\frac{1}{2}$ and let $x = \frac{1}{2}$ with probability $\frac{1}{2}$. Then $z^* = 0.149$, so that $q(x) = z^*/x < 1$ for both possible values of x .

chosen, the factor $1 + (1 - q(x))x$ drops out of equation (2.4) and the equilibrium condition (2.6) becomes

$$z = \beta \int_x \max(z, x) \lambda(dx). \quad (2.7)$$

The right side of (2.7) is a continuous function of z , positive at $z = 0$, with a slope between zero and one. Thus (2.7) has a unique solution $z^* > 0$. This solution will be less than one (and hence be interpretable as an equilibrium) if and only if:

$$\frac{\beta}{1 - \beta} \int_1^\infty (x - 1) \lambda(dx) < 1. \quad (2.8)$$

What does condition (2.8) mean, and why was nothing like it required for existence of a solution to the original Eq. (2.6)? Roughly speaking, if (2.8) is violated, it is because there is enough probability on the contingency that a large bond issue may make bonds such as bargain that, at constant goods prices, consumers gain by putting more cash into securities than any fraction $z < 1$. Technically, this means that no stationary equilibrium exists (though an equilibrium with rising prices, ruled out by definition in my analysis, might exist). In the formulation leading to (2.6), large bond issues are always associated with the prospect of a large influx of money, so low bond prices are needed to compensate for expected inflation and are not a bargain in this sense. In future sections, where only pure liquidity effects are studied in constant-money environments, some analogue to (2.8) will always be needed to guarantee the existence of a stationary equilibrium.

3. PURE LIQUIDITY EFFECTS: A GENERAL FRAMEWORK

The remainder of the paper will deal with variations on the second of the two examples in the last section, generalizing it to accommodate a wider variety of securities and more generally specified shocks to the supplies of these securities. Preferences and technology will be exactly as in the last section, as will the timing and nature of trading. I will assume that lump sum transfers maintain the money supply at a constant level, and normalize this level at unity. Hence all values will be expressed as relative to the existing quantity of money. In this section I will set up a general notation and work out some issues that are common to all the examples that are worked through later in the paper.

As in Section 2, the only shocks to the economy will be changes in the

supplies of securities. This suggests taking stocks of securities as the state variables of the system, but it turns out to be more convenient to specify the motion of the state more abstractly and to use functions defined on the state space to define various aspects of securities. Let $s_t \in S$, where (S, \mathcal{S}) is a measurable space, be a complete description of the state of the economy at the beginning of date t . Assume that the state follows a Markov process with the transition function P :

$$P(s, A) = \Pr\{s_{t+1} \in A \mid s_t = s\}, \quad s \in S, \quad A \in \mathcal{S}.$$

A given state s_t determines, via a given function $a: S \rightarrow D \subset \mathbb{R}^n$, a vector $a(s_t)$ of the supplies of n securities that are available prior to trading at date t . Think of s_{t+1} as being realized after cash is divided in period t , but before securities are traded. The states s_t and s_{t+1} together determine, via a function $\phi: S \times S \rightarrow D$, the stocks $\phi(s_t, s_{t+1})$ held at the end of trading on date t . I assume that end-of- t and beginning-of- $t+1$ holdings are linearly related, so that there is an $n \times n$ matrix B such that

$$a(s_{t+1}) = B\phi(s_t, s_{t+1}) \quad \text{for all } (s_t, s_{t+1}).$$

Finally, a holder of the vector $u \in D$ at the end of t receives the inner product $\pi(s_t, s_{t+1}) \cdot u$ in cash at the beginning of $t+1$, where $\pi: S \times S \rightarrow \mathbb{R}^n$ is another given function. This function π specifies the dollar payment the holder of the security is entitled to receive from the issuer at each state-date (s_t, s_{t+1}) combination. It represents a contractual obligation, not a market price. These three given functions a , ϕ , and π , together with the matrix B , define the set of securities assumed to be traded.

The purpose of this notation is to capture at the same time fixed maturity and infinite maturity securities. Thus if the only security in existence is a consol with a unit coupon, we would let $s_t \in \mathbb{R}$ be the stock of such consols, and define $a(s_t) = s_t$, $\phi(s_t, s_{t+1}) = s_{t+1}$, $\pi(s_t, s_{t+1}) = 1$, and let B be the 1×1 matrix $[1]$. But with fixed maturity securities, an end-of- t n -period security becomes a beginning-of- $(t+1)$ $(n-1)$ -period security or, if $n=1$, passes out of existence entirely. Thus in the one-period bond example studied in the last section, we would let $s_t = x_{t-1}$, $a(s_t) = 0$, $\phi(s_t, s_{t+1}) = s_{t+1} = x_t$, $\pi(s_t, s_{t+1}) = 1$ and let $B = [0]$. Later sections will provide other examples of particular specifications of these functions. For all the securities I will consider, B will be block-diagonal, with blocks equal to identity matrices or else having ones on the diagonal above the main diagonal and zeroes elsewhere ($b_{i,i+1} = 1$ for $i = 1, \dots, n-1$ and $b_{ij} = 0$ for $j \neq i+1$), but imposing this structure here would not simplify the discussion in this section.

In this setting, let $q(s, s')$ be the vector of securities prices when the current and next period states are (s, s') . Then the liquidity constraint for

a household that carries z units of cash into securities trading and trades from the portfolio a to the portfolio u takes the form

$$z \geq q(s, s') \cdot (u - a). \quad (3.1)$$

(Cash must cover net purchases.) This household will begin next period with cash balances of

$$m' = \pi_0(s, s') + z - q(s, s') \cdot (u - a) + \pi(s, s') \cdot u, \quad (3.2)$$

where $\pi_0(s, s')$ denotes net cash inflow from sources other than securities (uncapitalized endowment income plus subsidies less taxes).

The household's functional equation is

$$v(m, a, s) = \max_{0 \leq z \leq m} \left\{ U\left(\frac{m-z}{p(s)}\right) + \beta \int_S [\max_u v(m', Bu, s')] P(s, ds') \right\}, \quad (3.3)$$

where m' is defined by (3.2) and where the inner maximization is subject to the constraint (3.1).

An equilibrium is defined, then, as a value function $v: \mathbf{R}_+ \times D \times S \rightarrow \mathbf{R}$, a cash allocation function $z: S \rightarrow [0, 1]$, a securities purchase function $u: S \times S \rightarrow D$, a goods price function $p: S \rightarrow \mathbf{R}_{++}$, a securities price function $q: S \times S \rightarrow \mathbf{R}_+^n$, and a transfer function $\pi_0: S \times S \rightarrow \mathbf{R}$ such that

- (i) given p and q , v satisfies (3.3);
- (ii) z and u attain the right side of (3.3) at $(m, a, s) = (1, a(s), s)$;
- (iii) $1 - z(s) = p(s) y$ for all $s \in S$;
- (iv) $u(s, s') = \phi(s, s')$ for all $(s, s') \in S \times S$; and
- (v) $\pi_0(s, s') + z(s) - q(s, s') \cdot [\phi(s, s') - a(s)] + \pi(s, s') \cdot \phi(s, s') = 1$

for all $(s, s') \in S \times S$.

As in the last section, I will use the first-order and envelope conditions for problem (3.3) to characterize equilibria. The first-order conditions for the inner maximization in (3.3) are

$$\sum_{j=1}^n b_{ji} v_{a_j}(m', Bu, s') = v_m(m', Bu, s') [q_i(s, s') - \pi_i(s, s')] + \mu(s, s') q_i(s, s'), \quad i = 1, 2, \dots, n, \quad (3.4)$$

where $\mu(s, s')$ is the non-negative multiplier associated with (3.1). If $\mu(s, s') > 0$, then (3.1) must hold with equality. (Note that if the matrix B has the near-diagonal form that it will assume in all the examples later in the paper, at most one of the coefficients b_{ji} in the sum on the left of (3.4) will be non-zero.)

The first-order condition for the outer problem is

$$U' \left(\frac{m-z}{p(s)} \right) \frac{1}{p(s)} \geq \beta \int_S [v_m(m', Bu, s') + \mu(s, s')] P(s, ds'), \quad (3.5)$$

with equality if $z > 0$.

The envelope conditions are

$$v_m(m, a, s) = U' \left(\frac{m-z}{p(s)} \right) \frac{1}{p(s)} \quad (3.6)$$

\ and

$$v_{a_i}(m, a, s) = \beta \int_S [v_m(m', Bu, s') + \mu(s, s')] q_i(s, s') P(s, ds'), \quad i = 1, \dots, n. \quad (3.7)$$

In an equilibrium, it must be the case that $m = m' = 1$, $a = a(s)$, $Bu = a(s')$, $u = \phi(s, s')$, and $p(s)y = 1 - z(s)$. Then (3.6) implies that

$$v_m(1, a(s), s) = yU'(y) \frac{1}{1 - z(s)}.$$

Define the functions $\varphi_i: S \rightarrow \mathbf{R}$, $i = 1, \dots, n$ by $\varphi_i(s) = [yU'(y)]^{-1} v_{a_i}(1, a(s), s)$ and define $\theta: S \times S \rightarrow \mathbf{R}$ by $\theta(s, s') = [yU'(y)]^{-1} \mu(s, s')$. Making these substitutions in (3.4), (3.5), and (3.7) we obtain

$$\sum_{j=1}^n b_{ji} \varphi_j(s') = \frac{1}{1 - z(s')} [q_i(s, s') - \pi_i(s, s')] + \theta(s, s') q_i(s, s'), \quad i = 1, \dots, n, \quad (3.8)$$

$$\frac{1}{1 - z(s)} \geq \beta \int_S \left[\frac{1}{1 - z(s')} + \theta(s, s') \right] P(s, ds'), \quad (3.9)$$

with equality if $z(s) > 0$, and

$$\varphi_i(s) = \beta \int_S \left[\frac{1}{1 - z(s')} + \theta(s, s') \right] q_i(s, s') P(s, ds'), \quad i = 1, \dots, n. \quad (3.10)$$

The constraint (3.1) becomes, in equilibrium,

$$z(s) + q(s, s') \cdot (a(s) - \phi(s, s')) \geq 0, \quad (3.11)$$

with equality if $\theta(s, s') > 0$.

If we let $\varphi(s) = (\varphi_1(s), \dots, \varphi_n(s))$ then Eqs. (3.8) and (3.10) can be written more compactly as

$$\frac{1}{1-z(s')} [q(s, s') - \pi(s, s')] + \theta(s, s') q(s, s') = B^T \varphi(s') \quad (3.12)$$

and

$$\varphi(s) = \beta \int_s \left[\frac{1}{1-z(s')} \pi(s, s') + B^T \varphi(s') \right] P(s, ds'), \quad (3.13)$$

where B^T denotes the transpose of the matrix B .

Now using (3.12) to solve for the vector $q(s, s')$, inserting this expression into (3.11), and rearranging gives

$$\frac{1}{1-z(s')} + \theta(s, s') \geq \frac{1}{z(s)} \left[B^T \varphi(s') + \frac{1}{1-z(s')} \pi(s, s') \cdot [\phi(s, s') - a(s)] \right] \quad (3.14)$$

with equality if $\theta(s, s') > 0$. Hence (3.9) can be written

$$\frac{z(s)}{1-z(s)} = \beta \int_s \left\{ \max \left[\frac{z(s)}{1-z(s')}, \left[B^T \varphi(s') + \frac{1}{1-z(s')} \pi(s, s') \right] \cdot [\phi(s, s') - a(s)] \right] \right\} P(s, ds'). \quad (3.15)$$

(Note that (3.15) must hold with equality even if $z(s) = 0$, since the right side is non-negative.)

We can view (3.13) and (3.15) as $n+1$ equations in the unknown functions $z(s)$, $\varphi_1(s)$, ..., $\varphi_n(s)$. If they can be solved, then equilibrium asset prices are given, using (3.12), by

$$q(s, s') = \left[\frac{1}{1-z(s')} + \theta(s, s') \right]^{-1} \left[B^T \varphi(s') + \frac{1}{1-z(s')} \pi(s, s') \right], \quad (3.16)$$

where

$$\begin{aligned} & \frac{1}{1-z(s')} + \theta(s, s') \\ &= \max \left\{ \frac{1}{1-z(s')}, \frac{1}{z(s)} \left[B^T \varphi(s') + \frac{1}{1-z(s')} \pi(s, s') \right] \cdot [\phi(s, s') - a(s)] \right\}. \end{aligned} \quad (3.17)$$

The rest of this paper is concerned with using Eqs. (3.13) and (3.15) to characterize solutions for $z(s)$ and $\varphi(s)$ under various assumptions about

the nature of the securities being traded, and then using (3.16) and (3.17) to characterize equilibrium securities prices. The case of shocks with serially independent increments is treated in the next section, while Sections 5, 6, and 7 consider serially correlated shocks.

4. THE CASE OF INDEPENDENT SHOCKS

Consider the special case in which $a(s_t) = s_t$, so the state is just the stock of outstanding securities. Let $\phi(s_t, s_{t+1}) = s_t + x_t$, where $\{x_t\}$ is a sequence of independent shocks with common distribution λ on $X \subset \mathbf{R}^n$. Thus x_t has the interpretation as new issues at date t . Assume that the cash payout function π is constant. This is the case in which current issues give *no* information about the distribution of issues or cash payouts in the future. Intuition suggests that under this assumption the system (3.13) and (3.15) will have a constant solution (z^*, φ^*) . Why? Because this is a Modigliani–Miller–Ricardian-equivalence world, except for liquidity effects, so the outstanding stocks of securities should not matter unless they help to predict future liquidity effects.

Under these assumptions, and if the conjecture of a constant solution is correct, Eqs. (3.13) and (3.15) become

$$\varphi = \beta \int_X \left[\frac{1}{1-z} \pi + B^T \varphi \right] \lambda(dx), \quad (4.1)$$

$$\frac{z}{1-z} = \beta \int_X \left\{ \max \left[\frac{z}{1-z}, \left[\frac{\pi}{1-z} + B^T \varphi \right] \cdot x \right] \right\} \lambda(dx). \quad (4.2)$$

Assume that the matrix

$$[I - \beta B]^{-1} = \lim_{k \rightarrow \infty} [I + \beta B + \beta^2 B^2 + \dots + \beta^k B^k]$$

exists (as it will when B has the near-diagonal form used in all the examples studied below). Then (4.1) has the solution

$$\varphi = \beta \frac{1}{1-z} [I - \beta B^T]^{-1} \pi.$$

Substituting into (4.2) and cancelling the factor $(1-z)^{-1}$, we obtain

$$z = \beta \int_X \left\{ \max [z, [(I - \beta B^T)^{-1} \pi] \cdot x] \right\} \lambda(dx), \quad (4.3)$$

where the fact that $I + \beta B^T [I - \beta B^T]^{-1} = [I - \beta B^T]^{-1}$ has been applied. Solutions $z \in [0, 1)$ to (4.3) correspond to equilibria.

Define the function $\eta: X \rightarrow \mathbf{R}$ by $\eta(x) = [(I - \beta B')^{-1} \pi] \cdot x$. Then the scalar random variable $\omega = \eta(x)$ may be interpreted as a kind of total "value" of the vector x of new issues. Let μ denote the probability distribution of this random variable ω . Then (4.3) can be written

$$z = \beta \int \max(z, \omega) \mu(d\omega). \quad (4.4)$$

Equation (4.4) will have a unique solution $z^* \in [0, 1)$ if and only if

$$\int_1 [\omega - 1] \mu(d\omega) < \frac{1 - \beta}{\beta}, \quad (4.5)$$

which may be compared to (2.8). If $\int_0^\infty \omega \mu(d\omega) > 0$ (that is, if bonds are ever issued) then $z^* > 0$. We have shown, then, that if (4.5) holds a constant solution exists for this case of independent shocks. I do not know if there are other solutions to (3.13) and (3.15) for this case.

Let us characterize the constant solution for some even more specific sub-cases.

EXAMPLE 4.1: Consols. Let there be only one security in the system: a consol with the coupon payment $\pi = 1$. Then x_t is just the issue (possibly negative) of new consols at t and B is the matrix $[1]$. The random variable ω_t is equal to $(1 - \beta)^{-1} x_t$, which is of the order of the *value* of new consol issues at t . Hence the existence criterion (4.5) will be satisfied if the value of new issues cannot exceed the existing money supply.

As a check on units, suppose x_t is constant at the positive level x^* . Then the liquidity constraint is always just binding, so that (3.11) implies that $q = z^*/x^*$. Equation (4.4) implies $z^* = (1 - \beta)^{-1} \beta x^*$. When these facts are combined, the equilibrium price of a consol is $q^* = (1 - \beta)^{-1} \beta$. If we define the rate of time preference ρ in the usual way by $\beta = (1 + \rho)^{-1}$, then $q^* = 1/\rho$, which is just right as the price of a stream of \$1 payments starting one period hence, under my assumption that \$1 is not risky in real terms.

More generally, if (4.5) holds, so that (4.4) determines a unique equilibrium $z^* \in [0, 1)$, then the equilibrium bond price function $q(x)$ can be obtained from (3.12) and (3.14). Specialized to this example, these imply

$$q(x) = \left[\frac{1}{1 - z} + \theta(x) \right]^{-1} \left(\frac{1}{1 - z} \right) \left(\frac{1}{1 - \beta} \right),$$

where

$$\frac{1}{1 - z} + \theta(x) = \max \left\{ \frac{1}{1 - z}, \left(\frac{1}{1 - z} \right) \left(\frac{1}{1 - \beta} \right) \frac{x}{z} \right\}.$$

We conclude that

$$q(x) = \begin{cases} z/x & \text{if } x > (1 - \beta)z \\ \frac{1}{1 - \beta} & \text{if } x \leq (1 - \beta)z. \end{cases}$$

Note that in the case where the liquidity constraint is slack, $x \leq (1 - \beta)z$, the consol price is $(1 + \rho)\rho^{-1}$, not ρ^{-1} . It is as if the current one period rate is zero and all forward rates are ρ .

EXAMPLE 4.2: Fixed Maturity Bonds. As a second example, suppose that (pure discount) bonds are issued maturing in $1, 2, \dots, n$ periods, each bond entitling its holder to \$1 at maturity and nothing until then. Let $x_t = (x_{1t}, \dots, x_{nt})$ describe new issues at t , where $\{x_{it}\}$ are independent random variables with the common distribution λ on X . In this example, B is an $n \times n$ matrix with ones on the diagonal above the main diagonal and zeroes elsewhere: as n -period bond purchased at t becomes an $(n - 1)$ -period bond at $t + 1$ or, if $n = 1$, it becomes a dollar at $t + 1$. The payout function π is the vector $(1, 0, \dots, 0)$.

In this case, (4.1) becomes $\varphi_1 = \beta(1 - z)^{-1}$ and $\varphi_{i+1} = \beta\varphi_i$, $i = 1, \dots, n - 1$. This difference equation can be solved to obtain $\varphi_i = \beta^i(1 - z)^{-1}$, $i = 1, \dots, n$. Then the inner product appearing on the right of (4.2) is

$$\left[\frac{1}{1 - z} \pi + B^T \varphi \right] \cdot x = \frac{1}{1 - z} \sum_{i=1}^n \beta^{i-1} x_i = \frac{1}{1 - z} \omega,$$

where the second equality defines the random variable $\omega = \eta(x)$. Cancelling the factor $(1 - z)^{-1}$, (4.2) implies

$$z = \beta \int_X \max(z, \omega) \mu(d\omega). \quad (4.6)$$

Then again, (4.5) is a sufficient condition for there to exist an equilibrium $z^* \in [0, 1)$.

Note that the consol example 4.1 is just the limiting case of this example as $n \rightarrow \infty$ when the random variables x_1, x_2, \dots are all equal to a common value. The second example of Section 2 is obtained if $x_i = 0$ for $i \geq 2$.

When (3.12) and (3.14) are specialized to this case, equilibrium bond prices must satisfy

$$q_i(x) = \left[\frac{1}{1 - z} + \theta(x) \right]^{-1} \left(\frac{1}{1 - z} \right) \beta^{i-1}, \quad i = 1, \dots, n,$$

where

$$\frac{1}{1-z} + \theta(x) = \max \left\{ \frac{1}{1-z}, \left(\frac{1}{1-z} \right) \frac{1}{z} \sum_{i=1}^n \beta^{i-1} x_i \right\}.$$

These equations imply

$$q_i(\omega) = \begin{cases} \beta^{i-1}(z/\omega) & \text{if } z < \omega \\ \beta^{i-1} & \text{if } z \geq \omega. \end{cases}$$

Note that forward interest rates for $i \geq 2$ are always ρ (forward one period bond prices are always β) whatever the value of $\omega = \sum_{i=1}^n \beta^{i-1} x_i$. Moreover, it is immaterial what maturities of bonds are issued: only the "value-weighted" sum ω matters.

EXAMPLE 4.3: Equities. The two examples 4.1 and 4.2 can readily be combined, or other securities can be added, or both. Consider, for example, the situation where n -period bonds are traded and where an equity claim to the (normalized) dollar income stream $p(s)y$ is also traded. With independent shocks, $p(s)y = 1 - z$, so $\pi_e(x) = 1 - z$ for this added security. We consider the effect of this modification on the system (4.1)–(4.2).

Adding equities to the system alters the matrix B simply by adding another row and column with a one on the diagonal and zeroes elsewhere. This adds an independent equation to (4.1), which may be solved for the added marginal value term φ_e , say

$$\varphi_e = \frac{\beta}{1-\beta}.$$

The price of equities is, from (3.16), then

$$q_e(x) = \left[\frac{1}{1-z} + \theta(x) \right]^{-1} \left(\frac{1}{1-\beta} \right).$$

If the government does not trade in equities, the addition of this security does not affect the liquidity constraint and the determination of z and $\theta(x)$ is exactly as in example 4.2. In this case, equity prices are given by

$$q_e(x) = \begin{cases} \frac{1-z}{1-\beta} \frac{z}{\omega} & \text{if } \omega > z \\ \frac{1-z}{1-\beta} & \text{if } \omega \geq z. \end{cases}$$

Thus a large bond issue depresses equity prices, as it does bond prices. If the government does trade in equities (and this case is as easily imagined as the one discussed) then one would need to characterize these trades in terms of an additional component in the vector x .

5. A PSEUDO-CASE WITH CONSTANT GOODS PRICES

The case studied in the last section is simple because the assumption of independent shocks keeps the information structure simple: securities prices are subject to liquidity effects but are not affected by speculation about future liquidity effects. Now return to the more complicated situation described by Eqs. (3.13) and (3.15). The function $z(s)$ enters into these two equations in two ways: the factor $[1 - z(s)]^{-1}$ appears on the right side of (3.13) and on both sides of (3.15). In both cases, it represents the inverse of the equilibrium goods price level $p(s) = [1 - z(s)]/y$. If $z(s)$ is constant, as in the last section, these factors cancel from both sides of (3.15). In addition $z(s)$ appears on the right of (3.15), in its role as the amount of money, or of liquidity, in the securities market. This is the only role played by $z(s)$ in the last section.

In this section, we are interested in the case in which the current state of the system conveys information about future bond issues, so that money moves in or out of securities markets in response to changes in s . But if this is the case then cash spent on goods has to fluctuate as well: all the money in the system has to go somewhere. With a constant endowment of goods, this means the price level fluctuates, and these fluctuations imply changes in expected inflation rates that will affect interest rates for fundamental, Fisherian reasons. For present purposes, I think these price effects are just a nuisance, getting in the way of analyzing the more interesting and direct liquidity effects. Why not just assume them away by taking prices to be fixed and analyze the interest rate movements that result? That is exactly what I will do in this section. It leads to a very tractable system of equations that do not, unfortunately, exactly describe any economic equilibrium.

By the system (3.13) and (3.15) with constant prices I mean the equations

$$\psi(s) = \beta \int_S [\pi(s, s') + B^T \psi(s')] P(s, ds'), \quad (5.1)$$

$$z(s) = \beta \int_S \{ \max[z(s), [\pi(s, s') + B^T \psi(s')] \cdot [\phi(s, s') - a(s)]] \} P(s, ds'). \quad (5.2)$$

The function ψ is related to φ by $\psi(s) = (1 - z)\varphi(s)$, where $1 - z$ is the fixed price. The level at which it is fixed does not matter: this system has a kind of homogeneity property, corresponding to the fact that it is rates of inflation that affect interest rates, not price levels.

We have

LEMMA 1. *Let S be a metric space. Let π be continuous and bounded. Let the transition function P have the Feller property ($g: S \rightarrow \mathbf{R}$ is continuous implies $\int g(s')P(s, ds')$ is continuous). Let B be a matrix with entries either 0 or 1, with no column having more than one entry 1. Then (5.1) has a unique continuous bounded solution $\psi: S \rightarrow \mathbf{R}_+^n$.*

Proof. Let the right side of (5.1) define an operator V on the space C_n of continuous, bounded functions $f: S \rightarrow \mathbf{R}_+^n$. Norm C_n by $\|f\| = \max_i \sup_{s \in S} |f_i(s)|$. Then under the given assumptions $V: C_n \rightarrow C_n$ and V is a contraction with modulus β . Since C_n is a complete metric space, the conclusion follows.

Indeed, since (5.1) is linear, one can write out a formula for the unique solution ψ , just as we did for the solution to (4.1).

Given a solution ψ to (5.1), let $\eta(s, s')$ be the real-valued random variable defined by $\eta(s, s') = [\pi(s, s') + B^T \psi(s')] \cdot [\phi(s, s') - a(s)]$. Now define the operator T on C_1 by

$$(Tz)(s) = \beta \int_S \{\max[z(s), \eta(s, s')]\} P(s, ds'). \quad (5.3)$$

Then fixed points of this operator T coincide with solutions to (5.2). We have

LEMMA 2. *Let the hypotheses of Lemma 1 hold, and assume that ϕ and a are continuous and bounded. Then (5.3) has a unique continuous bounded solution z .*

Proof. Under the stated assumptions, $\eta(s, s')$ is bounded and continuous. Hence if z has these properties, so does $\max[z(s), \eta(s, s')]$. Then since P has the Feller property, Tz is continuous. Thus $T: C_1 \rightarrow C_1$. Since T is evidently a contraction with modulus β , the conclusion follows.

To interpret the fixed point z of T as a cash allocation function, we need $z: S \rightarrow [0, 1)$. Clearly $z(s) \geq 0$ implies $(Tz)(s) \geq 0$. A sufficient condition for $z(s) \leq 1$ to imply $(Tz)(s) < 1$ is that

$$\int_A [\eta(s, s') - 1] P(s, ds') < \frac{1 - \beta}{\beta} \quad \text{for all } s \in S, \quad (5.4)$$

where the set A is defined by

$$A = \{s' \in S: \eta(s, s') \geq 1\}.$$

Compare to (4.5).

If the function $\pi(s, s')$ is constant, as in the examples of Sections 2 and 4, then the solution ψ to (5.1) is constant, equal except for the factor $1 - z$ to the solutions for φ given in Section 4. Thus the determination of interest rates is not much altered if serial correlation is added the way I have done it here. If the only security is a consol, as in Example 4.1, with s_t interpreted as the stock and $s_{t+1} - s_t$ as new issues, the consol price is

$$q(s, s') = \begin{cases} z(s)/(s' - s) & \text{if } s' - s > (1 - \beta)z(s) \\ (1 - \beta)^{-1} & \text{if } s' - s \leq (1 - \beta)z(s). \end{cases}$$

If there are bonds of n different maturities, as in Example 4.2, where $s_t \in R^n$ is the vector of outstanding stocks, let $\eta(s, s') = \sum_{i=1}^n \beta^{i-1}(s'_i - s_i)$. Then the price of a bond maturing in i periods is

$$q_i(s, s') = \begin{cases} \beta^{i-1}z(s)/\eta(s, s') & \text{if } z(s) < \eta(s, s') \\ \beta^{i-1} & \text{if } z(s) \geq \eta(s, s'), \end{cases}$$

where $z(s)$ is the fixed point of T .

The forward interest rate at maturity i is just $-\ln[q_i(s, s')/q_{i-1}(s, s')]$, which equals the constant $-\ln(\beta) = \rho$ for all states (s, s') . Hence the theory, even with serially correlated shocks, does not offer the possibility of accounting for term structure fluctuations. On the other hand, complicated intertemporal patterns in interest rates generally, due to liquidity effects and the anticipation of such effects in the future, are possible.

6. THE CASE OF SERIALLY CORRELATED SHOCKS: FINITE STATE SPACE

The analysis of the last section was greatly simplified by the assumption of constant prices. Since this assumption is not tenable in the context of this model (except when shocks are independent in the sense of Section 4), the results of that analysis can be, at best, an approximation. Nevertheless, we will see that the methods used to arrive at these results are suggestive for the more general case introduced in Section 3. This analysis will be conducted under the assumption that the state space S is finite.

We return to (3.13) and (3.15). The givens in these equations are the characteristics of the securities being traded, defined by π , B , ϕ , a , S , and P . We impose the following assumptions on these characteristics.

(A1) S is finite.

(A2) The functions π , ϕ , and a are non-negative.

(A3) B has entries 0 or 1, with no more than one entry 1 in any column.

Under (A1), the coupon payments $\pi_i(s, s')$ are bounded. Let $\alpha = \max_i \max_{s, s' \in S} \pi_i(s, s')$. The last assumption serves the function of (2.8), (4.5), and (5.4).

(A4) There exists a number D with $(1 - \beta)D > 1$ such that for all $s \in S$

$$\int_{A(s)} \{D\alpha \mathbf{1} \cdot [\phi(s, s') - a(s)] - 1\} P(s, ds') \leq \frac{1 - \beta}{\beta},$$

where $A(s) = \{s' \in S : D\alpha \mathbf{1} \cdot [\phi(s, s') - a(s)] \geq 1\}$, where $\mathbf{1}$ denotes an n -vector of ones.

We will show (Theorem 1) that under (A1)–(A4) there exists an equilibrium cash allocation function $z(s)$ that is non-negative and strictly less than one. Our strategy, as in Sections 4 and 5, will be to solve (3.13) for φ in terms of z , substitute this solution into (3.15), and then to study the latter. We begin with

LEMMA 3. *Let z be a function on S with range $[0, 1 - 1/M]$ for some number $1 < M < \infty$. Let (A1)–(A3) hold. Then (3.13) has a unique solution φ_z with*

$$0 \leq \varphi_z(s) \leq \frac{\alpha\beta M}{1 - \beta} \mathbf{1}, \quad \text{all } s \in S. \quad (6.1)$$

Proof. The existence of a unique solution is an application of the Contraction Mapping Theorem, as in Lemma 1. To prove that the bounds (6.1) are satisfied, use an induction on the sequence $\{\varphi_n\}$ defined by $\varphi_{n+1} = V\varphi_n$, where V is the operator defined in the proof of Lemma 1 and where φ_0 is the zero vector. Every term in this sequence satisfies (6.1), and it converges to the unique solution to (3.13).

In view of Lemma 3, there is a solution φ_z to (3.13) corresponding to any function z on S with $1 \leq [1 - z(s)]^{-1} \leq M < \infty$. Let z be such a function, and consider the equation in the single variable γ

$$\frac{\gamma}{1 - \gamma} = \beta \int_S \left\{ \max \left[\frac{z(s)}{1 - z(s')}, K(s, s') \right] \right\} P(s, ds'), \quad (6.2)$$

where the real-valued function K is defined on $S \times S$ by

$$K(s, s') = \left[\frac{1}{1 - z(s')} \pi(s, s') + B^T \varphi_z(s') \right] \cdot [\phi(s, s') - a(s)].$$

We want to define an operator T on functions z by setting $(Tz)(s)$ equal to the unique γ -value satisfying (6.2). The next lemma justifies this.

LEMMA 4. *Let (A1)–(A3) hold. Let $z: S \rightarrow [0, 1 - 1/M]$ for some $M > 1$. Then for each $s \in S$ there is a unique $\gamma \in [0, 1)$ satisfying (6.2).*

Proof. For each fixed s and z let $B_z(s)$ denote the right side of (6.2). Since $0 \leq z(s) < 1$ for all s , $B_z(s) \geq 0$ for all s . Then $\gamma = [1 + B_z(s)]^{-1} B_z(s) \in [0, 1)$ is the unique solution to (6.2).

Call the solution to (6.2) $(Tz)(s)$. For any $M > 1$, this defines an operator on the set C_M of functions (n -vectors) $z: S \rightarrow [0, 1 - 1/M]$. The next result shows that M can be chosen so that T takes C_M into itself.

LEMMA 5. *Let (A1)–(A4) hold. Then there exists $M > 1$ such that $T: C_M \rightarrow C_M$.*

Proof. We need to find $M > 1$ such that if $z \in C_M$, then $(Tz)(s) = [1 + B_z(s)]^{-1} B_z(s) \leq 1 - 1/M$, where $B_z(s)$ is the right side of (6.2), as in the proof of the last lemma. Equivalently, we seek an M such that $z \in C_M$ implies $B_z(s) \leq M - 1$ or

$$\beta \int_S \left\{ \max \left[\frac{z(s)}{1 - z(s')}, K(s, s') \right] \right\} P(s, ds') \leq M - 1 \quad (6.3)$$

for all $s \in S$.

If $z \in C_M$, then $[1 - z(s')]^{-1} z(s) \leq M - 1$ for all s, s' . By Lemma 3 and (A3)

$$K(s, s') \leq \left(M\alpha + \frac{M\alpha\beta}{1 - \beta} \right) \mathbf{1} \cdot [\phi(s, s') - a(s)] \leq M \frac{\alpha}{1 - \beta} \mathbf{1} \cdot [\phi(s, s') - a(s)].$$

Thus (6.3) will hold for all $z \in C_M$ provided

$$\beta \int_S \max \left\{ 1, \frac{M}{M - 1} \frac{\alpha}{1 - \beta} \mathbf{1} \cdot [\phi(s, s') - a(s)] \right\} P(s, ds') \leq 1. \quad (6.4)$$

Now let D be as in (A4). Then if $M = [1 - 1/(1 - \beta)D]^{-1}$, (6.4) holds and the proof is complete.

We summarize the results of this section in:

THEOREM 1. *Under (A1)–(A4), there is a solution $(\varphi(s), z(s))$ to (3.13) and (3.15) with $0 \leq z(s) < 1$ for all $s \in S$.*

Proof. Choose D as in (A4) and M as in the proof of Lemma 5. By Lemmas 3 and 4, the operator T defines a function on the subset $C_M = [0, 1 - 1/M]^n$ of \mathbf{R}^n into \mathbf{R}^n . This function is evidently continuous. By Lemma 5, T takes $[0, 1 - 1/M]^n$ into itself. By Brouwer's Theorem, T has a fixed point z in this set and this z together with the function φ constructed from z in Lemma 3 satisfy (3.13) and (3.15).

7. NUMERICAL ILLUSTRATIONS

The examples in Section 4 provide, I hope, a good idea of the possibilities of the theory when shocks are independent. When shocks are serially correlated, as in the analysis of Sections 5 and 6, pencil-and-paper methods are of more limited usefulness. Accordingly, this section reports the results of some illustrative calculations on an example in which the only security priced is a one-period bond (as in Section 2).

In all the illustrations, the state of the system x takes on a finite numbers of values x_1, \dots, x_n , and these values are interpreted as the size of an issue of one period government bonds. There are no other securities. The transition matrix is $P = [p_{ij}]$, where p_{ij} is the probability that $x_{t+1} = x_j$ conditional on $x_t = x_i$. I will deal with the two equations

$$z_i = \beta \sum_{j=1}^n \max(z_i, x_j) p_{ij}, \quad i = 1, \dots, n \quad (7.1)$$

and

$$\frac{z_i}{1 - z_i} = \beta \sum_{j=1}^n \frac{1}{1 - z_j} \max(z_i, x_j) p_{ij}, \quad i = 1, \dots, n. \quad (7.2)$$

Equation (7.1) is a specialization of the system (5.1)–(5.2) that holds for the pseudo-case described in Section 5. Equation (7.2) is a specialization of the system studied in Section 6.

Let T_1 be the operator on \mathbf{R}^n such that $(T_1 z)_i$ is the right side of (7.1), so that solutions to (7.1) coincide with fixed points of T_1 , and fixed points with all coordinates in $[0, 1)$ can be interpreted as equilibria. The solutions of (7.1) tabulated below were obtained by iterating the operator T_1 on the indicated initial vector z_0 . As in Section 5, T_1 is a contraction, so this method locates the unique fixed point. Choosing the number of iterates m

so that $\|T_1^{m+1} - T_1^m\| \leq (1 - \beta)(.001)$ will yield answers accurate to the third decimal place.

Let T_2 be the operator on the subset of \mathbf{R}^n with coordinates less than one defined by $(T_2 z)_i = [1 + R_i(z)]^{-1} R_i(z)$, where $R_i(z)$ is the right side of (7.2). Then fixed points of T_2 coincide with solutions to (7.2), and solutions with coordinates in $[0, 1)$ have interpretations as equilibria. I calculated fixed points of T_2 by the method described in the preceding paragraph for T_1 . Theorem 1 in Section 6 gives sufficient condition for T_2 to satisfy the conditions of Brouwer's Theorem, but these conditions played no role in the calculations. In all cases, this iterative method located a fixed point, but Theorem 1 gives no assurance that this must always be the case, nor does it guarantee uniqueness of the fixed point when one is found.

The results from some of these calculations are reported in Tables I and II. I used $\beta = 0.995$, thinking of a *monthly* discount rate of 0.5 percent. The bond issue x takes on two values, .02 and .08, which are of the right order of magnitude for monthly U.S. government bond issues, relative to total reserves. Beyond selecting numbers of realistic orders of magnitude, I made no attempt to be realistic. To experiment with different degrees of positive and negative serial correlation, I used the transition matrix

$$P = \begin{pmatrix} \theta & 1 - \theta \\ 1 - \theta & \theta \end{pmatrix}.$$

Values of 0.001, 0.01, 0.1, 0.3, 0.5, 0.7, 0.9, 0.99, and 0.999 were used for θ . For all these values, P has the unique stationary distribution (0.5, 0.5) over the two issue-states 0.02 and 0.08, with an average issue of 0.05.

TABLE I
Solutions to (7.1) and (7.2)

θ	Equation (7.1)			Equation (7.2)		
	z_1	z_2	m	z_1	z_2	m
0.001	0.080	0.020	381	0.079	0.075	58
0.01	0.080	0.054	289	0.079	0.075	51
0.1	0.080	0.076	37	0.079	0.078	23
0.3	0.079	0.079	9	0.079	0.079	6
0.5	0.079	0.079	6	0.079	0.079	6
0.7	0.079	0.079	9	0.079	0.079	8
0.9	0.076	0.080	37	0.076	0.080	37
0.99	0.054	0.080	289	0.055	0.080	287
0.999	0.020	0.080	381	0.020	0.080	414

Note. Two states; $\beta = 0.995$; $(x_1, x_2) = (0.02, 0.08)$; $(z_1^0, z_2^0) = (0.079, 0.079)$.

TABLE II
Monthly Interest Rate Behavior Implied by (7.2)

θ	$E(r)$	$StD(r)$	$Pr\{r=0\}$	ρ_1	ρ_2	ρ_3	ρ_4
0.001	0.005	0.005	0.500	-0.922	0.920	-0.919	0.917
0.01	0.005	0.006	0.500	-0.567	0.556	-0.545	0.534
0.1	0.005	0.007	0.500	-0.185	0.148	-0.118	0.095
0.3	0.005	0.006	0.500	-0.076	0.030	-0.012	0.005
0.5	0.005	0.005	0.500	0.000	0.000	0.000	0.000
0.7	0.005	0.006	0.500	0.038	0.015	0.006	0.002
0.9	0.005	0.010	0.500	0.022	0.017	0.014	0.011
0.99	0.005	0.026	0.500	0.005	0.005	0.005	0.005
0.999	0.004	0.031	0.001	0.002	0.002	0.002	0.002

For $\theta = 0.5$, the shocks are independent and the solution can be obtained by hand as in Sections 2 or 4. The fixed point (z_1, z_2) has equal coordinates, with the common value 0.079. For the other θ -values, this vector z was taken as the initial value to which the operators T_1 and T_2 were applied.

Table I reports the fixed points of T_1 in the first two columns, and the iterations needed to meet the tolerance level $\|T_1^{m+1} - T_1^m\| = (0.005)(0.001) = 0.000005$ for the θ -values listed on the left. The next three columns in the table give the fixed points of T_2 and the iterations required to give the same tolerance level.

Table II describes the properties of equilibrium interest rates associated with the fixed points of T_2 , again for each θ -value. With serial correlation, bond prices and hence interest rates are functions of the current state, which determines x_t , and last period's state, which can affect z_t . Hence if x follows an n -state Markov process, interest rates follow an n^2 -state Markov process. The transition function for this latter process can be calculated from P alone. The values of the interest rate in each state are calculated using the fixed point z . The table reports the mean interest rate, the standard deviation, the probability of a zero rate, and the first four autocorrelation coefficients, with all moments taken with respect to the unique stationary distribution of the process. Again, the row corresponding to $\theta = 0.5$ is readily calculated by hand.

Table I is mainly interesting for the information it contains about the differences between (7.1) and (7.2). The solution (z_1, z_2) to (7.1) is a continuous function of the parameter θ on the interval $[0, 1]$. At $\theta = 1$ (the current state is always maintained), the solution is $z = \beta x = (0.995)(0.02, 0.08)$, which is equal to three decimals to the solution for $\theta = 0.999$ given in the table. Similarly, the solution given for $\theta = 0.001$ equals the solution at $\theta = 0$ (the current state is never maintained). But

away from these extremes, the solution to (7.1) is insensitive to changes in the degree of serial correlation, remaining almost constant on the interval $[0.1, 0.9]$.

The solution to (7.2) behaves in a very similar way, except at very low θ values where the second coordinates of the solutions z to (7.1) and (7.2) are very different. At this extreme, the price effects reflected in the terms $(1 - z)^{-1}$ have an important influence. When θ is very low, a system in state 2 will almost certainly move to state 1 next period, which means that unless z_2 falls below $x_2 = 0.02$, interest rates will almost certainly be zero. With no price effect (that is, if (7.1) holds), z_2 does fall, for just this reason. Suppose the same z_2 value were to occur when a price effect is operating (that is, if (7.2) holds). Then the price level in state 1 will rise (since less cash held for securities trading means more cash is spent on goods), but then the system almost certainly will return to state 2 the period after, with a return to a lower price level. Hence state 1 would be associated with a large expected deflation, and cash is an excellent security to hold. It is this expected deflation effect that keeps z_2 from falling to $(0.995)(0.02)$ near $\theta = 0$ in (7.2). Indeed, the solution for z_2 to (7.2) at $\theta = 0$ can be calculated theoretically: It is also 0.075.

Table II describes the interest rate behavior implied by Eq. (7.2). Obviously, except for very low θ values, Eq. (7.1) implies about the same behavior. Average interest rates are essentially given by consumers' rate of time preference. Recall that I have set the rate of money growth equal to zero, so one would expect nominal and real rates to be equal. Attitudes toward risk play no role in these liquidity effects, so interest rates do not change as the risk situation changes. The variability of interest rates is fairly stable, too, as well as fairly high: rates fluctuate between zero and very high levels. Serial correlation patterns are negligible, except at very low θ values where they reflect the assumed serial correlation pattern of the shocks in an obvious way.

I found these simulations informative, in an unexpected direction. If one were to apply a model of this type to explaining or predicting actual short-term interest rate series, one would do very well simply by calculating the constant equilibrium z -value for the i.i.d. case studied in Section 4, and assuming it holds for *any* time pattern of the shocks. The cash allocation is so insensitive to advance information on bond issues, even when this information is very sharp compared to what one would ever see in practice, that these information effects can as well be ignored. Perhaps one can think of shock processes where this would not be the case, but I was not able to do so. Another way of stating this conclusion is to say that Section 4 contains about 99% of what this paper has to say about the behavior of interest rates!

I carried out a number of calculations to check the sensitivity of the

results in Tables I and II to changes in assumptions. There were no surprises, so I will just summarize them briefly. Changes in the discount factor β had no systematic effects on the speed of convergence of the algorithm. Apparently the bounds implied by the Contraction Mapping Theorem are not approached in practice in (7.1). Changes in the initial guess z^0 in some cases increased the iterations required to over 1000 (in (7.2)) but the algorithm always converged and in no case was a fixed point found that differed from those reported in Table I. Increasing the number of shock-states to three, while retaining the symmetry of the two-state example, did not affect much the first two moments of the implied interest rate series.

8. CONCLUSIONS

The premise of this paper, as of the earlier contributions of Grossman and Weiss [2] and Rotemberg [13], is that at any time an economy's money is distributed over distinct locations, or markets, and that it takes time to move funds from one location to another. One implication of this premise is that an unanticipated change in the excess demand for cash in any one market will have different effects on prices and interest rates, depending on the way cash is distributed when the change occurs. To predict the consequences of such a change, one needs to know *where* money is as well as *how much* there is.

In order to model such effects in a tractable way, I followed [2] and [13] and imposed separate cash-in-advance or liquidity constraints on agents trading in distinct goods and securities markets. I departed from these earlier papers by taking these agents as being members of a single family, sharing a household utility function. This latter device greatly simplifies much of the analysis, permitting the analysis of a wide variety of stochastic (much wider, indeed, than I have explored here). It is also, in a sense, realistic. When we apply general equilibrium theory in the study of asset pricing, we typically consolidate accounts and impute as wealth to households the assets held by corporations in which they own shares, pension funds, and other institutions. This means that a given household's cash includes its own currency and bank accounts, plus the currency and bank accounts of its pension fund, of the financial intermediaries with which it deals, of the businesses of which it is part owner, and so on. All of this cash is properly viewed as included in the household's wealth, but it obviously cannot all be viewed as serving a common transactions purpose. I can pay for a cab ride with the currency I hold, but not with the money that TIAA-CREF holds on my behalf and, symmetrically, TIAA-CREF cannot use my demand deposits to acquire securities on my account, even when it would be in my interest for it to do so.

An immediate consequence of a financial liquidity constraint is that, at any time, there is a fixed demand curve for government securities along which the monetary authority can "peg" interest rates in a very literal sense. In this world, issuers of bonds can pick an interest rate at the beginning of a period and then conduct open market operations in such a way as to make it happen. This is the feature that the models of Grossman and Weiss and Rotemberg were designed to capture, and by building on their work, the models of this paper capture it too.

Beyond this, I have shown that there liquidity effects can induce a serially correlated stochastic component to equilibrium interest rates that need not bear any definite relationship to fundamentals in the sense of Irving Fisher. These liquidity shocks have the capacity to induce sudden, large drops in the prices of bonds and other securities. The right image is not a bubble popping, but getting one's wind knocked out: The return to fundamental levels should be quick. In the examples I have developed, these shocks are tightly linked to government bond issues that can be directly observed. In practice, I think shifts in the private sector's demand for cash balances are also an important source of liquidity effects, as I am using that term, so I would not be optimistic about an econometric test that treats the state of the system as fully observable.⁷

A more central prediction of the theory arises from its "one factor" character. Since the liquidity effect works through a single cash constraint, it has to affect all centrally traded securities at once, in more or less the same way. Thus the theory has no ability to account for changes in the term structure of interest rates or in the relative prices of bonds and equities. Technically, this prediction could be relaxed by assuming segmentation of securities markets, but I think this would move us farther from the kind of realism I seeking.

One feature of the theory that I find most unattractive is the fact that traders in securities will carry cash balances over only if short term interest rates are zero. The set-up does not get us far enough away from rate-of-return dominance. The example in the paper that comes closest to facing this issue is the case of consols (Example 4.1). Here, there is no maturity as short as one "period," so no security exactly dominates cash. Even so, the implicit short rate is zero if cash is carried over in this example too.⁸

There is a wealth of interesting data on flow of funds, turnover rates of various kinds of accounts, and so on that monetary theory ought to deal

⁷ Atkeson [1] analyzes a model similar to those in this paper in which private sector "churning" is the source of liquidity fluctuations.

⁸ See note 4. The modification Wallace suggested would imply that money need not be dominated by one-period interest-bearing bonds. In calculations based on the model in [10], however, I found that even with this modification interest rates equal zero with a non-negligible probability.

with but generally has not. To do so, we will need to get farther away from complete markets in our theory, just as labor economists have had to in their attempts to account for their interesting turnover series. If the theory of transactions demand for money is to move in this direction, it is clear that we will need formulations that place a smaller burden on the idea of a fixed period than do the models of this paper. I have in mind not so much explaining the crucial time lags in the monetary system (though that would be nice, too) but just describing them with free parameters that can be more easily varied to fit data than the period length in the usual discrete-time formulations.

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