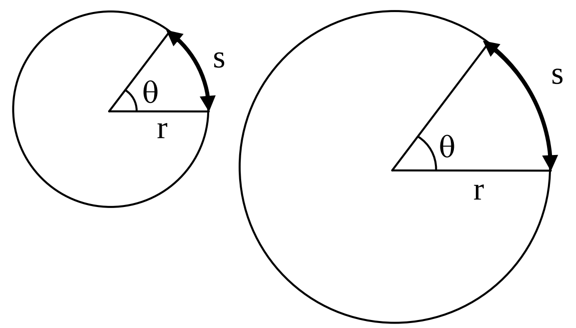
Rotational Motion

We are going to consider the motion of a rigid body about a fixed axis of rotation.

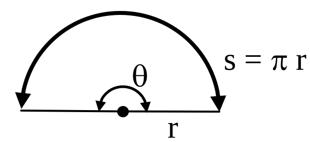
The angle of rotation is measured in radians: $\theta(\text{rads}) \equiv \dfrac{s}{r}$ (dimensionless)



Notice that for a given angle θ, the ratio s/r is independent of the size of the circle.

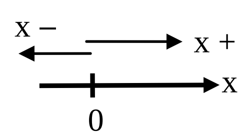
Example: How many radians in 180°? Circumference $C = 2\pi r$

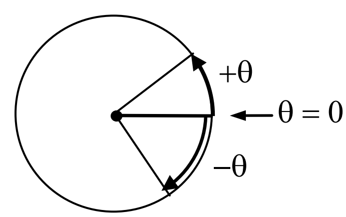
$$\theta = \frac{s}{r} = \frac{\pi r}{r} = \pi \text{ rads} \qquad \pi \text{ rads} = 180^\circ,\ 1 \text{ rad} = 57.3^\circ$$



Angle θ of a rigid object is measured relative to some reference orientation, just like 1D position x is measured relative to some reference position (the origin).

Angle θ is the "rotational position". Like position x in 1D, rotational position θ has a sign convention. Positive angles are CCW (counter-clockwise).

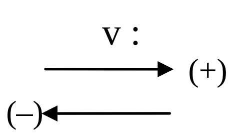


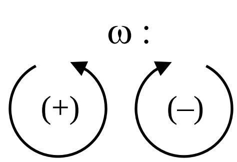


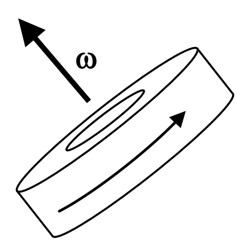
Definition of *angular velocity*: $\omega \equiv \dfrac{d\theta}{dt}, \quad \bar{\omega} = \dfrac{\Delta\theta}{\Delta t}$ (rad/s) $\left(\text{like } v \equiv \dfrac{dx}{dt},\ \bar{v} = \dfrac{\Delta x}{\Delta t}\right)$

units $[\omega] = \dfrac{\text{rad}}{\text{s}}$

In 1D, velocity v has a sign (+ or –) depending on direction. Likewise, for fixed-axis rotation, ω has a sign, depending on the sense of rotation.





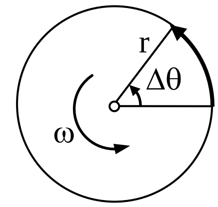


More generally, when the axis is not fixed, we define the **vector** angular velocity $\vec{\omega}$ with direction = the direction of the axis + "right hand rule". Curl fingers of right hand around rotation, thumb points in direction of vector.

For rotational motion, there is a relation between *tangential velocity* v (velocity along the rim) and angular velocity θ.

$$\Delta\theta = \frac{\Delta s}{r} \Rightarrow \Delta s = r\,\Delta\theta\ ,$$

$$v = \frac{\Delta s}{\Delta t} = \frac{r\,\Delta\theta}{\Delta t} = r\,\omega \qquad \boxed{v = r\,\omega}$$



Definition of *angular acceleration* : $\boxed{\alpha \equiv \frac{d\omega}{dt},\quad \bar{\alpha} = \frac{\Delta\omega}{\Delta t}\ (\text{rad/s}^2)}$

$\left(\text{like } a \equiv \frac{dv}{dt},\quad \bar{a} = \frac{\Delta v}{\Delta t}\right)$ Units: $[\alpha] = \frac{\text{rad}}{\text{s}^2}$

α = rate at which ω is changing.
ω = constant ⇔ α = 0 ⇒ speed v along rim = constant = r ω

Equations for constant α:

Recall from Chapter 2: We defined $v = \frac{dx}{dt}$, $a = \frac{dv}{dt}$,

and then showed that, if a = constant, $\begin{cases} v = v_0 + a\,t \\ x = x_0 + v_0 t + \frac{1}{2} a\,t^2 \\ v^2 = v_0^2 + 2a\,(x - x_0) \end{cases}$

Now, in Chapter 10, we define $\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt}$.

So, if α = constant, $\begin{cases} \omega = \omega_0 + \alpha\,t \\ \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha\,t^2 \\ \omega^2 = \omega_0^2 + 2\alpha\,(\theta - \theta_0) \end{cases}$

Same equations, just different symbols.

Example: Fast spinning wheel with ω_0 = 50 rad/s ($\omega_0 = 2\pi f \Rightarrow f \approx 8$ rev/s). Apply brake and wheel slows at α = –10 rad/s. How many revolutions before the wheel stops?

Use $\omega^2 = \omega_0^2 + 2\alpha\ \Delta\theta$, $\omega_{final} = 0 \Rightarrow\ 0 = \omega_0^2 + 2\alpha\ \Delta\theta \Rightarrow\ \Delta\theta = -\frac{\omega_0^2}{2\alpha} = -\frac{50^2}{2(-10)} = 125$ rad

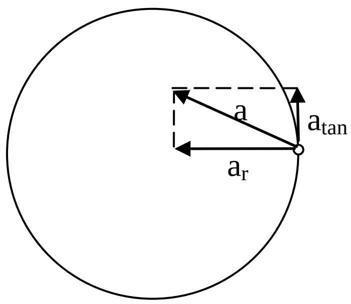
$$125\text{ rad} \times \frac{1\text{ rev}}{2\pi\text{ rad}} = 19.9\text{ rev}$$

Definition of *tangential acceleration* a_{tan} = rate at which speed v along rim is changing

$$a_{tan} \equiv \frac{dv}{dt} = \frac{d(r\,\omega)}{dt} = r\frac{d\omega}{dt} \quad \Rightarrow \quad \boxed{a_{tan} = r\,\alpha}$$

a_{tan} is different than the radial or centripetal acceleration $a_r = \dfrac{v^2}{r}$

a_r is due to change in *direction* of velocity **v**
a_{tan} is due to change in *magnitude* of velocity, speed v



a_{tan} and a_r are the tangential and radial components of the acceleration vector **a**.

$$|\vec{a}| = a = \sqrt{a_{tan}{}^2 + a_r{}^2}$$

Angular velocity ω also sometimes called angular frequency.
Difference between angular velocity ω and frequency f:

$$\omega = \frac{\#\text{ radians}}{\text{sec}} \quad , \quad f = \frac{\#\text{ revolutions}}{\text{sec}}$$

T = period = time for one complete revolution (or cycle or rev) ⇒

$$\omega = \frac{2\pi\text{ rad}}{T} = \frac{2\pi}{T}, \quad f = \frac{1\text{ rev}}{T} = \frac{1}{T} \quad \Rightarrow \quad \boxed{\omega = 2\,\pi\,f}$$

Units of frequency f = rev/s = hertz (Hz) . Units of angular velocity = rad /s = s^{-1}

Example: An old vinyl record disk with radius r = 6 in = 15.2 cm is spinning at 33.3 rpm (revolutions per minute).

• What is the period T?

$$\frac{33.3\text{ rev}}{1\text{ min}} = \frac{33.3\text{ rev}}{60\text{ s}} \Rightarrow \frac{60\text{ s}}{33.3\text{ rev}} = \frac{(60\,/\,33.3)\text{ s}}{1\text{ rev}} \cong 1.80\text{ s/rev}$$

⇒ period T = 1.80 s

• What is the frequency f ? f = 1 / T = 1 rev / (1.80 s) = 0.555 Hz

• What is the angular velocity ω ? $\omega = 2\,\pi\,f = 2\,\pi\,(0.555\text{ s}^{-1}) = 3.49\text{ rad}/\text{s}$

• What is the speed v of a bug hanging on to the rim of the disk?

$v = r\,\omega = (15.2\text{ cm})(3.49\text{ s}^{-1}) = 53.0\text{ cm/s}$

• What is the angular acceleration α of the bug? $\alpha = 0$, since ω = constant

• What is the magnitude of the acceleration of the bug? The acceleration has only a radial component a_r , since the tangential acceleration $a_{tan} = r\,\alpha = 0$.

$$a = a_r = \frac{v^2}{r} = \frac{(0.530\text{ m/s})^2}{0.152\text{ m}} = 1.84\text{ m/s}^2 \quad \text{(about 0.2 g's)}$$

For every quantity in linear (1D translational) motion, there is corresponding quantity in rotational motion:

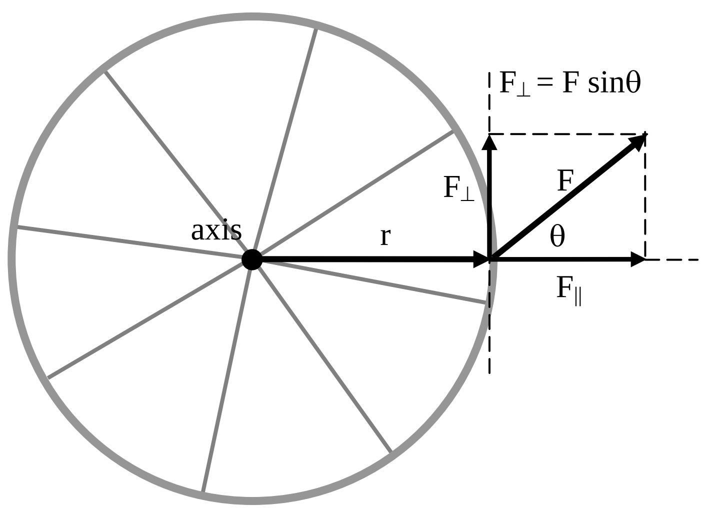
Translation	$\leftrightarrow$	**Rotation**
x	$\leftrightarrow$	θ
$v = \frac{dx}{dt}$	$\leftrightarrow$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt}$	$\leftrightarrow$	$\alpha = \frac{d\omega}{dt}$
F	$\leftrightarrow$	(?)
M	$\leftrightarrow$	(?)
F = Ma	$\leftrightarrow$	(?) = (?) α
KE = (1/2) m v^2	$\leftrightarrow$	KE = (1/2) (?) ω^2

The rotational analogue of force is *torque*.

Force F causes acceleration a $\leftrightarrow$

Torque τ causes angular acceleration α .

The torque (pronounced "tork") is a kind of "rotational force".



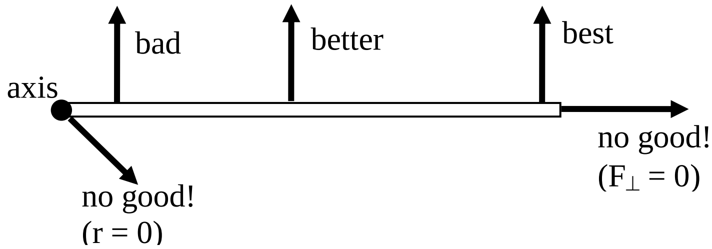
magnitude of torque: $\boxed{|\tau| \equiv r \cdot F_\perp = r\,F\sin\theta}$

$$[\tau] = [r][F] = \text{m N}$$

r = "lever arm" = distance from axis to point of application of force

$F_\perp$ = component of force perpendicular to lever arm

Example: Wheel on a fixed axis:
Notice that only the perpendicular component of the force **F** will rotate the wheel. The component of the force parallel to the lever arm ($F_{||}$) has no effect on the rotation of the wheel. If you want to easily rotate an object about an axis, you want a large lever arm r and a large perpendicular force $F_\perp$:

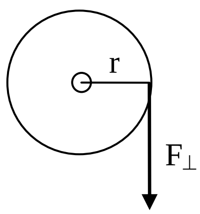


Example: Pull on a door handle a distance r = 0.8 m from the hinge with a force of magnitude F = 20 N at an angle θ = 30° from the plane of the door, like so:

$\tau = r\,F_\perp = r\,F\sin\theta =$

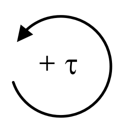
$(0.8\text{ m})(20\text{ N})(\sin 30^\circ) = 8.0\text{ m}\cdot\text{N}$

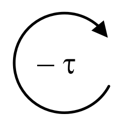
Another example:
a Pulley



$\tau = r\,F_\perp$

For fixed axis, torque has a sign (+ or –) :
Positive torque causes counter-clockwise CCW rotation.
Negative torque causes clockwise (CW) rotation.

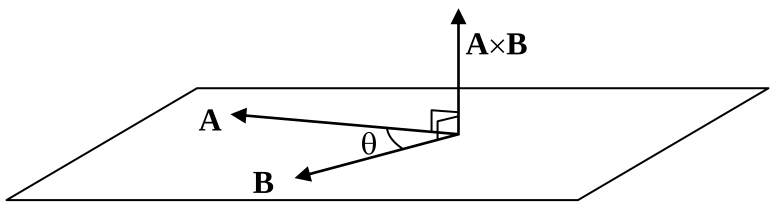




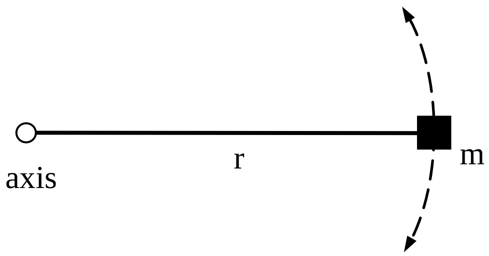
If several torques are applied, the net torque causes angular acceleration: $\tau_{net} = \sum \tau \propto \alpha$

Aside: Torque, like force, is a vector quantity. Torque has a direction.
Definition of *vector torque* : $\vec{\tau} = \vec{r}\times\vec{F}$ = cross product of **r** and **F**: "r cross F"

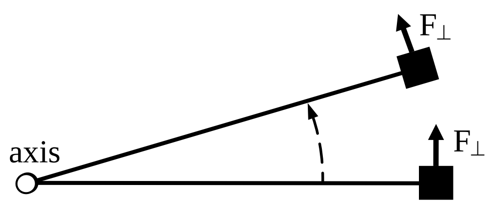
Vector Math interlude: The *cross-product* of two vectors is a third vector $\vec{A}\times\vec{B} = \vec{C}$ defined like this: The magnitude of $\vec{A}\times\vec{B}$ is $A\,B\sin\theta$. The direction of $\vec{A}\times\vec{B}$ is the direction perpendicular to the plane defined by the vectors **A** and **B** plus right-hand-rule. (Curl fingers from first vector **A** to second vector **B**, thumb points in direction of $\vec{A}\times\vec{B}$



To see the relation between torque τ and angular acceleration α , consider a mass *m* at the end of light rod of length *r*, pivoting on an axis like so:



Apply a force $F_\perp$ to the mass, keeping the force perpendicular to the lever arm r.



acceleration $a_{tan} = r\,\alpha$

Apply $F_{net} = m\,a$, along the tangential direction:
$F_\perp = m\,a_{tan} = m\,r\,\alpha$

Multiply both sides by r (to get torque in the game): $r\,F_\perp = (m\,r^2)\,\alpha$

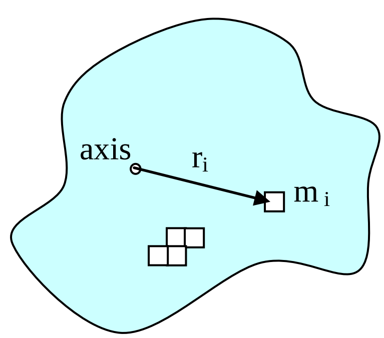
Define "moment of inertia" = $I = m\,r^2$

$\Rightarrow$ $\boxed{\tau = I \cdot \alpha}$ (like $F = m \cdot a$)

Can generalize definition of I:

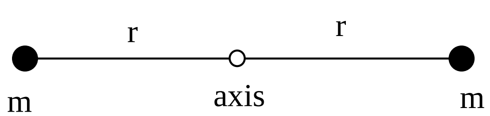
Definition of *moment of inertia* of an extended object about an axis of rotation:

$$I \equiv \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + \ldots$$



Examples:

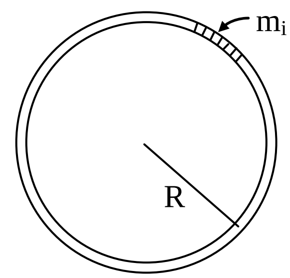
• 2 small masses on rods of length r:



$I = 2\,m\,r^2$

• A hoop of total mass M, radius R, with axis through the center, has $I_{hoop} = M R^2$

$$I = \sum_i m_i r_i^2 = \left(\sum_i m_i\right) R^2 = M R^2 \qquad (\text{since } r_i = R \text{ for all } i)$$



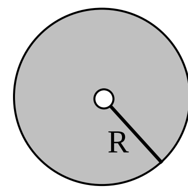
In detail:

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots = m_1 R^2 + m_2 R^2 + m_3 R^2 + \ldots$$
$$= (m_1 + m_2 + m_3 + \ldots) R^2 = M R^2$$

• A solid disk of mass M, radius R, with axis through the center:

$I_{disk} = (1/2)\, MR^2$ (need to do integral to prove this)

See Appendix for I's of various shapes.



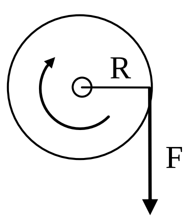
Moment of inertia I is "rotational mass".

Big I $\Rightarrow$ hard to get rotating (like Big M $\Rightarrow$ hard to get moving)

If I is big, need a big torque τ to produce angular acceleration according to

$$\tau_{net} = I \cdot \alpha \qquad (\text{like } F_{net} = m\,a)$$

Example: Apply a force F to a pulley consisting of solid disk of radius R, mass M. α = ?



$$\tau = I\,\alpha$$

$$R\,F = \left(\tfrac{1}{2} M R^2\right)\alpha \quad \Rightarrow \quad \alpha = \frac{2F}{MR}$$

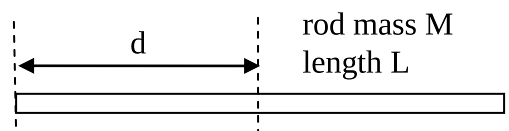
Parallel Axis Theorem

Relates I_{cm} (axis through center-of-mass) to I w.r.t. some other axis: $\boxed{I = I_{cm} + M\,d^2}$

(See proof in appendix.)

Example: Rod of length L, mass M

$$I_{CM} = \frac{1}{12} M R^2, \quad d = L/2 \Rightarrow$$



$$I_{end\,axis} = I_{CM} + M d^2 = \frac{1}{12} M L^2 + \frac{1}{4} M L^2 = \frac{1}{3} M L^2$$

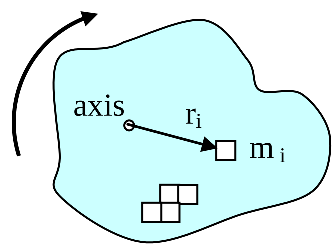
Rotational Kinetic Energy

How much KE in a rotating object? Answer: $\boxed{KE_{rot} = \frac{1}{2} I \omega^2}$ (like $KE_{trans} = \frac{1}{2} m v^2$)

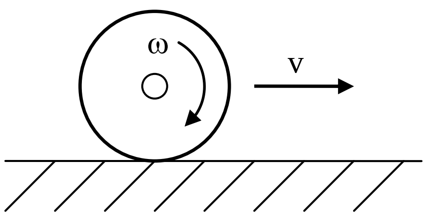
Proof: $KE_{tot} = \sum_i \left(\frac{1}{2} m_i v_i^{\,2}\right)$

$v = \omega r , \qquad v_i = \omega r_i$

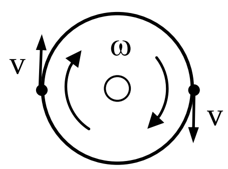
$$KE = \sum_i \left(\frac{1}{2} m_i \omega\, r_i^{\,2}\right) = \frac{1}{2}\left(\sum_i m_i r_i^{\,2}\right)\omega^2 = \frac{1}{2} I \omega^2$$



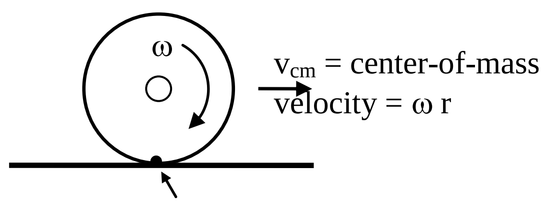
How much KE in a *rolling* wheel?



The formula v = r ω is true for a wheel spinning about a fixed axis, where v is speed of points on rim. A similar formulas v_{CM} = r ω works for a wheel rolling on the ground. Two very different situations, different v's: v = speed of rim vs. v_{cm} = speed of axis. But v = r ω true for both.

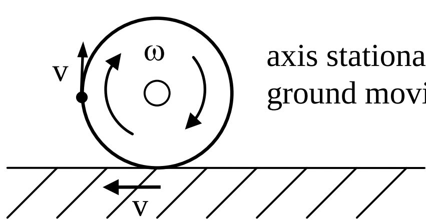


axis stationary: v = ω r



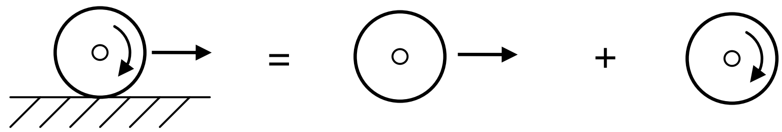
point touching ground
instantaneously at rest

To see why same formula works for both, look at situation from the bicyclist's point of view:



axis stationary,
ground moving

Rolling KE: Rolling wheel is simultaneously translating and rotating:



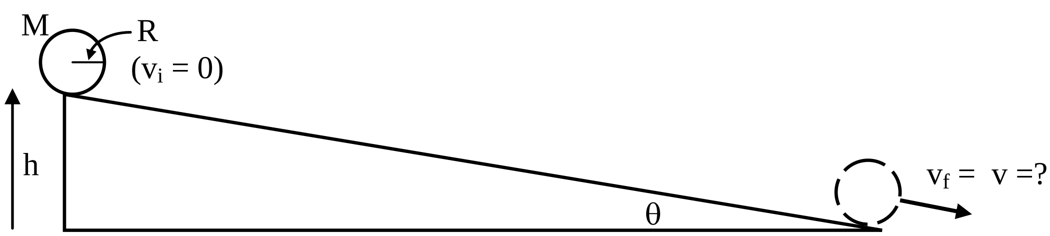
KE_{tot} = KE_{trans} + KE_{rot}

$$KE_{tot} = \tfrac{1}{2} M v^2 + \tfrac{1}{2} I \omega^2 \quad (v = V_{cm})$$

(See proof in appendix.)

Conservation of energy problem with rolling motion:

A sphere, a hoop, and a cylinder, each with mass M and radius R, all start from rest at the top of an inclined plane and roll down to the bottom. Which object reaches the bottom first?



Apply Conservation of Energy to determine v_{final}. Largest v_{final} will be the winner.

$$KE_i + PE_i = KE_f + PE_f$$

$$0 + Mgh = \underbrace{\tfrac{1}{2} M v^2}_{KE\,trans} + \underbrace{\tfrac{1}{2} I \omega^2}_{KE\,rot} + 0$$

Value of moment of inertia I depends on the shape of the rolling thing:

$I_{disk} = (1/2) M R^2$, $I_{hoop} = M R^2$, $I_{sphere} = (2/5) M R^2$ (Computing coefficient requires integral.)

Let's consider a disk, with $I = (1/2)MR^2$. For the disk, the rotational KE is

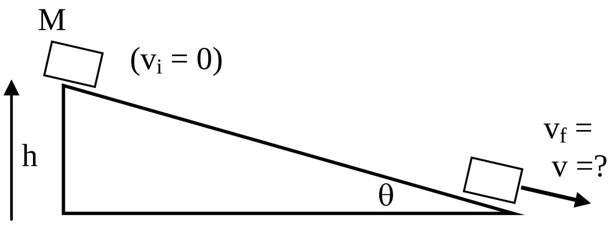
$$\tfrac{1}{2} I \omega^2 = \tfrac{1}{2}(\tfrac{1}{2} M R^2)\left(\frac{v}{R}\right)^2 = \tfrac{1}{4} M v^2 \qquad [\text{used } \omega = v/r]$$

$$\Rightarrow M g h = \tfrac{1}{2} M v^2 + \tfrac{1}{4} M v^2 = (\tfrac{1}{2} + \tfrac{1}{4}) M v^2 = \tfrac{3}{4} M v^2$$

$$g h = \tfrac{3}{4} v^2, \qquad v = \sqrt{\tfrac{4}{3} g h} \cong 1.16 \sqrt{g h}$$

Notice that final speed does not depend on M or R.

Let's compare to final speed of a mass M, *sliding* down the ramp (no rolling, no friction).



$M\,g\,h = \frac{1}{2} M\,v^2$ (M's cancel)

$\Rightarrow v = \sqrt{2\,g\,h} \cong 1.4\sqrt{g\,h}$

Sliding mass goes faster than rolling disk.

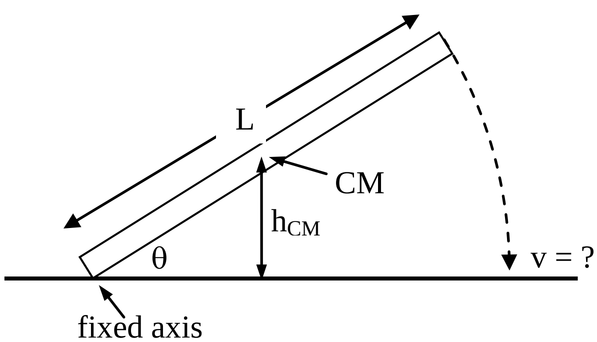
Why?

As the mass descends, PE is converted into KE. With a rolling object, $KE_{tot} = KE_{trans} + KE_{rot}$, so some of the PE is converted into KE_{rot} and less energy is left over for KE_{trans}. A smaller KE_{trans} means slower speed (since $KE_{trans} = (1/2)\,M\,v^2$). So rolling object goes slower than sliding object, because with rolling object some of the energy gets "tied up" in rotation, and less is available for translation.

Comparing rolling objects: $I_{hoop} > I_{disk} > I_{sphere} \Rightarrow$ Hoop has biggest $KE_{rot} = (1/2)\,I\,\omega^2$, $\Rightarrow$ hoop ends up with smallest $KE_{trans} \Rightarrow$ hoop rolls down slowest, sphere rolls down fastest.

Another conservation of rotational energy problem:

Rod of mass M, length L, one end stationary on ground, starts from rest at angle θ and falls. What is speed v of end of stick, when stick hits ground?



$I = \frac{1}{3} M L^2$ (axis at end)

Plan: Use conservation of energy to get ω, then $v = \omega\,r = \omega\,L$

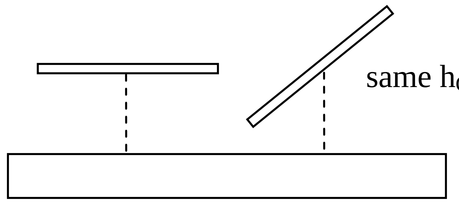
$E_i = E_f \Rightarrow M\,g\,h_{CM} = \frac{1}{2} I\,\omega^2$

Important point:

$PE_{grav} = Mgh$ where h = height of center-of-mass, independent of the orientation of the stick.

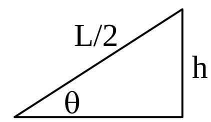
Proof: $PE_{grav} = \sum_i m_i\,g\,h_i = g\sum_i m_i\,h_i = g\,M\,Y_{CM} = M\,g\,h_{CM}$

(Have used definition of center-of-mass: $MY_{CM} = \sum_i m_i\, y_i$)



same h_{CM} , same PE_{grav}

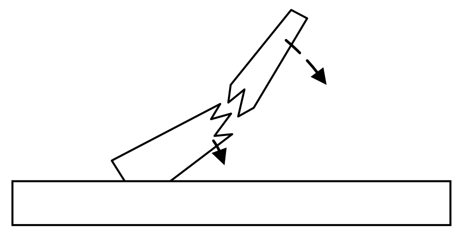
Back to the problem: $M g h_{CM} = \frac{1}{2} I \omega^2$, $h = \frac{1}{2} L \sin\theta$, $I = \dfrac{1}{3} M L^2$



$\cancel{M} g \frac{1}{\cancel{2}} \cancel{L} \sin\theta = \frac{1}{\cancel{2}} \frac{1}{3} \cancel{M} L^{\cancel{2}} \omega^2 \Rightarrow g \sin\theta = \frac{1}{3} L \omega^2$

Use $\omega = v/r = v/L$ to get: $3 g \sin\theta = L \dfrac{v^2}{L^2} = \dfrac{v^2}{L} \Rightarrow v = \sqrt{3 g L \sin\theta}$ Done.

The tip of the stick starts at height h_{tip} = L sinθ, but its final speed v is faster than the speed of an object that falls from that height h [$\frac{1}{2} m v^2 = m g h \Rightarrow v = \sqrt{2 g h}$]. The tip of the stick falls faster than it would in free-fall, because the central part of the rod pulls it down. This is why tall chimneys always break apart when toppled:



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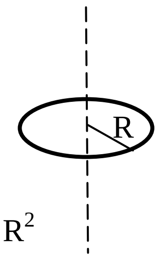
Let's Review: **Translation** ↔ **Rotation**

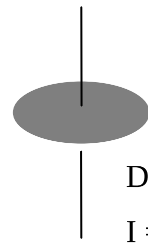
Translation		Rotation
x	↔	θ
$v = \dfrac{dx}{dt}$	↔	$\omega = \dfrac{d\theta}{dt}$
$a = \dfrac{dv}{dt}$	↔	$\alpha = \dfrac{d\omega}{dt}$
F	↔	τ
M	↔	I
F = Ma	↔	τ = I α
KE = (1/2) m v^2	↔	KE = (1/2) I ω^2

Appendix 1: Moments of Inertia for some shapes



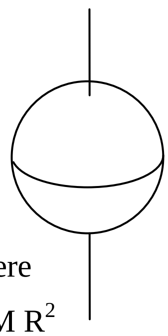
Hoop

$I = M R^2$



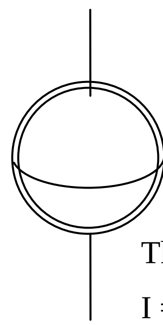
Disk

$I = (1/2) M R^2$



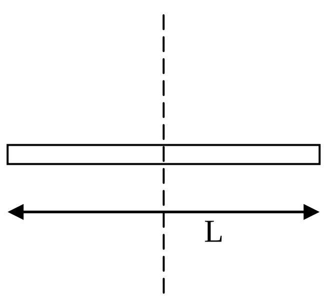
Solid sphere

$I = (2/5) M R^2$



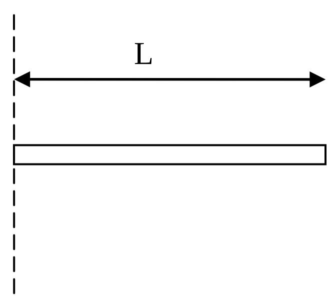
Thin spherical shell

$I = (2/3) M R^2$



Thin rod, axis thru center

$I = (1/12) M L^2$

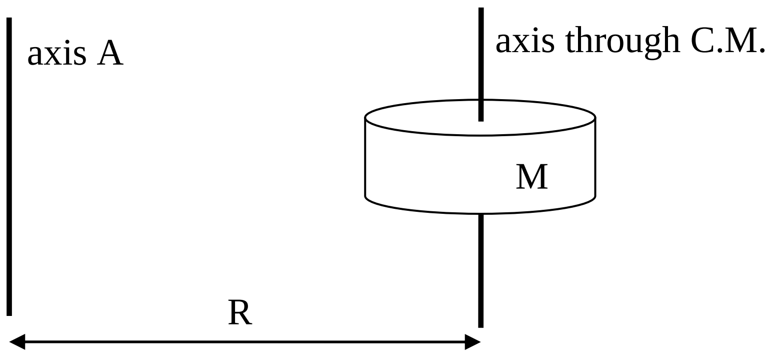


Thin rod, axis thru end

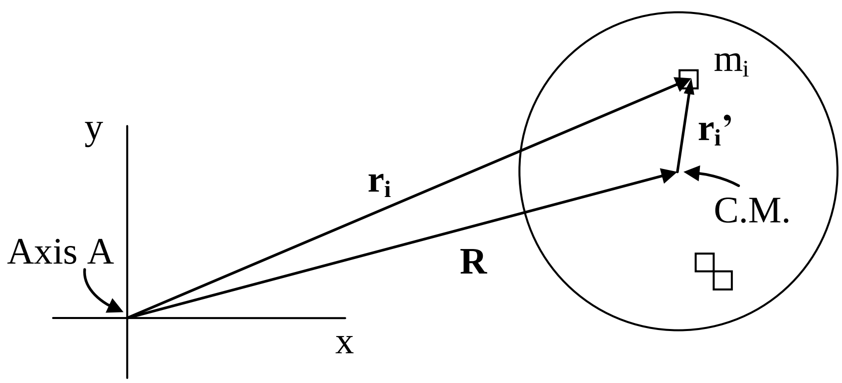
$I = (1/3) M L^2$

Appendix 2: Proof of Parallel Axis Theorem.

Consider an object with total mass M and with moment-of-inertia I_{cm} about an axis through the center-of-mass. Let axis A be any axis parallel to that center-of-mass axis. R is the distance between the axes.



Let's look at this situation from above the object, looking down along the axes. Let's place the origin of our xy coordinate system at the location of axis A.



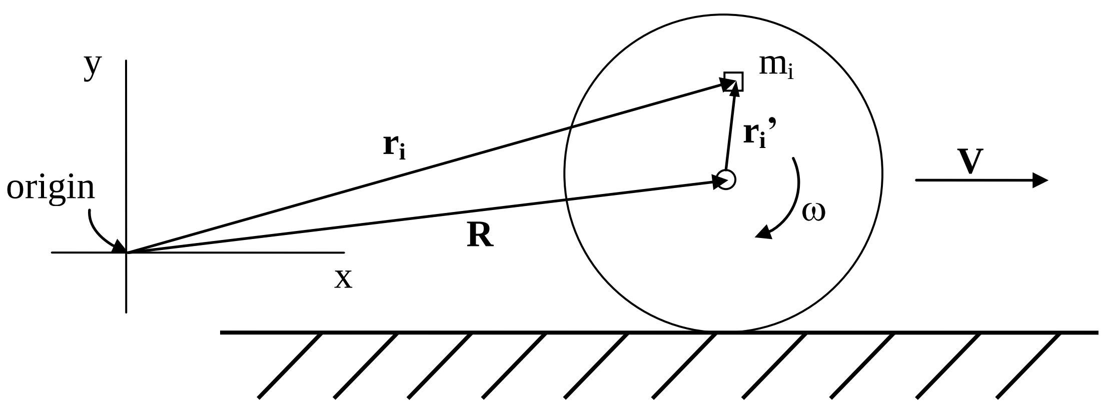
The position vector of the center-of-mass, relative to the origin, is **R.** We're using boldface type for vectors: **R** is the position vector. The magnitude of the vector **R** (bold) is R (not bold). A mass element m_i is a position $\mathbf{r_i}$, which can be written as $\mathbf{r_i = R + r_i'}$ **,** where the vector $\mathbf{r_i'}$ is the position of m_i, *relative to the center-of-mass*. Recall that the center-of-mass position vector **R** is defined by the equation $M\mathbf{R} = \sum_i m_i \mathbf{r}_i$. Notice that, if we had chosen the center-of-mass to be the location of the origin, then **R** would be zero, and $\mathbf{r_i}$ would be $\mathbf{r_i'}$**,** so $\sum_i m_i \mathbf{r_i'} = 0$ (we'll need this result in a moment).

Ok, the moment-of-inertia of the object about the axis A is $I = \sum_i m_i r_i^2$. For any position $\mathbf{r} = \mathbf{R} + \mathbf{r'}$, we have $r^2 = \mathbf{r} \cdot \mathbf{r} = (\mathbf{R} + \mathbf{r'}) \cdot (\mathbf{R} + \mathbf{r'}) = R^2 + r'^2 + 2\mathbf{R} \cdot \mathbf{r'}$. (Remember: bold is a vector, non-bold is the magnitude of the vector.) So we can write

$$I = \sum_i m_i r_i^2 = \sum_i m_i (R^2 + r_i'^2 + 2\mathbf{R} \cdot \mathbf{r}_i') = \underbrace{\left(\sum_i m_i\right)}_{M} R^2 + \underbrace{\sum_i m_i r_i'^2}_{Icm} + 2\mathbf{R} \cdot \underbrace{\sum_i m_i \mathbf{r}_i'}_{0}.$$

So $I = MR^2 + I_{CM}$. Done!

Appendix 3: Kinetic energy of a rolling wheel. Here we prove that the total kinetic energy of a rolling wheel (mass M, radius R, center-of-mass speed V) is the translational KE of the center-of-mass motion, $KE_{trans} = (1/2)MV^2$ plus the rotational KE about the C.M , $KE_{rot} = (1/2)I_{cm}\omega^2$. As in Appendix 2, we regard the position vector **r** of each piece of the wheel as the position **R** of the center-of-mass of the wheel plus the position **r'** of that piece, relative to the center-of-mass.



The vector equation relating the three vectors is: $\mathbf{r} = \mathbf{R} + \mathbf{r'}$. Taking the time derivative of this equation gives $\frac{d}{dt}\mathbf{r} = \frac{d}{dt}\mathbf{R} + \frac{d}{dt}\mathbf{r'}$, which is the same as $\mathbf{v} = \mathbf{V} + \mathbf{v'}$. In words: the velocity **v** of each particle of the rolling wheel is the velocity **V** of the center of the wheel, vector-added to the velocity **v'** of the particle *relative to the center of the wheel*. We showed in Appendix 2 that $\sum_i m_i \mathbf{r_i'} = 0$. Taking the time derivative of this equation gives $\sum_i m_i \mathbf{v_i'} = 0$, a result we will use below.

The total kinetic energy of the wheel is $KE_{tot} = \sum_i \frac{1}{2} m_i v_i^2$. Now we can write

$v^2 = \mathbf{v}\cdot\mathbf{v} = (\mathbf{V}+\mathbf{v'})\cdot(\mathbf{V}+\mathbf{v'}) = V^2 + v'^2 + 2\mathbf{V}\cdot\mathbf{v'}$ (remember: bold means vector, non-bold means magnitude.) So, we can rewrite the total KE as

$$KE_{tot} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\underbrace{\left(\sum_i m_i\right)}_{M} V^2 + \frac{1}{2}\left(\sum_i m_i v'^2_i\right) + \mathbf{V}\cdot\underbrace{\left(\sum_i m_i \mathbf{v'_i}\right)}_{0}.$$

Now recall that $v' = \omega r'$. So we have

$$KE_{tot} = \frac{1}{2} M V^2 + \frac{1}{2}\underbrace{\left(\sum_i m_i r'^2_i\right)}_{Icm}\omega^2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{cm}\,\omega^2 .$$

Done!

