Lecture 9: Linear Regression

Goals

- Develop basic concepts of linear regression from a probabilistic framework
- Estimating parameters and hypothesis testing with linear models
- Linear regression in R

Regression

- Technique used for the modeling and analysis of numerical data

- Exploits the relationship between two or more variables so that we can gain information about one of them through knowing values of the other

- Regression can be used for prediction, estimation, hypothesis testing, and modeling causal relationships

Regression Lingo

$$Y = X_1 + X_2 + X_3$$

Dependent Variable	Independent Variable
Outcome Variable	Predictor Variable
Response Variable	Explanatory Variable

Why Linear Regression?

- Suppose we want to model the dependent variable Y in terms of three predictors, X_1, X_2, X_3

$$\mathbf{Y = \mathit{f}(X_1, X_2, X_3)}$$

- Typically will not have enough data to try and directly estimate f

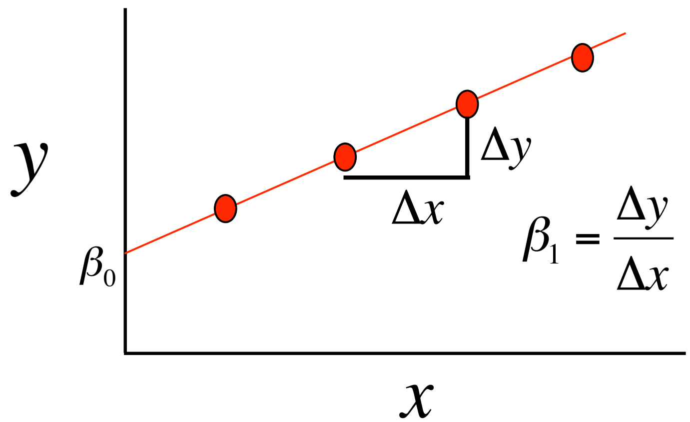
- Therefore, we usually have to assume that it has some restricted form, such as linear

$$\mathbf{Y = X_1 + X_2 + X_3}$$

Linear Regression is a Probabilistic Model

- Much of mathematics is devoted to studying variables that are deterministically related to one another

$$y = \beta_0 + \beta_1 x$$



- But we're interested in understanding the relationship between variables related in a nondeterministic fashion

A Linear Probabilistic Model

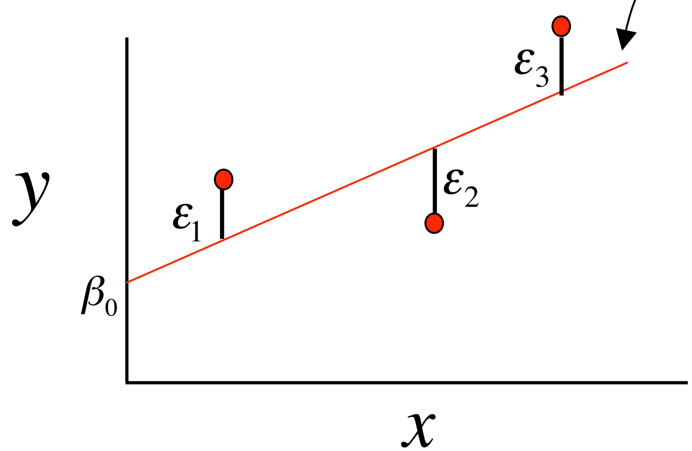
- Definition: There exists parameters β_0, β_1, and σ^2, such that for any fixed value of the independent variable x, the dependent variable is related to x through the model equation

$$y = \beta_0 + \beta_1 x + \varepsilon$$

- ε is a rv assumed to be $N(0, \sigma^2)$

True Regression Line

$$y = \beta_0 + \beta_1 x$$



Implications

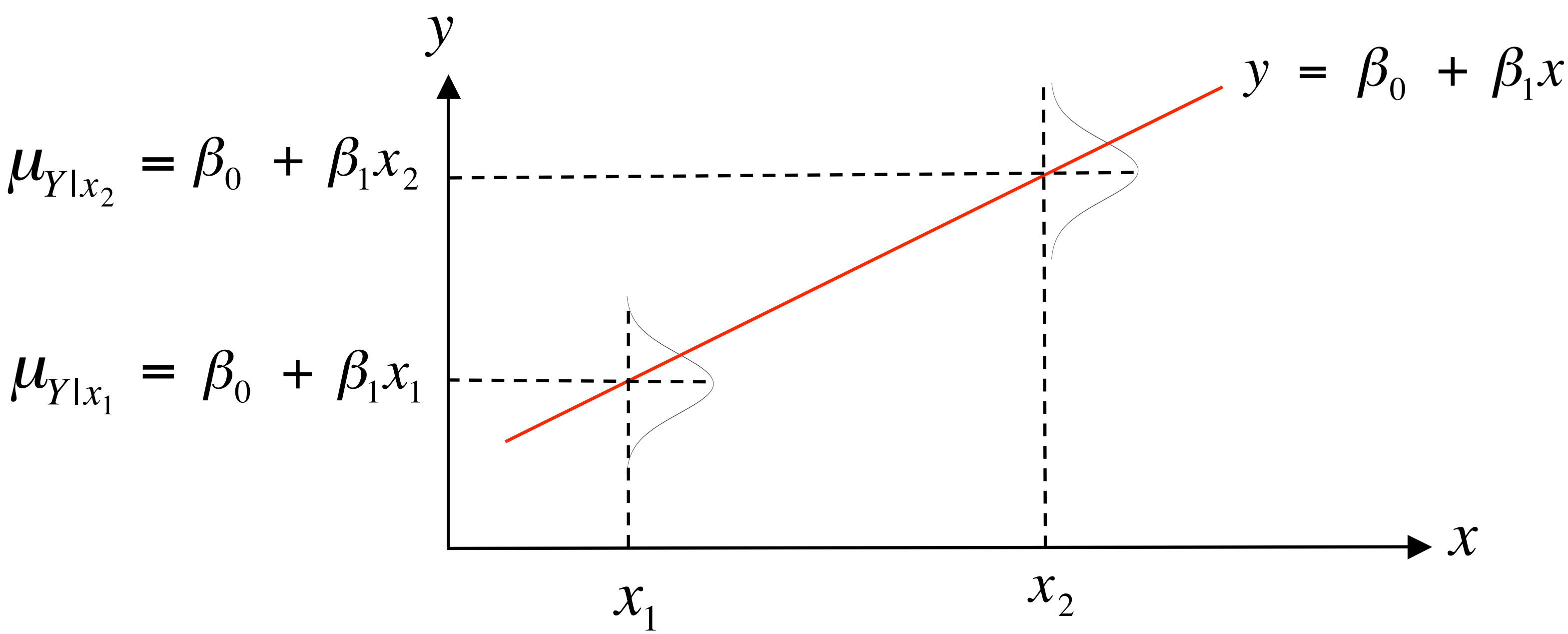
- The ***expected*** value of Y is a linear function of X, but for fixed x, the variable Y differs from its expected value by a random amount

- Formally, let x* denote a particular value of the independent variable x, then our linear probabilistic model says:

$$E(Y \mid x^*) = \mu_{\mathrm{Y|x*}} = \text{mean value of } Y \text{ when } x \text{ is } x^*$$

$$V(Y \mid x^*) = \sigma^2_{\mathrm{Y|x*}} = \text{variance of } Y \text{ when } x \text{ is } x^*$$

Graphical Interpretation



- For example, if x = height and y = weight then $\mu_{Y|x=60}$ is the average weight for all individuals 60 inches tall in the population

One More Example

Suppose the relationship between the independent variable height (x) and dependent variable weight (y) is described by a simple linear regression model with true regression line

y = 7.5 + 0.5x and $\sigma = 3$

- Q1: What is the interpretation of β_1 = 0.5?

  The expected change in height associated with a 1-unit increase in weight

- Q2: If x = 20 what is the expected value of Y?

$$\mu_{Y|\mathrm{x}=20} = 7.5 + 0.5(20) = 17.5$$

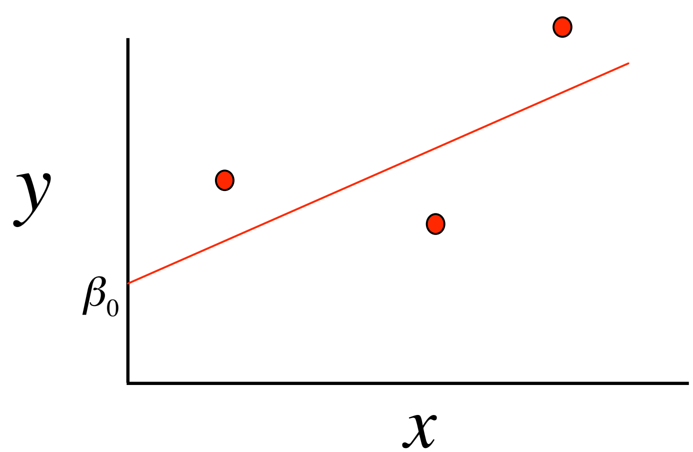
- Q3: If x = 20 what is P(Y > 22)?

$$P(\mathrm{Y} > 22 \,|\, \mathrm{x} = 20) = \mathrm{P}\left(\frac{22 - 17.5}{3}\right) = 1 - \phi(1.5) = 0.067$$

Estimating Model Parameters

- Point estimates of $\hat{\beta}_0$ and $\hat{\beta}_1$ are obtained by the principle of least squares

$$f(\beta_0, \beta_1) = \sum_{i=1}^{n} \left[y_i - (\beta_0 + \beta_1 x_i) \right]^2$$



- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

Predicted and Residual Values

- ***Predicted***, or fitted, values are values of y predicted by the least-squares regression line obtained by plugging in $x_1, x_2, \ldots, x_n$ into the estimated regression line

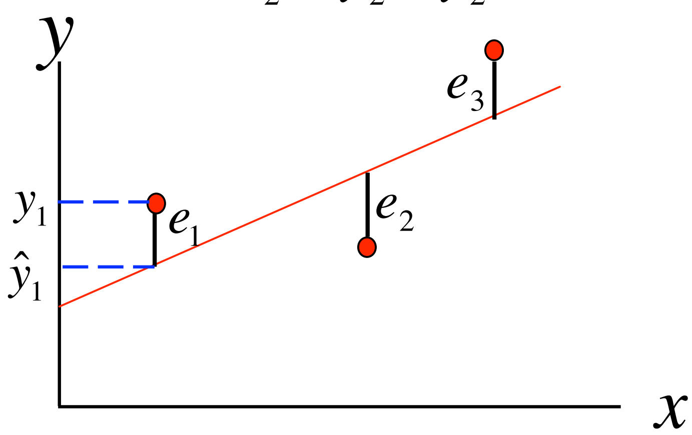
$$\hat{y}_1 = \hat{\beta}_0 - \hat{\beta}_1 x_1$$

$$\hat{y}_2 = \hat{\beta}_0 - \hat{\beta}_1 x_2$$

- ***Residuals*** are the deviations of observed and predicted values

$$e_1 = y_1 - \hat{y}_1$$

$$e_2 = y_2 - \hat{y}_2$$



Residuals Are Useful!

- They allow us to calculate the error sum of squares (SSE):

$$SSE = \sum_{i=1}^{n} (e_i)^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

- Which in turn allows us to estimate σ^2:

$$\hat{\sigma}^2 = \frac{SSE}{n-2}$$

- As well as an important statistic referred to as the coefficient of determination:

$$r^2 = 1 - \frac{SSE}{SST} \qquad SST = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

Multiple Linear Regression

- Extension of the simple linear regression model to two or more independent variables

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + \varepsilon$$

Expression = Baseline + Age + Tissue + Sex + Error

- Partial Regression Coefficients: β_i ≡ effect on the dependent variable when increasing the i^{th} independent variable by 1 unit, **holding all other predictors constant**

Categorical Independent Variables

- Qualitative variables are easily incorporated in regression framework through ***dummy variables***

- Simple example: sex can be coded as 0/1

- What if my categorical variable contains three levels:

$$x_i = \begin{cases} 0 \text{ if AA} \\ 1 \text{ if AG} \\ 2 \text{ if GG} \end{cases}$$

Categorical Independent Variables

- Previous coding would result in ***colinearity***
- Solution is to set up a series of dummy variable. In general for k levels you need k-1 dummy variables

$$x_1 = \begin{cases} 1 \text{ if AA} \\ 0 \text{ otherwise} \end{cases}$$

$$x_2 = \begin{cases} 1 \text{ if AG} \\ 0 \text{ otherwise} \end{cases}$$

	x_1	x_2
AA	1	0
AG	0	1
GG	0	0

Hypothesis Testing: Model Utility Test (or Omnibus Test)

- The first thing we want to know after fitting a model is whether any of the independent variables (X's) are significantly related to the dependent variable (Y):

$$\mathrm{H}_0 \ : \ \beta_1 = \beta_2 = \ldots = \beta_k = 0$$

$$\mathrm{H}_\mathrm{A} \ : \ \text{At least one } \beta_1 \neq 0$$

$$f = \frac{R^2}{(1-R^2)} \bullet \frac{k}{n-(k+1)}$$

$$\text{Rejection Region}: \ F_{\alpha,k,n-(k+1)}$$

Equivalent ANOVA Formulation of Omnibus Test

- We can also frame this in our now familiar ANOVA framework
  - partition total variation into two components: ***SSE*** (unexplained variation) and ***SSR*** (variation explained by linear model)

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Source of Variation	df	Sum of Squares	MS	F
Regression	k	$SSR = \sum (\hat{y}_i - \bar{y})^2$	$\frac{SSR}{k}$	$\frac{MS_R}{MS_E}$
Error	n-2	$SSE = \sum (y_i - \hat{y}_i)^2$	$\frac{SSE}{n-2}$	
Total	n-1	$SST = \sum (y_i - \bar{y})^2$		

$$\text{Rejection Region}: \; F_{\alpha,k,n-(k+1)}$$

F Test For Subsets of Independent Variables

- A powerful tool in multiple regression analyses is the ability to compare two models

- For instance say we want to compare:

$$\text{Full Model: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \varepsilon$$

$$\text{Reduced Model: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

- Again, another example of ANOVA:

SSE_R = error sum of squares for reduced model with l predictors

SSE_F = error sum of squares for full model with k predictors

$$f = \frac{(SSE_R - SSE_F)/(k-l)}{SSE_F/([n-(k+1)]}$$

Example of Model Comparison

- We have a quantitative trait and want to test the effects at two markers, M1 and M2.

Full Model: Trait = Mean + M1 + M2 + (M1*M2) + error

Reduced Model: Trait = Mean + M1 + M2 + error

$$f = \frac{(SSE_R - SSE_F)/(3-2)}{SSE_F/([100-(3+1)]} = \frac{(SSE_R - SSE_F)}{SSE_F/96}$$

$$\text{Rejection Region}: \; F_{a,\,1,\,96}$$

Hypothesis Tests of Individual Regression Coefficients

- Hypothesis tests for each $\hat{\beta}_i$ can be done by simple t-tests:

$$\mathrm{H}_0 \;:\; \hat{\beta}_i = 0$$

$$\mathrm{H}_\mathrm{A} \;:\; \hat{\beta}_i \neq 0$$

$$\mathrm{T} = \frac{\hat{\beta}_i - \beta_i}{se(\beta_i)}$$

$$\text{Critical value}:\; t_{\alpha/2,n-(k-1)}$$

- Confidence Intervals are equally easy to obtain:

$$\hat{\beta}_i \pm t_{\alpha/2,n-(k-1)} \bullet se(\hat{\beta}_i)$$

Checking Assumptions

- Critically important to examine data and check assumptions underlying the regression model
    - Outliers
    - Normality
    - Constant variance
    - Independence among residuals

- Standard diagnostic plots include:
    - scatter plots of y versus x_i (outliers)
    - qq plot of residuals (normality)
    - residuals versus fitted values (independence, constant variance)
    - residuals versus x_i (outliers, constant variance)

- We'll explore diagnostic plots in more detail in R

Fixed -vs- Random Effects Models

- In ANOVA and Regression analyses our independent variables can be treated as ***Fixed*** or ***Random***

- **Fixed Effects**: variables whose levels are either sampled exhaustively or are the only ones considered relevant to the experimenter

- **Random Effects**: variables whose levels are randomly sampled from a large population of levels

- Example from our recent AJHG paper:

Expression = Baseline + Population + Individual + Error