Compound interest, number e and natural logarithm

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- What is the difference between a bank account advertising 8% compounded annually and the one offering 8% compounded quarterly?
- Assume we deposit \$1000, find the balance B after t years (assume that the interest will not be withdrawn).

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- We call the 8% the **nominal rate** (nominal means "in name only").

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- **Extra question:** Write an expression for the balance in each bank after t years.

Using the Effective Annual Yield

If interest at an annual rate of r is compounded n times a year, i.e. r/n **times of the current balance** is added n times a year, then, with an initial deposit P, the balance t years later is

$$B = P\left(1 + \frac{r}{n}\right)^{nt}.$$

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- The difference is small (7.25056% and 7.25079%).

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- The values 1.0725082 is an upper bound that is approached as the frequency of compounding increase.
- When the effective annual rate is at this upper bound, we say that the interest is being **compounded continuously**.

Number e

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- The upper bound is called **Euler constant**, denoted by e.
- $$\left(1+\frac{0.07}{n}\right)^n \approx 1.0725082 \approx e^{0.07}$$
- If P is deposited at an annual rate 7% compounded continuously, the balance B after t year is $B = P(e^{0.07})^t$.

Number e

Definition

If the interest on an initial deposit P is compounded continuously at an annual rate r, the balance t years alter can be calculated using the formula

$$B = Pe^{rt}.$$

Natural Logarithm

Definition

The natural logarithm of x, written by $\ln x$, is the power of e needed to get x. In the other word,

$$\ln x = c \qquad \text{means} \qquad e^c = x.$$

The natural logarithm is sometimes written by $\log_e x$.

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- $e^{\ln x} = x$

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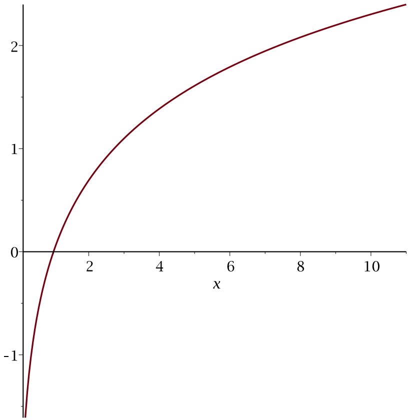

Solving equation using logarithms

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- $x = \frac{\ln 12}{\ln 4}$
- $x \approx 1.79248$

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- $t \ln(1.036) = \ln\left(\frac{5}{2.020}\right)$
- $t = \frac{\ln(5/2.020)}{\ln(1.036)} = 25.627$ years

Solving equation using logarithms

Problem 4. Find t such that $12 = 5e^{3t}$.

Exponential function with base e

Definition

Writing $a = e^k$, where $k = \ln a$, any exponential function can be written in two forms

$$P = P_0 a^t \qquad \text{or} \qquad P = P_0 e^{kt}.$$

- If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.

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- If $k > 0$, we have exponential growth; if $k < 0$, we have exponential decay.
- k is called the **continuous growth** or **continuous decay**.

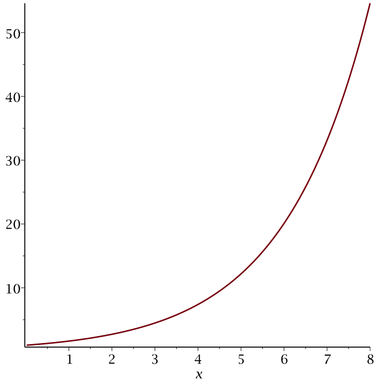
Exponential function with base e

- Convert the function $P = 1000e^{0.4t}$ to the form $P = P_0 a^t$.

Exponential function with base e

- Convert the function $P = 1000e^{0.4t}$ to the form $P = P_0 a^t$.
- Convert the function $P = 200(2.3)^t$ to the form $P = P_0 e^{kt}$.

Function $P = e^{0.5x}$



Function $P = 5e^{-0.2x}$