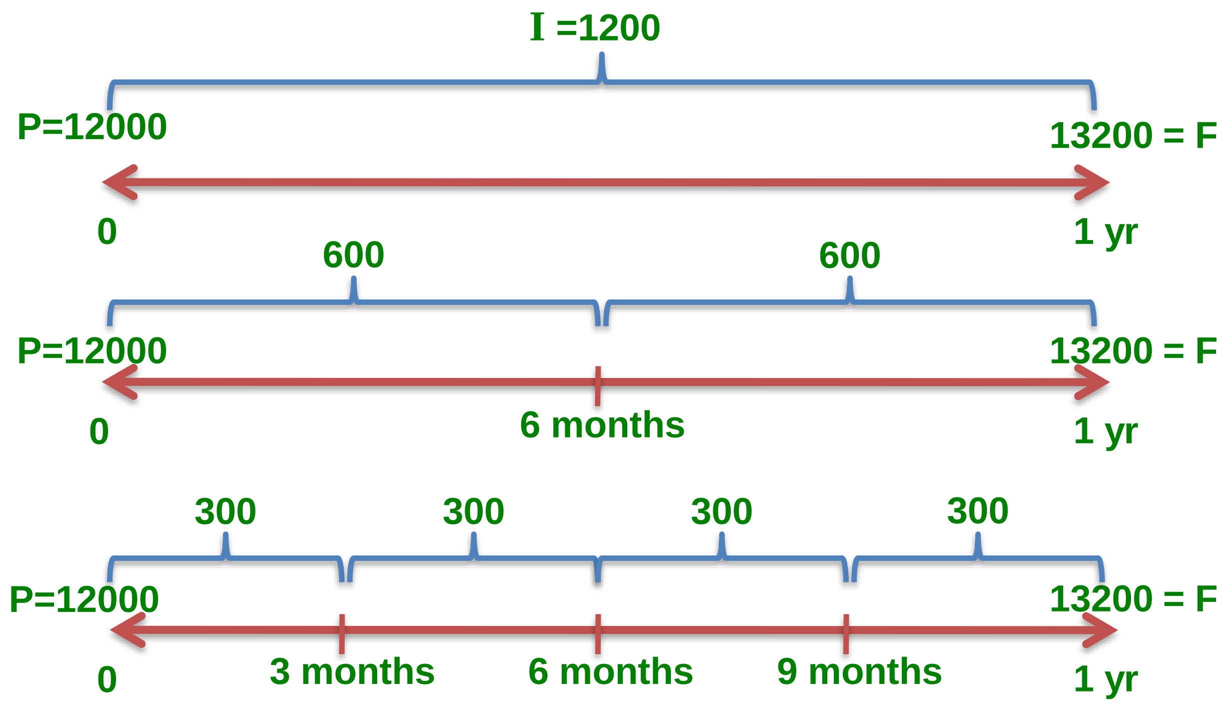
Compound Interest

The yield of simple interest is constant all throughout the investment or loan term.

$P = 12000$; $r = 10\% = 0.1$; $t = 1$ year ; $F = ?$; $I = ?$

$F = 12000\,(1 + (0.1)(1)) = 13200$ $I = F - P = 13200 - 12000 = 1200$



Note that the interest yield at a certain cut off date or time interval is constant all throughout the investment or loan term.

When interest yield or earned is added to the principal at regular time interval and the sum becomes the new principal then interest is said to be compounded or converted.

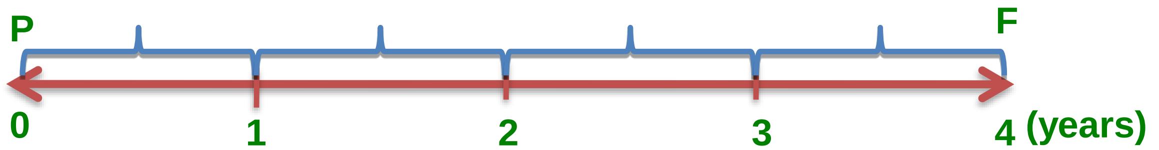
At compound interest, interest earned at a certain cut-off date is automatically reinvested to earn more interest.

When interest is being converted or compounded once or more than once a year, the time between successive conversions of interest is called a conversion or interest period or simply period.

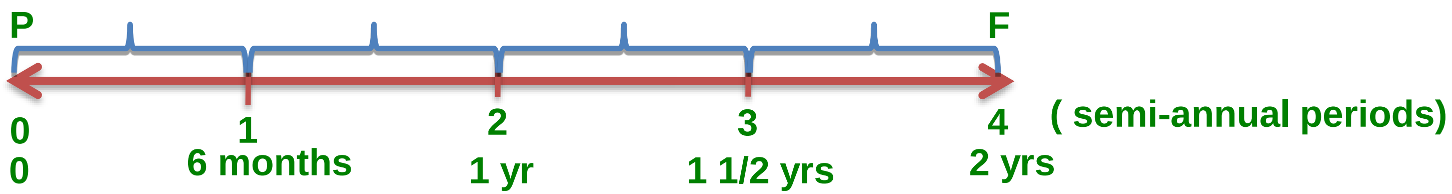
The number of conversion or interest periods in a year is called the frequency of conversion (m).

So interest may be compounded or converted

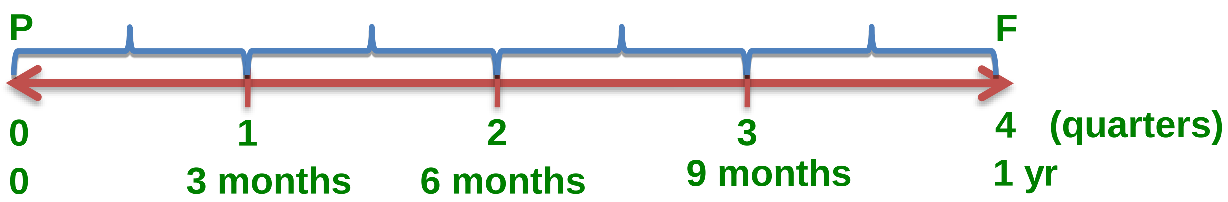
(i) **Annually** (once a year / every year) $m = 1$



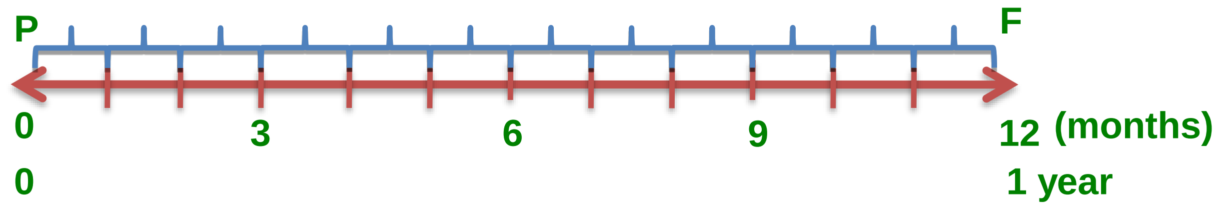
(ii) **Semi-annually** (twice a year / every 6 months) $m = 2$



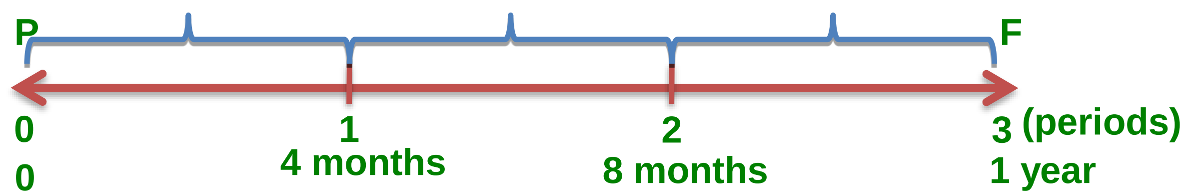
(iii) **Quarterly** (4 times a year / every 3 months) $m = 4$



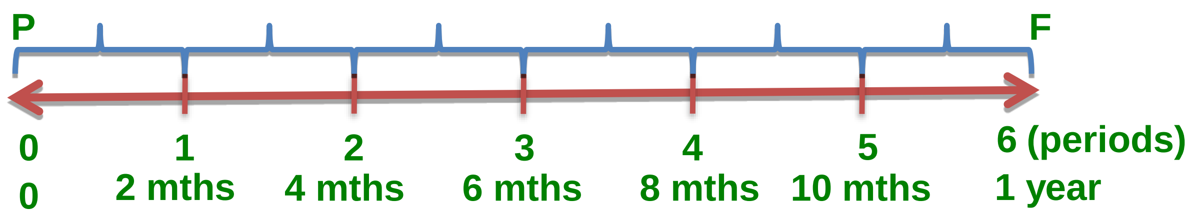
(iv) Monthly (12 times year / every month) $m = 12$



(v) Every 4 months (3 times year) $m = 3$



(vi) Every 2 months (6 times a year) $m = 6$



The stated annual rate of interest (converted *m* times a year) is called the <u>nominal rate</u> j.

The rate of interest per period is $i = j/m$ and the total number of conversion period is $n = t\,m$.

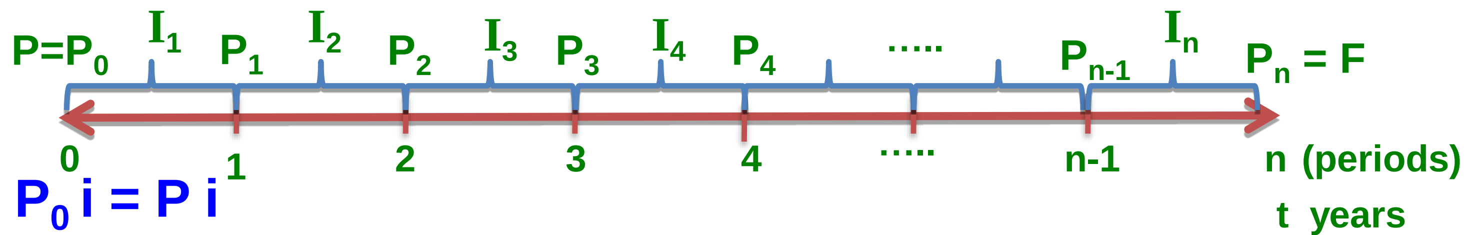
The final amount under compound interest is called the compound amount (F). The difference between compound amount F and original principal P is called the compound interest.

Find the compound amount F if P is invested at nominal rate j converted m times a year for a term of t years.

Let $P_0 = P$ (original principal) ; $i = j/m$; $n = t\,m$

I_k = interest earned at the end of the kth period

P_k = new principal at the end of the kth period = $P_{k-1} + I_k$



$$I_1 = P_0\, i = P\, i$$

$$P_1 = P_0 + I_1 = P + P\, i = P\,(1 + i\,)$$

$$\boxed{F = P\,(1 + i)^n}$$

$$I_2 = P_1\, i = P\,(1+i)\, i$$

$$P_2 = P_1 + I_2 = P(1+i)+P(1+i)i = P(1+i\,)(1+i) = P(1+i)^2$$

$$I_3 = P_2\, i = P(1+i)^2\, i$$

$$P_3 = P_2 + I_3 = P(1+i)^2+P(1+i)^2 i = P(1+i\,)^2(1+i) = P(1+i)^3$$

$$I_4 = P_3\, i = P(1+i)^3\, i$$

$$P_4 = P_3 + I_4 = P(1+i)^3+P(1+i)^3 i = P(1+i\,)^3(1+i) = P(1+i)^4$$

$\vdots$

$P_{n-1} = P(1+i)^{n-1}$ and $I_n = P_{n-1}\, i = P(1+i)^{n-1}\, i$

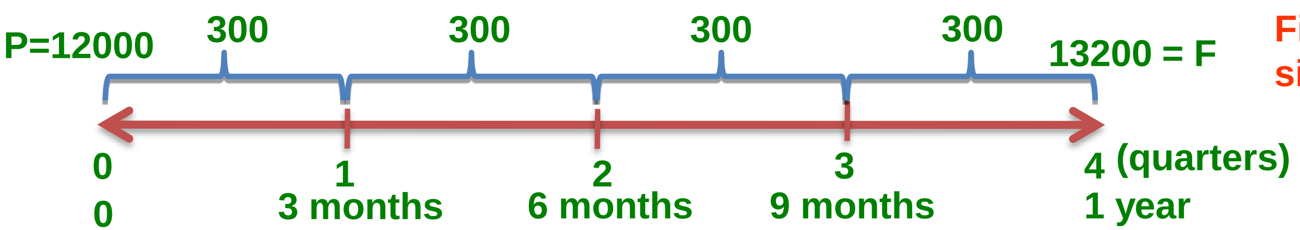
$$P_n = F = P_{n-1} + I_n = P(1+i)^{n-1} + P(1+i)^{n-1} i = P(1+i)^{n-1}(1+i) = P\,(1+i)^n$$

$P = 12000$; $j = 10\% = 0.10$ (compounded quarterly) ; $m = 4$

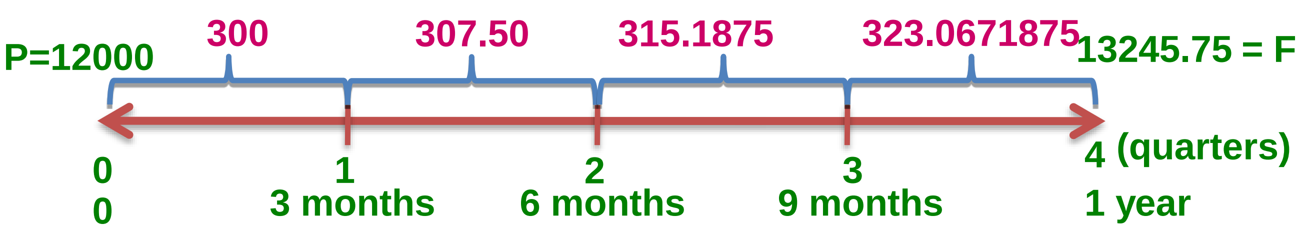
$i = j/m = 0.10/4 = 0.025$; $t = 1$ year ; $n = 1\,(4) = 4$

$F = 12000\,(1 + 0.025)^4 = 13245.75$ (compound amount)

$I = F - P = 13245.75 - 12000 = 1245.75$ (compound interest)



Final amount under simple interest



Final amount under compound interest

$I_1 = 12000(0.025) = 300$

$I_2 = 12300(0.025) = 307.50$

$I_3 = 12607.50(0.025) = 315.1875$

$I_4 = 12922.6875(0.025) = 323.0671875$

$P_1 = P + I_1 = 12000 + 300 = 12300$

$P_2 = P_1 + I_2 = 12300 + 307.50 = 12607.50$

$P_3 = P_2 + I_3 = 12607.50 + 315.1875 = 12922.6875$

$F = P_4 = P_3 + I_4 = 12922.6875 + 323.0671875$

$F = 13245.75469$

- **Formula for the compound amount F:**

$$F = P\boxed{(1+i)^n}$$ ***Accumulation factor***

P - original principal

j - rate of interest per year (nominal rate)

m - frequency of conversion

i - interest rate per priod; $i = \frac{j}{m}$

t - term in years

n - total number of conversion periods; $n = tm$

- **Table of the Frequency of Conversion**

Nominal Rate Converted	Frequency of Conversion (m)
Annually	1
Semi-annually	2
Quarterly	4
Monthly	12
Every 4 months	3
Every 2 months	6

$$\boxed{F = P\,\overbrace{(1+i)^n}^{\text{accumulation factor}}}$$

Compound amount F is the accumulated value of principal P at the end of n periods.

"To accumulate" means to find F.

Ex 2 Accumulate P80,000 for 7 years at 15% compounded every 4 months.

$P = 80{,}000$; $m = 3$; $j = 15\% = 0.15$; $i = 0.15/3 = 0.05$

$t = 7$ years ; $n = 7(3) = 21$; $F = ?$

$$F = P(1+i)^n$$

$$F = 80000\,(1+0.05)^{21}$$

$$F = 222{,}877.01$$

Ex3. Find the compound amount and interest at the end of 6 years if P80,00 is invested at $12\frac{1}{3}\%$ compounded a) semi-annually b) monthly .

a) P=80,000 t = 6 yr j = $\frac{37}{300}$ m = 2

$i = \frac{\frac{37}{300}}{2} = \frac{37}{600}$ n = (6)(2)=12

$$F = P(1+i)^n = 80000(1+\frac{37}{600})^{12} = 164{,}039.40$$

$$I = F - P = 164039.40 - 80000 = 84{,}039.40$$

b) m = 12 $i = \frac{\frac{37}{300}}{12} = \frac{37}{3600}$ n = (6)(12) = 72

$$F = P(1+i)^n = 80000(1+\frac{37}{3600})^{72} = 167{,}042.74$$

$$I = F - P = 167042.74 - 80000 = 87{,}042.74$$

$$\underbrace{(1+i)^{-n}}_{\text{discount factor}}$$

$$\boxed{P = F\,\underbrace{(1+i)^{-n}}_{}} \longleftarrow \boxed{F = P\,(1+i)^{n}}$$

Present value of an amount F due in n periods is the value P (principal) which is invested **now** at a given nominal rate j.

"To discount F" means to find its present value P at n periods before F is due.

$$P = F(1+i)^{-n}$$ *Discount factor*

Ex 1 **A man needs P500,000 in 3 years to start a small business. How much money should he place in an account <u>now</u> that gives 4.02% compounded semi-annually so he can start the business by then?**

$F = 500{,}000$; $m = 2$; $j = 4.02\% = 0.0402$; $i = 0.0402/2 = 0.0201$

$t = 3$ years ; $n = 3(2) = 6$; $P = ?$

$P = F(1+i)^{-n} \longrightarrow P = 500000\,(1+0.0201)^{-6}$

$P = 443{,}724.61$

Ex 2 **In purchasing a unit of I-phone 6S, Hans makes a down payment of P5000 and agrees to pay P50,000 15 months later. Find the cash value of the I-phone if money is worth 9% compounded monthly.**

Cash value (CV) = Down payment (D) + Present Value (P)

$F = 50{,}000$; $m = 12$; $j = 9\% = 0.09$; $i = 0.09/12 = 0.0075$

$t = 15$ months $= 15/12$ years ; $n = (15/12)(12) = 15$

$D = 5000$; $P = ?$; $CV = ?$

$P = F(1+i)^{-n}$ $\longrightarrow$ $P = 50000(1+0.0075)^{-15}$

$$\boxed{P = 44{,}698.63}$$

$$CV = D + P$$

$$CV = 5000 + 44698.63$$

$$\boxed{CV = 49{,}698.63}$$

Ex 3 On her 18th birthday, Liza receives P20,000 as gift from her parents. If she invests this money in a bank that gives 3% interest converted every 2 months, how much money will she have on her 25th birthday? How much interest will she earn?

$P = 20000$; $t = 7$ years ; $m = 6$; $n = 7(6) = 42$; $i = 0.03/6 = 0.005$

Ans: $F = 20000\,(1+0.005)^{42} = 24{,}660.65$; $I = 4660.65$

Ex 4 The buyer of a car pays P150,000 down payment and the balance of P500,000 to be paid two years later. What is the cash price of the car if money is worth 12% compounded annually?

$D = 150{,}000$; $F = 500{,}000$; $m = 1$; $t = 2$ yrs ; $n = 2$; $j = 0.12$; $i = 0.12$

Ans: $P = 500000\,(1+0.12)^{-2} = 398{,}596.94$

$CV = CP = 150{,}000 + 398{,}596.94 = 548{,}596.94$

Ex 5 What is the maturity value of a 75,000-peso, three-year investment earning 5% compounded monthly?

$$P = 75{,}000 \; ; \; m = 12 \; ; \; j = 0.05 \; ; \; i = \tfrac{0.05}{12} \; ; \; n = 3(12) = 36$$

$$F = 75{,}000\left(1 + \tfrac{0.05}{12}\right)^{36}$$

$$F = 87{,}110.42$$

Do this if i is not a terminating decimal

Ex 6 Find the compound amount after 5 years and 9 months if the principal is P150,000 and the rate is 7% compounded quarterly.

$$P = 150{,}000 \quad ; \quad t = 5 + \tfrac{9}{12} = 5 + \tfrac{3}{4} = \tfrac{23}{4}\text{ yrs}; \quad m = 4; \quad n = \tfrac{23}{4}(4) = 23$$

$$j = 0.07 \;\; ; \;\; i = \tfrac{0.07}{4} = 0.0175$$

$$F = 150{,}000\left(1 + \tfrac{0.07}{4}\right)^{23} = 223{,}554.22$$

Finding Interest Rate (Compound Interest)

$$j = ? \qquad i = \frac{j}{m} \quad \Rightarrow \quad j = i(m)$$

$$F = P(1+i)^n$$

$$\frac{F}{P} = (1+i)^n \Rightarrow \left[\frac{F}{P} = (1+i)^n\right]^{\frac{1}{n}} \Rightarrow \left(\frac{F}{P}\right)^{\frac{1}{n}} = 1+i$$

$$i = \left[\left(\frac{F}{P}\right)^{\frac{1}{n}} - 1\right] \Rightarrow$$ ***Nominal rate*** $$j = i(m)$$

Ex 1 At what nominal rate compounded quarterly will P30,000 amount to P45,000 in 3 years?

$$P = 30{,}000 \quad ; \quad F = 45{,}000 \;;\; t = 3\text{yrs} \;;\; m = 4; \;\; n = 3(4) = 12$$

$$j = i(4) \;;\; i = ?$$

$$45{,}000 = 30{,}000(1+i)^{12} \Rightarrow \left[\frac{45000}{30000} = (1+i)^{12}\right]^{\frac{1}{12}}$$

$$\left(\frac{45000}{30000}\right)^{\frac{1}{12}} - 1 = i \;\; \Rightarrow \;\; i = 0.034366083$$

$$j = i(4) = 0.137464332 = 0.1375 = 13.75\%$$

Ex 2 Allan borrows P135,000 and agrees to pay P142,000 for a debt in 1 year and 3 months from now. At what rate compounded monthly is he paying interest ?

$$P = 135{,}000 \quad ; \quad F = 142{,}000$$

$$t = 1 + \tfrac{3}{12} = 1 + \tfrac{1}{4} = \tfrac{5}{4} \ \text{yrs}; \ \ m = 12; \ \ n = \tfrac{5}{4}(12) = 15$$

$$j = i(12) \ ; \ \ i = ?$$

$$142{,}000 = 135{,}000(1+i)^{15} \Rightarrow \left[\tfrac{142000}{135000} = (1+i)^{15}\right]^{\frac{1}{15}}$$

$$\left(\tfrac{142000}{135000}\right)^{\frac{1}{15}} - 1 = i \ \ \Rightarrow \ \ i = 0.00337583729$$

$$j = i(12) = 0.040510047 = 0.0405 = 4.05\%$$

Ex 3 If Bobby get P56,471.27 at the end of 4 years and 6 months for investing P25,000 now. At what rate compounded semi-annually is he earning interest ?

$$P = 25{,}000 \quad ; \quad F = 56{,}471.27 \; ; \; t = 4 + \tfrac{6}{12} = 4 + \tfrac{1}{2} = \tfrac{9}{2}\text{ yrs} \; ;$$

$$m = 2 \; ; \; n = \tfrac{9}{2}(2) = 9$$

$$j = i(2) \; ; \; i = ?$$

$$56{,}471.27 = 25{,}000(1+i)^9 \Rightarrow \left[\tfrac{56471.27}{25000} = (1+i)^9\right]^{\frac{1}{9}}$$

$$\left(\tfrac{56471.27}{25000}\right)^{\frac{1}{9}} - 1 = i \;\; \Rightarrow \;\; i = 0.094764833$$

$$j = i(2) = 0.189529667 = 0.1895 = 18.95\%$$

Ex 4 **On June 30, 2010, Cyril invested P30,000 in a bank that pays interest converted quarterly . If she wants her money to be 4 times as large on Dec 30, 2016, at what rate should her money earn interest ?**

$$P = 30{,}000 \quad ; \quad F = 4(30{,}000) \;;\; m = 4$$

$$\text{O.D.} = 6/30/10 \quad ; \quad \text{M.D.} = 12/30/16 \qquad t = 6 + \tfrac{6}{12} = 6 + \tfrac{1}{2} = \tfrac{13}{2}\text{ yrs} \rightarrow n = \tfrac{13}{2}(4) = 26$$

$$6/30/10 \text{ to } 6/30/16 \rightarrow 6 \text{ yrs}$$

$$6/30/16 \text{ to } 12/30/16 \rightarrow 6 \text{ months}$$

$$j = i(4) \;;\; i = ?$$

$$4(30{,}000) = 30{,}000(1+i)^{26} \Rightarrow \left[\frac{4(30000)}{30000} = 4 = (1+i)^{26}\right]^{\frac{1}{26}}$$

$$(4)^{\frac{1}{26}} - 1 = i \;\Rightarrow\; i = 0.054766076$$

$$j = i(4) = 0.219064305 = 0.2191 = 21.91\%$$

Properties of Logarithm or Laws of Logarithm

1) $\log_b(MN) = \log_b M + \log_b N \rightarrow \ln(MN) = \ln M + \ln N$

2) $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \rightarrow \ln\left(\frac{M}{N}\right) = \ln M - \ln N$

3) $\log_b N^r = r\log_b N \rightarrow \ln N^r = r\ln N$

4) $\log_b b = 1 \rightarrow \ln e = 1$

5) $\log_b 1 = 0 \rightarrow \ln 1 = 0$

6) $\log_b b^x = x \rightarrow \ln e^x = x$

EX. start with 3 $\rightarrow 2^3 = 8 \rightarrow \log_2\left(2^3\right) = \log_2(8) = 3$

7) $b^{\log_b x} = x \rightarrow e^{\ln x} = x$

EX. start with 8 $\rightarrow \log_2(8) = 3 \rightarrow 2^{\log_2 8} = 2^3 = 8$

Ex 1 How long will it take P50,000 to accumulate to P58,000 at 12% converted every 2 months?

$$P = 50{,}000 \;\; ; \;\; F = 58{,}000 \; ; \;\; m = 6 \; ; \;\; j = 0.12 \; ; \;\; i = \tfrac{0.12}{6} = 0.02$$

$$t = \tfrac{n}{6}; \;\; n = ?$$

$$58{,}000 = 50{,}000\left(1 + \tfrac{0.12}{6}\right)^{n} \Rightarrow \log\left(\tfrac{58000}{50000}\right) = \log\left(1 + 0.02\right)^{n} = n\log(1.02)$$

$$n = \frac{\log\left(\frac{58000}{50000}\right)}{\log(1.02)} \Rightarrow \;\; n = 7.494965335$$

$$t = \frac{n}{6} = 1.249160889 = 1.25 \;\; \text{yrs}$$

Ex 2 On March 15, 2013, a man invested P50,000 in a bank that gives 15% interest compounded every 4 months. If he decided to withdraw his money when it accumulated to P60,000, when did he make his withdrawal?

$$P = 50{,}000 \quad ; \quad F = 60{,}000 \;; \quad m = 3 \;; \quad j = 0.15 \;; \quad i = \tfrac{0.15}{3} = 0.05$$

$$t = \tfrac{n}{3}; \quad n = ? \quad \rightarrow \text{possible date?}$$

$$60{,}000 = 50{,}000\left(1 + \tfrac{0.15}{3}\right)^n \Rightarrow \log\left(\tfrac{60000}{50000}\right) = \log(1 + 0.05)^n = n\log(1.05)$$

$$n = \frac{\log\left(\frac{60000}{50000}\right)}{\log(1.05)} \Rightarrow \quad n = 3.736850652$$

$$t = \frac{n}{3} = 1.245616884 = 1.25 \text{ yrs} \rightarrow 1 \text{ yr } \& \text{ 3 months} \approx \text{June 15, 2014}$$

Ex 3 If P80,000 is invested at the rate of 6 ½% compounded annually, when will it earn interest of P15,000?

$$P = 80{,}000 \;;\; I = 15{,}000 \;\rightarrow F = 95{,}000 \;; m = 1 \;;\; j = \tfrac{13}{2}\% = \tfrac{13}{200} \;;\; i = \tfrac{13}{200} = 0.065$$

$$t = \tfrac{n}{1};\;\; n = ?$$

$$95{,}000 = 80{,}000\left(1 + \tfrac{13}{200}\right)^n \Rightarrow \log\left(\tfrac{95000}{80000}\right) = \log(1 + 0.065)^n = n\log(1.065)$$

$$n = \frac{\log\left(\frac{95000}{80000}\right)}{\log(1.065)} \Rightarrow \;\; n = 2.728873442$$

$$t = \frac{n}{1} = 2.73 \text{ yrs}$$

Ex 4 On April 15, 2011, Justin borrowed P1.4M. He agreed to pay the principal and the interest at 8% compounded semi-annually on Oct. 15, 2016. How much will he pay then?

$$P = 1{,}400{,}000 \;\; ; \;\; m = 2 \; ; \; j = 0.08 \; ; \; i = \tfrac{0.08}{2} = 0.04$$

$$\text{O.D.} = 4/15/11 \; ; \; \text{M.D.} = 7/15/16 \;\; \to 5 \text{ years} \,\&\, 6 \text{ months}$$

$$t = 5 + \tfrac{6}{12} = 5 + \tfrac{1}{2} = \tfrac{11}{2} \to n = \tfrac{11}{2}(2) = 11$$

$$F = ?$$

$$F = 1{,}400{,}000(1 + \tfrac{0.08}{2})^{11} = 2{,}155{,}235.68$$

CONTINUOUS COMPOUNDING

Interest may be converted very frequently like weekly, daily or hourly.

Let us observe the value of P1000 after 1 year at nominal rate of 5% at different frequencies of conversion m.

$$P = 1000 \;\; ; \quad j = 0.05 \;\; ; \;\; t = 1 \;\; \text{year}$$

		m = n	i	F	increase
1	annually	1	0.05	1050	
2	semi-annually	2	0.05/2	1050.625	0.625000
3	quarterly	4	0.05/4	1050.945337	0.320337
4	monthly	12	0.05/12	1051.161898	0.216561
5	weekly	52	0.05/52	1051.245842	0.083944
6	daily	365	0.05/365	1051.267496	0.021655
7	hourly	8760	0.05/8760	1051.270946	0.003450

Frequent compounding will only increase interest earned very slightly. Thus when interest is being compounded very frequently we say it is being compounded continuously.

When interest is being compounded continuously, we use e^{jt} as accumulation factor instead of $(1+i)^n$. That is ,

$$F = P\ e^{jt}$$

And consequently, $P = F\ e^{-jt}$

$$\text{So } F = 1000\ e^{0.05(1)} = 1051.271096$$

	m = n	i	F	increase
daily	365	0.05/365	1051.267496	
hourly	8760	0.05/8760	1051.270946	0.003450
CONTINUOUSLY			1051.271096	0.000150

Ex 1 How much should be invested now in order to have P50,000 in 3 ¼ years if it is invested at 6 2/3 % compounded continuously?

$$F = 50{,}000 \;\; ; \;\; t = 3\tfrac{1}{4} = \tfrac{13}{4}\,\text{yrs} \;\; ; \;\; j = 6\tfrac{2}{3}\,\% = \tfrac{20}{3}\,\% = \tfrac{20}{300}$$

$$\mathbf{P = 50{,}000\;\; e^{-\left(\frac{20}{300}\right)\left(\frac{13}{4}\right)} = 40{,}259.92}$$

Ex 2 How much is the accumulated value of P93,450 after 5 years if it earns 2.25% compounded continuously ?

$$P = 93{,}450;\; t = 5\;\text{ yrs} \;\; ; \;\; \text{continuously} \;\; ; \;\; j = 0.0225$$

$$F = 93{,}450\, e^{(0.0225)(5)} = 104{,}577.3024 = 104{,}577.30$$