5 Mathematics of Finance

Buying a car usually requires both some savings for a down payment and a loan for the balance. An exercise in Section 2 calculates the regular deposits that would be needed to save up the full purchase price, and other exercises and examples in this chapter compute the payments required to amortize a loan.

Teaching Tip: Chapter 5 is full of symbols and formulas. Students will need to become familiar with the notation and know which formula is appropriate for a given problem. Section 5.1 ends with a summary of formulas.

Everybody uses money. Sometimes you work for your money and other times your money works for you. For example, unless you are attending college on a full scholarship, it is very likely that you and your family have either saved money or borrowed money, or both, to pay for your education. When we borrow money, we normally have to pay interest for that privilege. When we save money, for a future purchase or retirement, we are lending money to a financial institution and we expect to earn interest on our investment. We will develop the mathematics in this chapter to understand better the principles of borrowing and saving. These ideas will then be used to compare different financial opportunities and make informed decisions.

5.1 Simple and Compound Interest

APPLY IT

If you can borrow money at 8% interest compounded annually or at 7.9% compounded monthly, which loan would cost less?

In this section we will learn how to compare different interest rates with different compounding periods. The question above will be answered in Example 7.

Simple Interest

Interest on loans of a year or less is frequently calculated as **simple interest**, a type of interest that is charged (or paid) only on the amount borrowed (or invested) and not on past interest. The amount borrowed is called the **principal**. The **rate** of interest is given as a percentage per year, expressed as a decimal. For example, $6\% = 0.06$ and $11\frac{1}{2}\% = 0.115$. The **time** the money is earning interest is calculated in years. One year's interest is calculated by multiplying the principal times the interest rate, or Pr. If the time that the money earns interest is other than one year, we multiply the interest for one year by the number of years, or Prt.

Simple Interest

$$I = Prt$$

where

P is the principal;

r is the annual interest rate (expressed as a decimal);

t is the time in years.

EXAMPLE 1 Simple Interest

To buy furniture for a new apartment, Pamela Shipley borrowed \$5000 at 8% simple interest for 11 months. How much interest will she pay?

SOLUTION Since 8% is the yearly interest rate, we need to know the time of the loan in years. We can convert 11 months into years by dividing 11 months by 12 (the number of months per year). Use the formula $I = Prt$, with $P = 5000$, $r = 0.08$, and $t = 11/12$ (in years). The total interest she will pay is

$$I = 5000(0.08)(11/12) \approx 366.67,$$

or \$366.67.

A deposit of P dollars today at a rate of interest r for t years produces interest of $I = Prt$. The interest, added to the original principal P, gives

$$P + Prt = P(1 + rt).$$

This amount is called the *future value* of P dollars at an interest rate r for time t in years. When loans are involved, the future value is often called the *maturity value* of the loan. This idea is summarized as follows.

Future or Maturity Value for Simple Interest

The **future** or **maturity value** A of P dollars at a simple interest rate r for t years is

$$\boldsymbol{A = P(1 + rt)}.$$

EXAMPLE 2 Maturity Values

Find the maturity value for each loan at simple interest.

(a) A loan of \$2500 to be repaid in 8 months with interest of 4.3%

SOLUTION The loan is for 8 months, or $8/12 = 2/3$ of a year. The maturity value is

$$\begin{aligned} A &= P(1 + rt) \\ &= 2500\left[1 + 0.043\left(\frac{2}{3}\right)\right] \qquad P = 2500,\ r = 0.043,\ t = 2/3 \\ &\approx 2500(1 + 0.028667) = 2571.67, \end{aligned}$$

or \$2571.67. (The answer is rounded to the nearest cent, as is customary in financial problems.) Of this maturity value,

$$I = A - P = \$2571.67 - \$2500 = \$71.67$$

represents interest.

(b) A loan of \$11,280 for 85 days at 7% interest

SOLUTION It is common to assume 360 days in a year when working with simple interest. We shall usually make such an assumption in this book. Using $P = 11{,}280$, $r = 0.07$, and $t = 85/360$, the maturity value in this example is

$$A = 11{,}280\left[1 + 0.07\left(\frac{85}{360}\right)\right] \approx 11{,}466.43,$$

or \$11,466.43. TRY YOUR TURN 1

YOUR TURN 1 Find the maturity value for a \$3000 loan at 5.8% interest for 100 days.

CAUTION When using the formula for future value, as well as all other formulas in this chapter, we often neglect the fact that in real life, money amounts are rounded to the nearest penny. As a consequence, when the amounts are rounded, their values may differ by a few cents from the amounts given by these formulas. For instance, in Example 2(a), the interest in each monthly payment would be $\$2500(0.043/12) \approx \$8.96$, rounded to the nearest penny. After 8 months, the total is $8(\$8.96) = \$71.68$, which is 1¢ more than we computed in the example.

In part (b) of Example 2 we assumed 360 days in a year. Historically, to simplify calculations, it was often assumed that each year had twelve 30-day months, making a year 360 days long. Treasury bills sold by the U.S. government assume a 360-day year in calculating interest. Interest found using a 360-day year is called *ordinary interest*, and interest found using a 365-day year is called *exact interest*.

The formula for future value has four variables, P, r, t, and A. We can use the formula to find any of the quantities that these variables represent, as illustrated in the next example.

EXAMPLE 3 Simple Interest Rate

Alicia Rinke wants to borrow \$8000 from Robyn Martin. She is willing to pay back \$8180 in 6 months. What interest rate will she pay?

SOLUTION Use the formula for future value, with $A = 8180$, $P = 8000$, $t = 6/12 = 0.5$, and solve for r.

$$\begin{aligned} A &= P(1 + rt) \\ 8180 &= 8000(1 + 0.5r) \\ 8180 &= 8000 + 4000r && \text{Distributive property} \\ 180 &= 4000r && \text{Subtract 8000.} \\ r &= 0.045 && \text{Divide by 4000.} \end{aligned}$$

Thus, the interest rate is 4.5% (written as a percent). TRY YOUR TURN 2

YOUR TURN 2 Find the interest rate if \$5000 is borrowed, and \$5243.75 is paid back 9 months later.

When you deposit money in the bank and earn interest, it is as if the bank borrowed the money from you. Reversing the scenario in Example 3, if you put \$8000 in a bank account that pays simple interest at a rate of 4.5% annually, you will have accumulated \$8180 after 6 months.

Compound Interest

As mentioned earlier, simple interest is normally used for loans or investments of a year or less. For longer periods compound interest is used. With **compound interest**, interest is charged (or paid) on interest as well as on principal. For example, if \$1000 is deposited at 5% interest for 1 year, at the end of the year the interest is $\$1000(0.05)(1) = \$50$. The balance in the account is $\$1000 + \$50 = \$1050$. If this amount is left at 5% interest for another year, the interest is calculated on \$1050 instead of the original \$1000, so the amount in the account at the end of the second year is $\$1050 + \$1050(0.05)(1) = \$1102.50$. Note that simple interest would produce a total amount of only

$$\$1000[1 + (0.05)(2)] = \$1100.$$

The additional \$2.50 is the interest on \$50 at 5% for one year.

To find a formula for compound interest, first suppose that P dollars is deposited at a rate of interest r per year. The amount on deposit at the end of the first year is found by the simple interest formula, with $t = 1$.

$$A = P(1 + r \cdot 1) = P(1 + r)$$

If the deposit earns compound interest, the interest earned during the second year is paid on the total amount on deposit at the end of the first year. Using the formula $A = P(1 + rt)$ again, with P replaced by $P(1 + r)$ and $t = 1$, gives the total amount on deposit at the end of the second year.

$$A = [P(1 + r)](1 + r \cdot 1) = P(1 + r)^2$$

In the same way, the total amount on deposit at the end of the third year is

$$P(1 + r)^3.$$

Generalizing, if P is the initial deposit, in t years the total amount on deposit is

$$A = P(1 + r)^t,$$

called the **compound amount**.

NOTE Compare this formula for compound interest with the formula for simple interest.

Compound interest	$A = P(1 + r)^t$
Simple interest	$A = P(1 + rt)$

The important distinction between the two formulas is that in the compound interest formula, the number of years, t, is an *exponent*, so that money grows much more rapidly when interest is compounded.

Interest can be compounded more than once per year. Common compounding periods include *semiannually* (two periods per year), *quarterly* (four periods per year), *monthly* (twelve periods per year), or *daily* (usually 365 periods per year). The *interest rate per period*, i, is found by dividing the annual interest rate, r, by the number of compounding periods, m, per year. To find the total number of compounding periods, n, we multiply the number of years, t, by the number of compounding periods per year, m. The following formula can be derived in the same way as the previous formula.

Compound Amount

$$\boldsymbol{A = P(1 + i)^n}$$

where $i = \frac{r}{m}$ and $n = mt$,

- A is the future (maturity) value;
- P is the principal;
- r is the annual interest rate;
- m is the number of compounding periods per year;
- t is the number of years;
- n is the number of compounding periods;
- i is the interest rate per period.

EXAMPLE 4 Compound Interest

Suppose \$1000 is deposited for 6 years in an account paying 4.25% per year compounded annually.

(a) Find the compound amount.

SOLUTION Since interest is compounded annually, the number of compounding periods per year is $m = 1$. The interest rate per period is $i = r/m = 0.0425/1 = 0.0425$ and the number of compounding periods is $n = mt = 1(6) = 6$. (Notice that when interest is compounded annually, $i = r$ and $n = t$.)

Using the formula for the compound amount with $P = 1000$, $i = 0.0425$, and $n = 6$ gives

$$\begin{aligned} A &= P(1 + i)^n \\ &= 1000(1 + 0.0425)^6 \\ &= 1000(1.0425)^6 \\ &\approx 1283.68, \end{aligned}$$

or \$1283.68.

(b) Find the amount of interest earned.

SOLUTION Subtract the initial deposit from the compound amount.

$$I = A - P = \$1283.68 - \$1000 = \$283.68$$

EXAMPLE 5 Compound Interest

Find the amount of interest earned by a deposit of \$2450 for 6.5 years at 5.25% compounded quarterly.

SOLUTION The principal is $P = 2450$, the annual interest rate is $r = 0.0525$, and the number of years is $t = 6.5$ years. Interest is compounded quarterly, so the number of compounding periods per year is $m = 4$. In 6.5 years, there are $n = mt = 4(6.5) = 26$ compounding periods. The interest rate per quarter is $i = r/m = 0.0525/4$. Now use the formula for compound amount.

$$\begin{aligned} A &= P(1+i)^n \\ &= 2450(1+0.0525/4)^{26} \\ &\approx 3438.78 \end{aligned}$$

YOUR TURN 3 Find the amount of interest earned by a deposit of \$1600 for 7 years at 4.2% compounded monthly.

Rounded to the nearest cent, the compound amount is \$3438.78. The interest earned is

$$\begin{aligned} I &= A - P \\ &= \$3438.78 - \$2450 \\ &= \$988.78. \end{aligned}$$

TRY YOUR TURN 3

TECHNOLOGY NOTE

Graphing calculators can be used to find the future value (compound amount) of an investment. On the TI-84 Plus C, select `APPS`, then `Finance`, then `TVM Solver`. Enter the following values (no entry can be left blank).

`N` = Total number of compounding periods.
`I%` = Annual interest rate (as a percentage)
`PV` = Present value
`PMT` = Payment
`FV` = Future value
`P/Y` = Payments per year
`C/Y` = Compounding periods per year

The TVM Solver uses the cash flow sign convention, which indicates the direction of the cash flow. Cash inflows are entered as positive numbers, while cash outflows are entered as negative numbers. If you invest money, the present value is the amount you invest and is considered an outflow (negative value). The future value is money you will receive at the end of the investment, so it is an inflow (positive value). On the other hand, if you borrow money, the present value is money you will receive, which is an inflow (positive value). The future (or maturity) value is money you must pay back, so it is an outflow (negative value).

For the investment in Example 5, we would enter `N` = 26 and `I%` = 5.25. For `PV`, we enter −2450. (`PV` is an outflow.) We let `PMT` and `FV` equal 0. Both `P/Y` and `C/Y` are equal to 4. See Figure 1(a). To find the future value, move the cursor to the `FV` line and enter `ALPHA`, then `SOLVE` (the `ENTER` button). The rounded future value is \$3438.78, as shown in Figure 1(b). For more information on using the TVM solver, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

```
N=26
I%=5.25
PV=-2450
PMT=0
FV=0
P/Y=4
C/Y=4
PMT: END BEGIN
```

(a)

```
N=26
I%=5.25
PV=-2450
PMT=0
▪FV=3438.78419
P/Y=4
C/Y=4
PMT: END BEGIN
```

(b)

FIGURE 1

CAUTION As shown in Example 5, compound interest problems involve two rates—the annual rate r and the rate per compounding period i. Be sure you understand the distinction between them. When interest is compounded annually, these rates are the same. In all other cases, $i \neq r$. Similarly, there are two quantities for time: the number of years t and the number of compounding periods n. When interest is compounded annually, these variables have the same value. In all other cases, $n \neq t$.

It is interesting to compare loans at the same rate when simple or compound interest is used. Figure 2 shows the graphs of the simple interest and compound interest formulas with $P = 1000$ at an annual rate of 10% from 0 to 20 years. The future value after 15 years is shown for each graph. After 15 years of compound interest, \$1000 grows to \$4177.25, whereas with simple interest, it amounts to \$2500.00, a difference of \$1677.25.

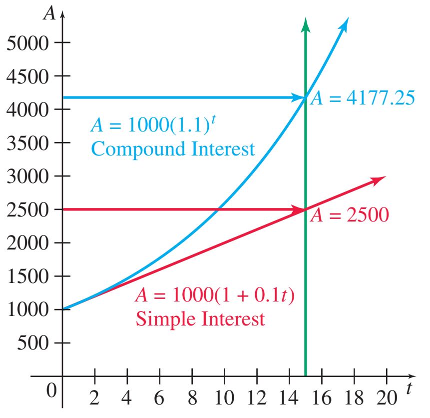


FIGURE 2

TECHNOLOGY NOTE

Spreadsheets are ideal for performing financial calculations. Figure 3 shows a Microsoft Excel spreadsheet with the formulas for compound and simple interest used to create columns B and C, respectively, when \$1000 is invested at an annual rate of 10%. Compare row 16 with Figure 2. For more details on the use of spreadsheets in the mathematics of finance, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

	A	B	C
1	period	compound	simple
2	1	1100	1100
3	2	1210	1200
4	3	1331	1300
5	4	1464.1	1400
6	5	1610.51	1500
7	6	1771.561	1600
8	7	1948.7171	1700
9	8	2143.58881	1800
10	9	2357.947691	1900
11	10	2593.74246	2000
12	11	2853.116706	2100
13	12	3138.428377	2200
14	13	3452.27124	2300
15	14	3797.498336	2400
16	15	4177.248169	2500
17	16	4594.972986	2600
18	17	5054.470285	2700
19	18	5559.917313	2800
20	19	6115.909045	2900
21	20	6727.499949	3000

FIGURE 3

We can also solve the compound amount formula for the interest rate, as in the following example.

EXAMPLE 6 Compound Interest Rate

Suppose Susan Nassy invested \$5000 in a savings account that paid quarterly interest. After 6 years the money had accumulated to \$6539.96. What was the annual interest rate?

SOLUTION The principal is $P = 5000$, the number of years is $t = 6$, and the compound amount is $A = 6539.96$. We are asked to find the interest rate, r.

Since the account paid quarterly interest, $m = 4$. The number of compounding periods is $n = 4(6) = 24$. The interest rate per period can be written as $i = r/4$. Use these values in the formula for compound amount, and then solve for r.

$$\begin{aligned} P(1+i)^n &= A \\ 5000(1+r/4)^{24} &= 6539.96 \\ (1+r/4)^{24} &= 1.30799 && \text{Divide both sides by 5000.} \\ 1+r/4 &= 1.30799^{1/24} \approx 1.01125 && \text{Take both sides to the 1/24 power.} \\ r/4 &= 0.01125 && \text{Subtract 1 from both sides.} \\ r &= 0.045 && \text{Multiply both sides by 4.} \end{aligned}$$

As a percent, the annual interest rate was 4.5%. TRY YOUR TURN 4

YOUR TURN 4 Find the annual interest rate if \$6500 is worth \$8665.69 after being invested for 8 years in an account that compounded interest monthly.

Effective Rate

Suppose \$1 is deposited at 6% compounded semiannually. Here, $i = r/m = 0.06/2 = 0.03$ for $m = 2$ periods. At the end of one year, the compound amount is $A = 1(1 + 0.06/2)^2 \approx 1.06090$. This shows that \$1 will increase to \$1.06090, an actual increase of 6.09%.

The actual increase of 6.09% in the money is somewhat higher than the stated increase of 6%. To differentiate between these two numbers, 6% is called the **nominal** or **stated rate** of interest, while 6.09% is called the *effective rate*.* To avoid confusion between stated rates and effective rates, we shall continue to use r for the stated rate and we will use r_E for the effective rate.

Generalizing from this example, the effective rate of interest is given by the following formula.

Effective Rate

The **effective rate** corresponding to a stated rate of interest r compounded m times per year is

$$r_E = \left(1 + \frac{r}{m}\right)^m - 1.$$

EXAMPLE 7 Effective Rate

Joe Vetere needs to borrow money. His neighborhood bank charges 8% interest compounded semiannually. An Internet bank charges 7.9% interest compounded monthly. At which bank will Joe pay the lesser amount of interest?

*When applied to consumer finance, the effective rate is called the annual percentage rate, APR, or annual percentage yield, APY.

APPLY IT

SOLUTION Compare the effective rates.

$$\text{Neighborhood bank:} \quad r_E = \left(1 + \frac{0.08}{2}\right)^2 - 1 = 0.0816 = 8.16\%$$

$$\text{Internet bank:} \quad r_E = \left(1 + \frac{0.079}{12}\right)^{12} - 1 \approx 0.081924 \approx 8.19\%$$

The neighborhood bank has the lower effective rate, although it has a higher stated rate.

TRY YOUR TURN 5

YOUR TURN 5 Find the effective rate for an account that pays 2.7% compounded monthly.

Present Value

The formula for compound interest, $A = P(1 + i)^n$, has four variables: A, P, i, and n. Given the values of any three of these variables, the value of the fourth can be found. In particular, if A (the future amount), i, and n are known, then P can be found. Here P is the amount that should be deposited today to produce A dollars in n periods.

EXAMPLE 8 Present Value

Stacey Sveum must pay a lump sum of \$6000 in 5 years. What amount deposited today at 6.2% compounded annually will amount to \$6000 in 5 years?

SOLUTION Here $A = 6000$, $i = r = 0.062$, $n = t = 5$, and P is unknown. Substituting these values into the formula for the compound amount gives

$$A = P(1 + i)^n$$

$$6000 = P(1.062)^5$$

$$P = \frac{6000}{(1.062)^5} \approx 4441.49,$$

or \$4441.49. If Stacey leaves \$4441.49 for 5 years in an account paying 6.2% compounded annually, she will have \$6000 when she needs it. To check your work, use the compound interest formula with $P = \$4441.49$, $i = 0.062$, and $n = 5$. You should get $A = \$6000.00$.

As Example 8 shows, \$6000 in 5 years is approximately the same as \$4441.49 today (if money can be deposited at 6.2% compounded annually). An amount that can be deposited today to yield a given sum in the future is called the *present value* of the future sum. Generalizing from Example 8, by solving $A = P(1 + i)^n$ for P, we get the following formula for present value.

NOTE When computing the present value, you may have to round the principal up to the nearest cent to ensure that enough money is accumulated in the given amount of time. Always check your solution.

Present Value for Compound Interest

The **present value** of A dollars compounded at an interest rate i per period for n periods is

$$P = \frac{A}{(1 + i)^n} \quad \text{or} \quad P = A(1 + i)^{-n}.$$

EXAMPLE 9 Present Value

Find the present value of \$16,000 in 9 years if money can be deposited at 2% compounded semiannually.

SOLUTION In 9 years there are $2 \cdot 9 = 18$ semiannual periods. A rate of 2% per year is 1% in each semiannual period. Apply the formula with $A = 16{,}000$, $i = 0.01$, and $n = 18$.

$$P = \frac{A}{(1 + i)^n} = \frac{16{,}000}{(1.01)^{18}} \approx 13{,}376.28$$

A deposit of \$13,376.28 today, at 2% compounded semiannually, will produce a total of \$16,000 in 9 years.

TRY YOUR TURN 6

YOUR TURN 6 Find the present value of \$10,000 in 7 years if money can be deposited at 4.25% compounded quarterly.

We can solve the compound amount formula for n also, as the following example shows.

Teaching Tip: The material on logarithms is optional, and you may want to skip this if your students are not familiar with logarithms. Material on logarithms is in Section 10.5 of *Finite Mathematics and Calculus with Applications* and Appendix B of *Finite Mathematics.*

EXAMPLE 10 Compounding Time

Suppose the $2450 from Example 5 is deposited at 5.25% compounded quarterly until it reaches at least $10,000. How much time is required?

SOLUTION The principal is $P = 2450$ and the interest rate per quarter is $i = 0.0525/4$. We are looking for the number of quarters (n) until the compound amount reaches at least $10,000, that is, until

$$A = 10{,}000,$$

or

$$2450(1 + 0.0525/4)^n = 10{,}000. \qquad A = P(1 + i)^n$$

Notice that the variable n is in the exponent. We will solve this equation for n with two different methods: graphically and algebraically (with logarithms).

SOLUTION

Method 1 Graphing Calculator

Graph the functions $y = 2450(1 + 0.0525/4)^x$ and $y = 10{,}000$ in the same window, and then find the point of intersection. As Figure 4 shows, the functions intersect at $x = 107.8634$. Note, however, that interest is added to the account only every quarter, so we must wait 108 quarters, or $108/4 = 27$ years, for the money to be worth at least $10,000.

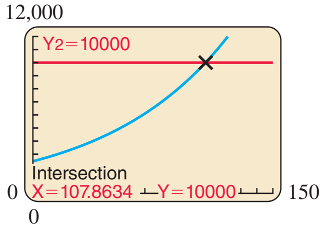


FIGURE 4

Method 2 Using Logarithms (Optional)

The goal is to find n by solving the equation

$$2450(1 + 0.0525/4)^n = 10{,}000.$$

Divide both sides by 2450, and simplify the expression in parentheses to get

$$1.013125^n = \frac{10{,}000}{2450}.$$

Now take the logarithm (either base 10 or base e) of both sides to get

$$\log(1.013125^n) = \log(10{,}000/2450)$$

$$n \log(1.013125) = \log(10{,}000/2450) \qquad \text{Use logarithm property } \log x^r = r \log x.$$

$$n = \frac{\log(10{,}000/2450)}{\log(1.013125)} \qquad \text{Divide both sides by } \log(1.013125).$$

$$\approx 107.86.$$

As in Method 1, this means that we must wait 108 quarters, or $108/4 = 27$ years, for the money to be worth at least $10,000.

TRY YOUR TURN 7

FOR REVIEW

For a review of logarithmic functions, refer to Appendix B if you are using *Finite Mathematics*, or to Section 10.5 if you are using *Finite Mathematics and Calculus with Applications*. The only property of logarithms needed here is $\log x^r = r \log x$. Logarithms may be used in base 10, using the LOG button on a calculator, or in base e, using the LN button.

YOUR TURN 7 Find the time needed for $3800 deposited at 3.5% compounded semiannually to be worth at least $7000.

TECHNOLOGY NOTE

We can use the TVM Solver on the TI-84 Plus C calculator to find the value of any of the variables in the compound amount formula.

To find the nominal interest rate in Example 6, we would enter N = 24, PV = −5000, FV = 6539.96, and P/Y = 4. Placing the cursor on the I% line and pressing SOLVE yields 4.5% (rounded).

To find the compounding time in Example 10, we would enter I% = 5.25, PV = −2450, FV = 10000, and P/Y = 4. Placing the cursor on the N line and pressing SOLVE yields 107.86 (rounded) quarters.

EXAMPLE 11 Price Doubling

Suppose the general level of inflation in the economy averages 8% per year. Find the number of years it would take for the overall level of prices to double.

SOLUTION We are looking for the number of years, with an annual inflation rate of 8%, it takes for overall prices to double. We can use the formula for compound amount with an arbitrary value for P. If we let $P = \$1$, we are looking for the number of years until this amount doubles to $A = \$2$. Since the inflation rate is an annual rate, $i = r = 0.08$, and we find n in the equation

$$\begin{aligned} P(1+i)^n &= A \\ 1(1+0.08)^n &= 2 \qquad P = 1, i = 0.08, \text{ and } A = 2 \\ (1.08)^n &= 2. \end{aligned}$$

Solving this equation using either a graphing calculator or logarithms, as in Example 10, shows that $n = 9.00647$. Thus, the overall level of prices will double in about 9 years.

You can quickly estimate how long it takes a sum of money to double, when compounded annually, by using either the **rule of 70** or the **rule of 72**. The rule of 70 (used for small rates of growth) says that for $0.001 \le r < 0.05$, the value of $70/100r$ gives a good approximation of the doubling time. The rule of 72 (used for larger rates of growth) says that for $0.05 \le r \le 0.12$, the value of $72/100r$ approximates the doubling time well. In Example 11, the inflation rate is 8%, so the doubling time is approximately $72/8 = 9$ years.*

Continuous Compounding

Suppose that a bank, in order to attract more business, offers to not just compound interest every quarter, or every month, or every day, or even every hour, but constantly. This type of compound interest, in which the number of times a year that the interest is compounded becomes infinite, is known as **continuous compounding**. To see how it works, look back at Example 5, where we found that \$2450, when deposited for 6.5 years at 5.25% compounded quarterly, resulted in a compound amount of \$3438.78. We can find the compound amount if we compound more often by putting different values of n in the formula $A = 2450(1 + 0.0525/n)^{6.5n}$, as shown in the following table.

Teaching Tip: Continuous compounding is not needed for the rest of Chapter 5 and may be skipped. It is a nice way to introduce the number e and was once a popular gimmick with banks but is now rare among financial institutions.

Compounding n Times Annually		
n	Type of Compounding	Compound Amount
4	quarterly	\$3438.78
12	monthly	\$3443.86
360	daily	\$3446.34
8640	every hour	\$3446.42

Notice that as n becomes larger, the compound amount also becomes larger, but by a smaller and smaller amount. In this example, increasing the number of compounding periods a year from 360 to 8640 earns only 8¢ more. It is shown in calculus that as n becomes infinitely large, $P(1 + r/n)^{nt}$ gets closer and closer to Pe^{rt}, where e is a very important irrational number whose approximate value is 2.718281828. To calculate interest with continuous compounding, use the e^x button on your calculator. You will learn more about the number e if you study calculus, where e plays as important a role as π does in geometry.

*To see where the rule of 70 and the rule of 72 come from, see the section on Taylor Series in *Calculus with Applications* by Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey, Pearson, 2016.

Continuous Compounding

If a deposit of P dollars is invested at a rate of interest r compounded continuously for t years, the compound amount is

$$A = Pe^{rt} \text{ dollars.}$$

EXAMPLE 12 Continuous Compounding

Suppose that \$2450 is deposited at 5.25% compounded continuously.

(a) Find the compound amount and the interest earned after 6.5 years.

SOLUTION Using the formula for continuous compounding with $P = 2450$, $r = 0.0525$, and $t = 6.5$, the compound amount is

$$A = 2450e^{0.0525(6.5)} \approx 3446.43.$$

The compound amount is \$3446.43, which is just a penny more than if it had been compounded hourly, or 9¢ more than daily compounding. Because it makes so little difference, continuous compounding has dropped in popularity in recent years. The interest in this case is $\$3446.43 - 2450 = \$996.43$, or \$7.65 more than if it were compounded quarterly, as in Example 5.

(b) Find the effective rate.

SOLUTION As in Example 7, the effective rate is just the amount of interest that \$1 would earn in one year. If $P = 1$ and $t = 1$, then $A = 1e^{1 \cdot 0.0525} = e^{0.0525}$. The amount of interest earned in one year is

$$e^{0.0525} - 1 \approx 0.0539,$$

or an increase of 5.39%. In general, the effective rate for interest compounded continuously at a rate r is $r_E = e^r - 1$.

(c) Find the time required for the original \$2450 to grow to \$10,000.

SOLUTION

Method 1 Graphing Calculator

As in Example 10, use a graphing calculator to find the intersection of $y = 2450e^{0.0525x}$ and $y = 10{,}000$. The answer is 26.79 years. Notice that, unlike in Example 10, you don't need to wait until the next compounding period to reach this amount, because interest is being added to the account continuously.

Method 2 Using Logarithms (Optional)

Similar to the process in Example 10 (Method 2), we let $P = 2450$, $A = 10{,}000$, and $r = 0.0525$ and solve for t. Using logarithms,

$$\begin{aligned}
A &= Pe^{rt} \\
10{,}000 &= 2450e^{0.0525t} \\
10{,}000/2450 &= e^{0.0525t} && \text{Divide both sides by 2450.} \\
\ln(10{,}000/2450) &= \ln(e^{0.0525t}) && \text{Take natural logarithm of both sides.} \\
\ln(10{,}000/2450) &= 0.0525t && \text{Use the logarithm property } \ln(e^x) = x. \\
\frac{\ln(10{,}000/2450)}{0.0525} &= t && \text{Divide both sides by 0.0525.} \\
t &\approx 26.79,
\end{aligned}$$

or 26.79 years.

TRY YOUR TURN 8

YOUR TURN 8 Find the interest earned on \$5000 deposited at 3.8% compounded continuously for 9 years.

At this point, it seems helpful to summarize the notation and the most important formulas for simple and compound interest. We use the following variables.

P = principal or present value
A = future or maturity value
r = annual (stated or nominal) interest rate
t = number of years
m = number of compounding periods per year
i = interest rate per period $\quad \boldsymbol{i = r/m}$
n = total number of compounding periods $\quad \boldsymbol{n = tm}$
r_E = effective rate

Simple Interest	Compound Interest	Continuous Compounding
$A = P(1 + rt)$	$A = P(1 + i)^n$	$A = Pe^{rt}$
$P = \dfrac{A}{1 + rt}$	$P = \dfrac{A}{(1 + i)^n} = A(1 + i)^{-n}$	$P = Ae^{-rt}$
	$r_E = \left(1 + \dfrac{r}{m}\right)^m - 1$	$r_E = e^r - 1$

5.1 WARM-UP EXERCISES

Evaluate the following expressions. Round decimals to the nearest hundredth. *(Sec R.1)*

W1. $1000(1 + 0.017 \cdot 3)$ 1051

W2. $1500(1 + 0.05)^6$ 2010.14

W3. $(1 + 0.04)^8 - 1$ 0.37

W4. $\dfrac{6000}{(1 + 0.03)^5}$ 5175.65

5.1 EXERCISES

1. What factors determine the amount of interest earned on a fixed principal? The interest rate and number of compounding periods

2. In your own words, describe the *maturity value* of a loan.

3. What is meant by the *present value* of money?

4. We calculated the loan in Example 2(b) assuming 360 days in a year. Find the maturity value using 365 days in a year. Which is more advantageous to the borrower? \$11,463.88; 360

Find the simple interest.

5. \$25,000 at 3% for 9 months \$562.50

6. \$4289 at 4.5% for 35 weeks \$129.91

7. \$1974 at 6.3% for 25 weeks \$59.79

8. \$6125 at 1.25% for 6 months \$38.28

Find the simple interest. Assume a 360-day year.

9. \$8192.17 at 3.1% for 72 days \$50.79

10. \$7236.15 at 4.25% for 30 days \$25.63

Find the maturity value and the amount of simple interest earned.

11. \$3125 at 2.85% for 7 months \$3176.95; \$51.95

12. \$12,000 at 5.3% for 11 months \$12,583.00; \$583

13. If \$1500 earned simple interest of \$56.25 in 6 months, what was the simple interest rate? 7.5%

14. If \$23,500 earned simple interest of \$1057.50 in 9 months, what was the simple interest rate? 6%

15. Explain the difference between simple interest and compound interest.

16. What is the difference between r and i? r is the interest rate per year, while i is the interest rate per compounding period.

17. What is the difference between t and n? t is the number of years, while n is the number of compounding periods.

18. In Figure 2, one line is straight and the other is curved. Explain why this is, and which represents each type of interest.

Find the compound amount for each deposit and the amount of interest earned.

19. \$1000 at 6% compounded annually for 8 years \$1593.85; \$593.85

20. \$1000 at 4.5% compounded annually for 6 years \$1302.26; \$302.26

21. \$470 at 5.4% compounded semiannually for 12 years \$890.82; \$420.82

22. \$15,000 at 6% compounded monthly for 10 years \$27,290.95; \$12,290.95

23. \$8500 at 8% compounded quarterly for 5 years \$12,630.55; \$4130.55

24. \$9100 at 6.4% compounded quarterly for 9 years \$16,114.43; \$7014.43

Find the interest rate for each deposit and compound amount.

25. \$8000 accumulating to \$11,672.12, compounded quarterly for 8 years 4.75%

26. \$12,500 accumulating to \$20,077.43, compounded quarterly for 9 years 5.3%

27. \$4500 accumulating to \$5994.79, compounded monthly for 5 years 5.75%

28. \$6725 accumulating to \$10,353.47, compounded monthly for 7 years 6.18%

Find the effective rate corresponding to each nominal rate.

29. 4% compounded quarterly 4.06%

30. 6% compounded quarterly 6.14%

31. 7.25% compounded semiannually 7.38%

32. 6.25% compounded semiannually 6.35%

Find the present value (the amount that should be invested now to accumulate the following amount) if the money is compounded as indicated.

33. \$12,820.77 at 4.8% compounded annually for 6 years \$9677.13

34. \$36,527.13 at 5.3% compounded annually for 10 years \$21,793.75

35. \$2000 at 6% compounded semiannually for 8 years \$1246.34

36. \$2000 at 7% compounded semiannually for 8 years \$1153.42

37. \$8800 at 5% compounded quarterly for 5 years \$6864.08

38. \$7500 at 5.5% compounded quarterly for 9 years \$4587.23

39. How do the nominal or stated interest rate and the effective interest rate differ?

40. If interest is compounded more than once per year, which rate is higher, the stated rate or the effective rate? The effective rate

Using either logarithms or a graphing calculator, find the time required for each initial amount to be at least equal to the final amount.

41. \$5000, deposited at 4% compounded quarterly, to reach at least \$9000 15 years

42. \$8000, deposited at 3% compounded quarterly, to reach at least \$23,000 35.5 years

43. \$4500, deposited at 3.6% compounded monthly, to reach at least \$11,000 24 years, 11 months

44. \$6800, deposited at 5.4% compounded monthly, to reach at least \$15,000 14 years, 9 months

Find the doubling time for each of the following levels of inflation using (a) logarithms or a graphing calculator, and (b) the rule of 70 or 72, whichever is appropriate.

45. 3.3% **(a)** 21.35 years **(b)** 21.21 years

46. 6.25% **(a)** 11.43 years old **(b)** 11.52 years

For each of the following amounts at the given interest rate compounded continuously, find (a) the future value after 9 years, (b) the effective rate, and (c) the time to reach \$10,000.

47. \$5500 at 3.1% **(a)** \$7269.94 **(b)** 3.15% **(c)** 19.29 years

48. \$4700 at 4.65% **(a)** \$7142.50 **(b)** 4.76% **(c)** 16.24 years

APPLICATIONS

Business and Economics

49. **Loan Repayment** Celeste Nossiter borrowed \$7200 from her father to buy a used car. She repaid him after 9 months, at an annual interest rate of 6.2%. Find the total amount she repaid. How much of this amount is interest? \$7534.80; \$334.80

50. **Delinquent Taxes** An accountant for a corporation forgot to pay the firm's income tax of \$321,812.85 on time. The government charged a penalty based on an annual interest rate of 13.4% for the 29 days the money was late. Find the total amount (tax and penalty) that was paid. (Use a 365-day year.) \$325,239.05

51. **Savings** A \$1500 certificate of deposit held for 75 days was worth \$1521.25. To the nearest tenth of a percent, what interest rate was earned? Assume a 360-day year. 6.8%

52. **Bond Interest** A bond with a face value of \$10,000 in 10 years can be purchased now for \$5988.02. What is the simple interest rate? 6.7%

53. **Cash Advance Fees** According to an advertisement, if you need a cash advance before payday, a national company can get you approved for an in-store cash advance at any of their locations. For a \$300 cash advance, a single repayment of \$378.59 is required fourteen days later. Determine the simple interest rate to the nearest hundredth of a percent. Use a 365-day year. 682.98%

54. **Stock Growth** A stock that sold for \$22 at the beginning of the year was selling for \$24 at the end of the year. If the stock paid a dividend of \$0.50 per share, what is the simple interest rate on an investment in this stock? (*Hint:* Consider the interest to be the increase in value plus the dividend.) 11.4%

55. **Investments** Suppose \$10,000 is invested at an annual rate of 5% for 10 years. Find the future value if interest is compounded as follows.

(a) Annually \$16,288.95 **(b)** Quarterly \$16,436.19

(c) Monthly \$16,470.09 **(d)** Daily (365 days) \$16,486.65

(e) Continuously \$16,487.21

56. **Investments** In Exercise 55, notice that as the money is compounded more often, the compound amount becomes larger and larger. Is it possible to compound often enough so that the compound amount is \$17,000 after 10 years? Explain.

57. Comparing Investments According to the Federal Reserve, from 1971 until 2014, the U.S. benchmark interest rate averaged 6.05%. ***Source: Federal Reserve***.

(a) Suppose $1000 is invested for 1 year in a CD earning 6.05% interest, compounded monthly. Find the future value of the account. $1062.21

(b) In March of 1980, the benchmark interest rate reached a high of 20%. Suppose the $1000 from part (a) was invested in a 1-year CD earning 20% interest, compounded monthly. Find the future value of the account. $1219.39

(c) In December of 2009, the benchmark interest rate reached a low of 0.25%. Suppose the $1000 from part (a) was invested in a 1-year CD earning 0.25% interest, compounded monthly. Find the future value of the account. $1002.50

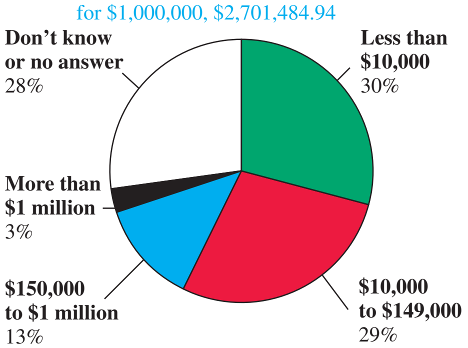
(d) Discuss how changes in interest rates over the past years have affected the savings and the purchasing power of average Americans.

58. Investments Shaun Murie borrowed $5200 from his friend Ian Desrosiers to pay for remodeling work on his house. He repaid the loan 10 months later with simple interest at 3%. Ian then invested the proceeds (original principal plus interest) in a 5-year certificate of deposit paying 3.3% compounded quarterly. How much will he have at the end of 5 years? (*Hint:* You need to use both simple and compound interest.) $6281.91

59. Student Loan Upon graduation from college, Warren Roberge was able to defer payment on his $40,000 subsidized Stafford student loan for 6 months. Since the interest will no longer be paid on his behalf, it will be added to the principal until payments begin. If the interest is 4.66% compounded monthly, what will the principal amount be when he must begin repaying his loan? ***Source: U.S. Department of Education***. $40,941.10

60. Comparing Investments Two partners agree to invest equal amounts in their business. One will contribute $10,000 immediately. The other plans to contribute an equivalent amount in 3 years, when she expects to acquire a large sum of money. How much should she contribute at that time to match her partner's investment now, assuming an interest rate of 6% compounded semiannually? $11,940.52

61. Retirement Savings The pie graph below shows the percent of adults aged 46–49 who said they had investments with a total value as shown in each category. ***Source: The New York Times***. For $10,000, $27,014.85; for $149,000, $402,521.26; for $1,000,000, $2,701,484.94



Figures add to more than 100% because of rounding.

Note that 30% have saved less than $10,000. Assume the money is invested at an average rate of 5% compounded quarterly. What will the top numbers in each category amount to in 20 years, when this age group will be ready for retirement?

62. Wealth A 1997 article in *The New York Times* discussed how long it would take for Bill Gates, the world's second richest person at the time (behind the Sultan of Brunei), to become the world's first trillionaire. His birthday is October 28, 1955, and on July 16, 1997, he was worth $42 billion. (*Note:* A trillion dollars is 1000 billion dollars.) ***Source: The New York Times***.

(a) Assume that Bill Gates's fortune grows at an annual rate of 58%, the historical growth rate through 1997 of Microsoft stock, which made up most of his wealth in 1997. Find the age at which he becomes a trillionaire. (*Hint:* Use the formula for compound amount, $A = P(1 + i)^n$, with $P = 42$ and $A = 1000$. Solve for n using one of the methods described in Example 10.) 48

(b) Repeat part (a) using 10.5% growth, the average return on all stocks since 1926. ***Source: CNN***. 73

(c) What rate of growth would be necessary for Bill Gates to become a trillionaire by the time he is eligible for Social Security on January 1, 2022, after he has turned 66? 13.84%

(d) *Forbes* magazine's listings of billionaires for 2006 and 2014 have given Bill Gates's worth as roughly $50.0 billion and $79.1 billion, respectively. What was the rate of growth of his wealth between 2006 and 2014? ***Source: Forbes***. 5.90%

Negative Interest Under certain conditions, Swiss banks pay negative interest: They charge you. (You didn't think all that secrecy was free?) Suppose a bank "pays" −2.4% interest compounded annually. Find the compound amount for a deposit of $150,000 after each period.

63. 4 years $136,110.16

64. 8 years $123,506.50

+ **65. Savings** On January 1, 2010, Jack deposited $1000 into bank X to earn interest at a rate of j per annum compounded semiannually. On January 1, 2015, he transferred his account to bank Y to earn interest at the rate of k per annum compounded quarterly. On January 1, 2018, the balance of bank Y is $1990.76. If Jack could have earned interest at the rate of k per annum compounded quarterly from January 1, 2010, through January 1, 2018, his balance would have been $2203.76. Calculate the ratio k/j. ***Source: Society of Actuaries***. 5/4

+ **66. Savings** Eric deposits $100 into a savings account at time 0, which pays interest at a nominal rate of i, compounded semiannually. Mike deposits $200 into a different savings account at time 0, which pays simple interest, at an annual rate of i. Eric and Mike earn the same amount of interest during the last 6 months of the 8th year. Calculate i. Choose one of the following. ***Source: Society of Actuaries***. (c)

(a) 9.06% **(b)** 9.26%

(c) 9.46% **(d)** 9.66%

(e) 9.86%

+ indicates more challenging problem.

+ **67. Interest** Bruce and Robbie each open new bank accounts at time 0. Bruce deposits $100 into his bank account, and Robbie deposits $50 into his. Each account earns the same annual effective interest rate. The amount of interest earned in Bruce's account during the 11th year is equal to X. The amount of interest earned in Robbie's account during the 17th year is also equal to X. Calculate X. Choose one of the following. ***Source:* *Society of Actuaries*.** (e)

(a) 28.0 **(b)** 31.3 **(c)** 34.6 **(d)** 36.7 **(e)** 38.9

68. Interest Rate In 1995, O. G. McClain of Houston, Texas, mailed a $100 check to a descendant of Texas independence hero Sam Houston to repay a $100 debt of McClain's great-great-grandfather, who died in 1835, to Sam Houston. A bank estimated the interest on the loan to be $420 million for the 160 years it was due. Find the interest rate the bank was using, assuming interest is compounded annually. ***Source: The New York Times*.** 10.00%

69. Effective Rate In 2014, the Home Savings Bank paid 1% interest, compounded daily, on a 1-year CD, while the Palladian Private Bank paid 1% compounded quarterly. ***Source: Bankrate.com*.**

(a) What are the effective rates, rounded to the nearest thousandth percent, for the two CDs? Use a 365-day year. 1.005%, 1.004%

(b) Suppose $1000 was invested in each of these accounts. Find the compound amount, rounded to the nearest penny, after one year for each account. $1010.05, $1010.04

70. Effective Rate Determine the effective rates for the following nominal rates when interest is compounded daily and compounded quarterly. Use a 365-day year. Round to the nearest thousandth percent.

(a) 2% 2.020%, 2.015%

(b) 5% 5.127%, 5.095%

(c) 10% 10.516%, 10.381%

(d) 20% 22.134%, 21.551%

(e) Use your answers from parts (a)–(d), along with Exercise 69, to discuss the effect of the frequency of compounding on different interest rates.

71. Comparing CD Rates Synchrony Bank offered the following special CD rates for deposits above $25,000. The rates are annual percentage yields, or effective rates, which are higher than the corresponding nominal rates. Interest is compounded daily. Using a 365-day year, solve for r to approximate the corresponding nominal rates to the nearest hundredth. ***Source: Synchrony Bank*.** 0.65%, 1.09%, 1.29%, 2.27%

Term	6 mo	12 mo	3 yr	5 yr
APY%	0.65	1.10	1.30	2.30

72. Savings A department has ordered 8 new Apple computers at a cost of $2309 each. The order will not be delivered for 6 months. What amount could the department deposit in a special 6-month CD paying 4.79% compounded monthly to have enough to pay for the machines at time of delivery? $18,035.71

73. Buying a House Cara Tilley wants to have $30,000 available in 5 years for a down payment on a house. She has inherited $25,000. How much of the inheritance should she invest now to accumulate $30,000, if she can get an interest rate of 5.5% compounded quarterly? $22,829.90

74. Rule of 70 On the day of their first grandchild's birth, a new set of grandparents invested $10,000 in a trust fund earning 4.5% compounded monthly.

(a) Use the rule of 70 to estimate how old the grandchild will be when the trust fund is worth $20,000. 16 years old

(b) Use your answer to part (a) to determine the actual amount that will be in the trust fund at that time. How close was your estimate in part (a)? $20,516.69

Doubling Time Use the ideas from Example 11 to find the time it would take for the general level of prices in the economy to double at each average annual inflation rate.

75. 4% About 18 years

76. 5% 14 years

77. Doubling Time The consumption of electricity has increased historically at 6% per year. If it continues to increase at this rate indefinitely, find the number of years before the electric utilities will need to double their generating capacity. About 12 years

78. Doubling Time Suppose a conservation campaign coupled with higher rates causes the demand for electricity to increase at only 2% per year, as it has recently. Find the number of years before the utilities will need to double generating capacity. 35 years

79. Mitt Romney According to *The New York Times*, "During the fourteen years [Mitt Romney] ran it, Bain Capital's investments reportedly earned an annual rate of return of over 100 percent, potentially turning an initial investment of $1 million into more than $14 million by the time he left in 1998." ***Source: The New York Times*.**

(a) What rate of return, compounded annually, would turn $1 million into $14 million by 1998? 20.7%

(b) The actual rate of return of Bain Capital during the 14 years that Romney ran it was 113%. How much would $1 million, compounded annually at this rate, be worth after 14 years? ***Source: The American*.** $39.6 billion

80. Investment In the New Testament, Jesus commends a widow who contributed 2 mites to the temple treasury (Mark 12: 42–44). A mite was worth roughly 1/8 of a cent. Suppose the temple invested those 2 mites at 4% interest compounded quarterly. How much would the money be worth 2000 years later? $\$9.31 \times 10^{31}$

YOUR TURN ANSWERS

1. $3048.33	**2.** 6.5%
3. $545.75	**4.** 3.6%
5. 2.73%	**6.** $7438.39
7. 18 years	**8.** $2038.80

5.2 Future Value of an Annuity

APPLY IT **If you deposit \$1500 each year for 6 years in an account paying 8% interest compounded annually, how much will be in your account at the end of this period?**

We will use geometric sequences to develop a formula for the future value of such periodic payments in Example 3.

Teaching Tip: The material on geometric sequences may seem more abstract than the rest of the material in this chapter, but it is the fundamental tool for deriving formulas for annuities.

CAUTION

Do not confuse r, the ratio of two successive terms in a geometric series, with r, the annual interest rate. Different letters might have been helpful, but the usage in both cases is almost universal.

Geometric Sequences

If a and r are nonzero real numbers, the infinite list of numbers $a, ar, ar^2, ar^3, ar^4, \ldots, ar^n, \ldots$ is called a **geometric sequence**. For example, if $a = 3$ and $r = -2$, we have the sequence

$$3, 3(-2), 3(-2)^2, 3(-2)^3, 3(-2)^4, \ldots,$$

or

$$3, -6, 12, -24, 48, \ldots.$$

In the sequence $a, ar, ar^2, ar^3, ar^4, \ldots$, the number a is called the **first term** of the sequence, ar is the **second term**, ar^2 is the **third term**, and so on. Thus, for any $n \geq 1$,

ar^{n-1} is the nth term of the sequence.

Each term in the sequence is r times the preceding term. The number r is called the **common ratio** of the sequence.

EXAMPLE 1 Geometric Sequence

Find the seventh term of the geometric sequence 5, 20, 80, 320, ...

SOLUTION The first term in the sequence is 5, so $a = 5$. The common ratio, found by dividing the second term by the first, is $r = 20/5 = 4$. We want the seventh term, so $n = 7$. Use ar^{n-1}, with $a = 5$, $r = 4$, and $n = 7$.

$$ar^{n-1} = (5)(4)^{7-1} = 5(4)^6 = 20{,}480$$

Next, we need to find the sum S_n of the first n terms of a geometric sequence, where

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}. \qquad (1)$$

If $r = 1$, then

$$S_n = \underbrace{a + a + a + a + \cdots + a}_{n \text{ terms}} = na.$$

If $r \neq 1$, multiply both sides of equation (1) by r to get

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n. \qquad (2)$$

Now subtract corresponding sides of equation (1) from equation (2).

$$\begin{aligned}
rS_n &= ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\
-S_n &= -(a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}) \\
\hline
rS_n - S_n &= -a + ar^n \\
S_n(r - 1) &= a(r^n - 1) && \text{Factor.} \\
S_n &= \frac{a(r^n - 1)}{r - 1} && \text{Divide both sides by } r - 1.
\end{aligned}$$

This result is summarized on the next page.

Sum of Terms

If a geometric sequence has first term a and common ratio r, then the sum S_n of the first n terms is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1.$$

EXAMPLE 2 Sum of a Geometric Sequence

Find the sum of the first six terms of the geometric sequence 3, 12, 48, ...

SOLUTION Here $a = 3$, $r = 4$, and $n = 6$. Find S_6 by the formula above.

$$S_6 = \frac{3(4^6 - 1)}{4 - 1} \qquad n = 6, a = 3, r = 4.$$

$$= \frac{3(4096 - 1)}{3}$$

$$= 4095$$

TRY YOUR TURN 1

YOUR TURN 1 Find the sum of the first 9 terms of the geometric series 4, 12, 36, ...

Ordinary Annuities

A sequence of equal payments made at equal periods of time is called an **annuity**. If the payments are made at the end of the time period, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity**. The time between payments is the **payment period**, and the time from the beginning of the first payment period to the end of the last period is called the **term** of the annuity. The **future value of the annuity**, the final sum on deposit, is defined as the sum of the compound amounts of all the payments, compounded to the end of the term.

Two common uses of annuities are to accumulate funds for some goal or to withdraw funds from an account. For example, an annuity may be used to save money for a large purchase, such as an automobile, a college education, or a down payment on a home. An annuity also may be used to provide monthly payments for retirement. We explore these options in this and the next section.

EXAMPLE 3 Ordinary Annuity

Suppose \$1500 is deposited at the end of each year for the next 6 years in an account paying 8% per year compounded annually. How much will be in the account at the end of this period?

APPLY IT

SOLUTION Figure 5 shows this annuity. To find the future value of the annuity, look separately at each of the \$1500 payments. The first of these payments will produce a compound amount of

$$1500(1 + 0.08)^5 = 1500(1.08)^5.$$

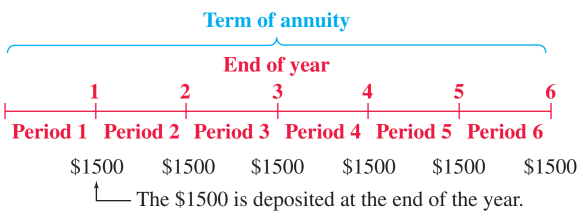


FIGURE 5

Use 5 as the exponent instead of 6, since the money is deposited at the *end* of the first year and earns interest for only 5 years. The second payment of \$1500 will produce a compound amount of $1500(1.08)^4$. As shown in Figure 6, the future value of the annuity is

$$1500(1.08)^5 + 1500(1.08)^4 + 1500(1.08)^3 + 1500(1.08)^2 + 1500(1.08)^1 + 1500.$$

(The last payment earns no interest at all.)

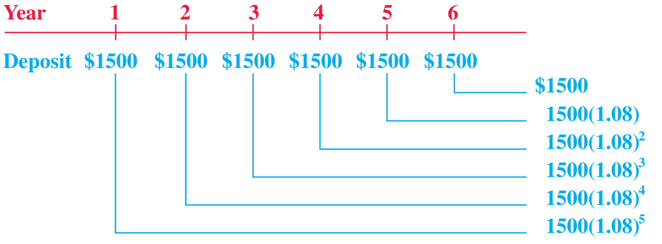


FIGURE 6

Reading this sum in reverse order, we see that it is the sum of the first six terms of a geometric sequence, with $a = 1500$, $r = 1.08$, and $n = 6$. Thus, the sum equals

$$\frac{a(r^n - 1)}{r - 1} = \frac{1500[(1.08)^6 - 1]}{1.08 - 1} \approx \$11{,}003.89.$$

To generalize this result, suppose that payments of R dollars each are deposited into an account at the end of each period for n periods, at a rate of interest i per period. The first payment of R dollars will produce a compound amount of $R(1 + i)^{n-1}$ dollars, the second payment will produce $R(1 + i)^{n-2}$ dollars, and so on; the final payment earns no interest and contributes just R dollars to the total. If S represents the future value (or sum) of the annuity, then (as shown in Figure 7),

$$S = R(1 + i)^{n-1} + R(1 + i)^{n-2} + R(1 + i)^{n-3} + \cdots + R(1 + i) + R,$$

or, written in reverse order,

$$S = R + R(1 + i)^1 + R(1 + i)^2 + \cdots + R(1 + i)^{n-1}.$$

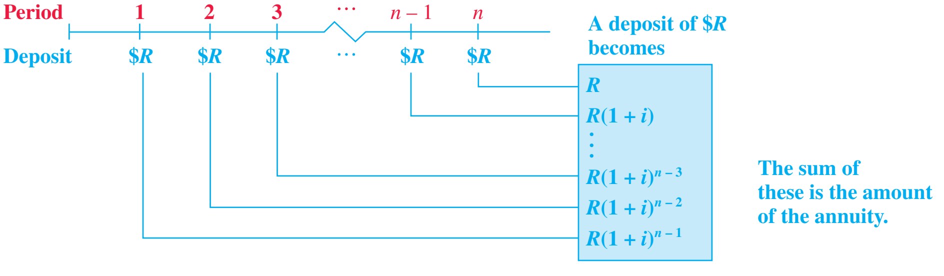


FIGURE 7

This result is the sum of the first n terms of the geometric sequence having first term R and common ratio $1 + i$. Using the formula for the sum of the first n terms of a geometric sequence,

$$S = \frac{R[(1 + i)^n - 1]}{(1 + i) - 1} = \frac{R[(1 + i)^n - 1]}{i} = R\left[\frac{(1 + i)^n - 1}{i}\right].$$

The quantity in brackets is commonly written $s_{\overline{n}|i}$ (read "s-angle-n at i"), so that

$$S = R \cdot s_{\overline{n}|i}.$$

Values of $s_{\overline{n}|i}$ can be found with a calculator.

A formula for the future value of an annuity S of n payments of R dollars each at the end of each consecutive interest period, with interest compounded at a rate i per period, follows. Recall that this type of annuity, with payments at the *end* of each time period, is called an ordinary annuity.

Future Value of an Ordinary Annuity

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] \quad \text{or} \quad S = Rs_{\overline{n}|i}$$

where

S is the future value;

R is the periodic payment;

i is the interest rate per period;

n is the number of periods.

NOTE We use S for the future value of an annuity, instead of A as in the compound interest formula, to help avoid confusing the two formulas. Some texts use FV for the future value, PV for the present value, and PMT for the payment. Although these multiple-letter variables may be easier to identify, in mathematics, a string of variables usually indicates multiplication. (In other words, the symbol FV would indicate the multiplication of the variables F and V.) Therefore, to avoid confusion, we will continue to use S for the future value of an annuity, P for the present value, and R for the periodic payment.

EXAMPLE 4 Ordinary Annuity

Leslie Mitchell is an athlete who believes that her playing career will last 7 years.

(a) To prepare for her future, she deposits \$24,000 at the end of each year for 7 years in an account paying 6% compounded annually. How much will she have on deposit after 7 years?

SOLUTION Her yearly payments form an ordinary annuity with $R = 24{,}000$. Since the interest is compounded annually, $m = 1$. The number of periods is $n = mt = 1(7) = 7$ and the interest rate per period is $i = r/m = 0.06/1 = 0.06$. Using the formula for the future value of an annuity,

$$S = 24{,}000 \cdot s_{\overline{7}|0.06} = 24{,}000\left[\frac{(1.06)^7 - 1}{0.06}\right] \approx 201{,}452.10,$$

or \$201,452.10. Note that she made 7 payments of \$24,000, or \$168,000. The interest that she earned is \$201,452.10 − \$168,000 = \$33,452.10.

(b) Instead of investing \$24,000 at the end of each year, suppose Leslie deposits \$2000 at the end of each month for 7 years in an account paying 6% compounded monthly. How much will she have on deposit after 7 years?

SOLUTION Interest is compounded monthly, so $m = 12$, $n = 12(7) = 84$, and $i = 0.06/12 = 0.005$. Using the formula for the future value of an annuity with $R = 2000$ gives

$$S = 2000 \cdot s_{\overline{84}|0.005} = 2000\left[\frac{(1.005)^{84} - 1}{0.005}\right] \approx 208{,}147.85,$$

or \$208,147.85. Note that Leslie made 84 payments of \$2000, so she still invested \$168,000. With this investment, her interest is \$40,147.85. **TRY YOUR TURN 2**

YOUR TURN 2 Find the accumulated amount after 11 years if \$250 is deposited every month in an account paying 3.3% interest compounded monthly.

TECHNOLOGY NOTE

As we saw in Section 5.1, the TI-84 Plus C graphing calculator can be used to find the future value of many investments. Select `APPS`, then `Finance`, then `TVM Solver`. For the investment in Example 4(b), we would enter 84 for `N` (number of payments), 6 for `I%` (interest rate), 0 for `PV` (present value), −2000 for `PMT` (negative because payments are considered an outflow), 0 for `FV` (future value), and 12 for `P/Y` (payments per year). To find the future value, move the cursor to the `FV` line and enter `ALPHA`, then `SOLVE` (the `ENTER` button). The (rounded) future value is $208147.85, as shown in Figure 8. For more information on using the TVM solver, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

```
N=84
I%=6
PV=0
PMT=-2000
▪FV=208147.8544
P/Y=12
C/Y=12
PMT: END BEGIN
```

FIGURE 8

Sinking Funds

A fund set up to receive periodic payments as in Example 4 is called a **sinking fund**. The periodic payments, together with the interest earned by the payments, are designed to produce a given sum at some time in the future. For example, a sinking fund might be set up to receive money that will be needed to pay off the principal on a loan at some future time. If the payments are all the same amount and are made at the end of a regular time period, they form an ordinary annuity.

EXAMPLE 5 Sinking Fund

Experts say that the baby boom generation (Americans born between 1946 and 1960) cannot count on a company pension or Social Security to provide a comfortable retirement, as their parents did. It is recommended that they start to save early and regularly. Beth Hudacky, a baby boomer, has decided to deposit $200 each month for 20 years in an account that pays interest of 7.2% compounded monthly.

(a) How much will be in the account at the end of 20 years?

SOLUTION This savings plan is an annuity with $R = 200$, $i = 0.072/12 = 0.006$, and $n = 12(20) = 240$. The future value is

$$S = 200\left[\frac{(1 + 0.006)^{240} - 1}{0.006}\right] \approx 106{,}752.47,$$

or $106,752.47.

(b) Beth believes she needs to accumulate $130,000 in the 20-year period to have enough for retirement. What interest rate would provide that amount?

SOLUTION We are looking for the annual interest rate, r, that would yield a future value of $130,000 when compounded monthly. One way to determine this rate is to let $i = r/12$ and $S = 130{,}000$, and then solve the future value equation for r:

$$130{,}000 = 200\left[\frac{(1 + r/12)^{240} - 1}{r/12}\right]$$

This is a difficult equation to solve. Although trial and error could be used, we will use a graphing calculator.

Method 1
TVM Solver

Using the TVM Solver on the TI-84 C calculator, enter 240 for `N` (the number of periods), 0 for `PV` (present value), 0 for `I%` (annual interest rate), −200 for `PMT` (negative because the money is being paid out), 130000 for `FV` (future value), and 12 for `P/Y` (payments per year). Put the cursor on the `I%` line and press `SOLVE`. The result, shown in Figure 9, indicates that an interest rate of 8.79% (rounded) is needed.

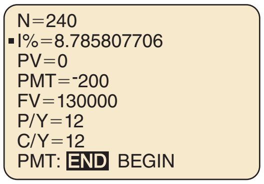


FIGURE 9

Method 2
Graphing Calculator

Graph the equations $y = 200((1 + x/12)^{240} - 1)/(x/12)$ and $y = 130{,}000$ on the same window, and then find the point of intersection. The annual interest rate, rounded to the nearest hundredth, is 8.79%. See Figure 10.

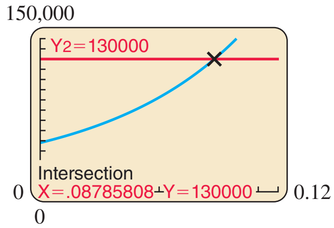


FIGURE 10

In Example 5 we used sinking fund calculations to determine the amount of money that accumulates over time through monthly payments and interest. We can also use this formula to determine the amount of money necessary to periodically invest at a given interest rate to reach a particular goal. Start with the annuity formula

$$S = R\left[\frac{(1+i)^n - 1}{i}\right],$$

and multiply both sides by $i/[(1+i)^n - 1]$ to derive the following formula.

NOTE
When calculating the sinking fund payment, you may have to round the periodic payment up to the nearest cent to ensure that enough money is accumulated by the specified time in the future. Always check your solution.

Sinking Fund Payment

$$R = \frac{Si}{(1+i)^n - 1} \quad \text{or} \quad R = \frac{S}{s_{\overline{n}|i}}$$

where

R is the periodic payment;

S is the future value;

i is the interest rate per period;

n is the number of periods.

EXAMPLE 6 Sinking Fund Payment

Suppose Beth, in Example 5, cannot get the higher interest rate to produce \$130,000 in 20 years. To meet that goal, she must increase her monthly payment. What payment should she make each month?

SOLUTION Beth's goal is to accumulate \$130,000 in 20 years at 7.2% compounded monthly. Therefore, the future value is $S = 130{,}000$, the monthly interest rate is $i = 0.072/12 = 0.006$, and the number of periods is $n = 12(20) = 240$. Use the sinking fund payment formula to find the payment R.

$$R = \frac{(130{,}000)(0.006)}{(1 + 0.006)^{240} - 1} \approx 243.5540887$$

Beth will need payments of \$243.56 each month for 20 years to accumulate at least \$130,000. Notice that \$243.55 is not quite enough, so round up here. **TRY YOUR TURN 3**

YOUR TURN 3 Find the quarterly payment needed to produce \$13,500 in 14 years at 3.75% interest compounded quarterly.

TECHNOLOGY NOTE

To use the TVM Solver on the TI-84 Plus C calculator to find the periodic payment in Example 6, enter `N` = 240, `I%` = 7.2, `PV` = 0, `PMT` = 0, `FV` = 130000, and `P/Y` = 12. Put the cursor on `PMT` and press `SOLVE`. See Figure 11. The payment is negative because the money is an outflow. The monthly payment, rounded up to the nearest cent, is \$243.55.

```
N=240
I%=7.2
PV=0
▪PMT=⁻243.5540887
FV=130000
P/Y=12
C/Y=12
PMT: END BEGIN
```

FIGURE 11

TECHNOLOGY NOTE

We can also use a graphing calculator or spreadsheet to make a table of the amount in a sinking fund. In the formula for future value of an annuity, simply let n be a variable with values from 1 to the total number of payments. Figure 12(a) shows the beginning of such a table generated on a TI-84 Plus C for Example 6. Figure 12(b) shows the beginning of the same table using Microsoft Excel.

X	Y1
1	243.55
2	488.56
3	735.04
4	983
5	1232.5
6	1483.4
7	1735.8
8	1989.8
9	2245.3
10	2502.3
11	2760.9

X=1

(a)

n	Amount in Fund
1	243.55
2	488.56
3	735.04
4	983.00
5	1232.45
6	1483.40
7	1735.85
8	1989.81
9	2245.30
10	2502.32
11	2760.89
12	3021.00

(b)

FIGURE 12

Annuities Due The future value formula developed in this section is for *ordinary annuities*—those with payments made at the *end* of each time period. These results can be modified slightly to apply to **annuities due**—annuities in which payments are made at the *beginning* of each time period. To find the future value of an annuity due, treat each payment as if it were made at the *end* of the *preceding* period. That is, find $s_{\overline{n}|i}$ for *one additional period*; to compensate for this, subtract the amount of one payment.

Thus, the **future value of an annuity due** of n payments of R dollars each at the beginning of consecutive interest periods, with interest compounded at the rate of i per period, is

$$S = R\left[\frac{(1+i)^{n+1}-1}{i}\right] - R \quad \text{or} \quad S = Rs_{\overline{n+1}|i} - R.$$

TECHNOLOGY NOTE The finance feature of the TI-84 Plus C can be used to find the future value of an annuity due. Proceed as you did with an ordinary annuity, but on the last line of the solver, select `BEGIN` for `PMT`.

EXAMPLE 7 Future Value of an Annuity Due

Find the future value of an annuity due if payments of $500 are made at the beginning of each quarter for 7 years, in an account paying 6% compounded quarterly.

SOLUTION In 7 years, there are $n = 28$ quarterly periods. Add one period to get $n + 1 = 29$, and use the formula with $i = 0.06/4 = 0.015$.

$$S = 500\left[\frac{(1.015)^{29}-1}{0.015}\right] - 500 \approx 17{,}499.35$$

The account will have a total of $17,499.35 after 7 years. **TRY YOUR TURN 4**

YOUR TURN 4 Find the future value of an annuity due with $325 made at the beginning of each month for 5 years in an account paying 3.3% compounded monthly.

5.2 WARM-UP EXERCISES

Find the compound amount for each deposit and the amount of interest earned. ***(Sec. 5.1)***

W1. $1500 at 3.2% compounded quarterly for 6 years. $1816.12; $316.12

W2. $800 at 4.8% compounded monthly for 3 years. $923.64; $123.64

5.2 EXERCISES

Find the fifth term of each geometric sequence.

1. $a = 3;\quad r = 2$ 48

2. $a = 7;\quad r = 5$ 4375

3. $a = -8;\quad r = 3$ −648

4. $a = -6;\quad r = 2$ −96

5. $a = 1;\quad r = -3$ 81

6. $a = 12;\quad r = -2$ 192

7. $a = 256;\quad r = \frac{1}{4}$ 1

8. $a = 729;\quad r = \frac{1}{3}$ 9

Find the sum of the first four terms for each geometric sequence.

9. $a = 1;\quad r = 2$ 15

10. $a = 4;\quad r = 4$ 340

11. $a = 5;\quad r = \frac{1}{5}$ 156/25

12. $a = 6;\quad r = \frac{1}{2}$ 45/4

13. $a = 128;\quad r = -\frac{3}{2}$ −208

14. $a = 64;\quad r = -\frac{3}{4}$ 25

15. Explain how a geometric sequence is related to an ordinary annuity.

16. Explain the difference between an ordinary annuity and an annuity due.

Find the future value of each ordinary annuity. Interest is compounded annually.

17. $R = 100;\quad i = 0.06;\quad n = 4$ $437.46

18. $R = 1000;\quad i = 0.03;\quad n = 5$ $5309.14

19. $R = 25{,}000;\quad i = 0.045;\quad n = 36$ $2,154,099.15

20. $R = 29{,}500;\quad i = 0.058;\quad n = 15$ $676,272.05

Find the future value of each ordinary annuity, if payments are made and interest is compounded as given. Then determine how much of this value is from contributions and how much is from interest.

21. $R = 9200$; 10% interest compounded semiannually for 7 years $180,307.41; $128,800; $51,507.41

22. $R = 1250$; 5% interest compounded semiannually for 18 years $71,626.77; $45,000; $26,626.77

23. $R = 800$; 6.51% interest compounded semiannually for 12 years $28,438.21; $19,200; $9238.21

24. $R = 4600$; 8.73% interest compounded quarterly for 9 years $247,752.70; $165,600; $82,152.70

25. $R = 12{,}000$; 4.8% interest compounded quarterly for 16 years $1,145,619.96; $768,000; $377,619.96

26. $R = 42{,}000$; 10.05% interest compounded semiannually for 12 years $1,875,230.74; $1,008,000; $867,230.74

27. What is meant by a sinking fund? Give an example of a sinking fund.

28. List some reasons for establishing a sinking fund.

Determine the interest rate needed to accumulate the following amounts in a sinking fund, with monthly payments as given.

29. Accumulate $56,000, monthly payments of $300 over 12 years 4.19%

30. Accumulate $120,000, monthly payments of $500 over 15 years 3.69%

Find the periodic payment that will amount to each given sum under the given conditions.

31. $S = \$10{,}000$; interest is 5% compounded annually; payments are made at the end of each year for 12 years. $628.26

32. $S = \$150{,}000$; interest is 6% compounded semiannually; payments are made at the end of each semiannual period for 11 years. $4912.11

Find the amount of each payment to be made into a sinking fund so that enough will be present to accumulate the following amounts. Payments are made at the end of each period.

33. $8500; money earns 8% compounded annually; there are 7 annual payments. $952.62

34. $2750; money earns 5% compounded annually; there are 5 annual payments. $497.68

35. $75,000; money earns 6% compounded semiannually for $4\frac{1}{2}$ years. $7382.54

36. $25,000; money earns 5.7% compounded quarterly for $3\frac{1}{2}$ years. $1626.16

37. $65,000; money earns 7.5% compounded quarterly for $2\frac{1}{2}$ years. $5970.23

38. $9000; money earns 4.8% compounded monthly for $2\frac{1}{2}$ years. $282.96

Find the future value of each annuity due. Assume that interest is compounded annually.

39. $R = 600$; $i = 0.06$; $n = 8$ $6294.79

40. $R = 1700$; $i = 0.04$; $n = 15$ $35,401.70

41. $R = 16{,}000$; $i = 0.05$; $n = 7$ $136,785.74

42. $R = 4000$; $i = 0.06$; $n = 11$ $63,479.76

Find the future value of each annuity due. Then determine how much of this value is from contributions and how much is from interest.

43. Payments of $1000 made at the beginning of each semiannual period for 9 years at 8.15% compounded semiannually $26,874.97; $18,000; $8874.97

44. $750 deposited at the beginning of each month for 15 years at 5.9% compounded monthly $217,328.08; $135,000; $82,328.08

45. $250 deposited at the beginning of each quarter for 12 years at 4.2% compounded quarterly $15,662.40; $12,000; $3662.40

46. $1500 deposited at the beginning of each semiannual period for 11 years at 5.6% compounded semiannually $46,034.09; $33,000; $13,034.09

APPLICATIONS

Business and Economics

47. Comparing Accounts April Peel deposits $12,000 at the end of each year for 9 years in an account paying 8% interest compounded annually.

(a) Find the final amount she will have on deposit. $149,850.69

(b) April's brother-in-law works in a bank that pays 6% compounded annually. If she deposits money in this bank instead of the one above, how much will she have in her account? $137,895.79

(c) How much would April lose over 9 years by using her brother-in-law's bank? $11,954.90

48. Savings Boyd Shepherd is saving for an Ultra HDTV. At the end of each month he puts $100 in a savings account that pays 2.25% interest compounded monthly. How much is in the account after 2 years? How much did Boyd deposit? How much interest did he earn? $2452.47; $2400; $52.47

49. Savings A typical pack-a-day smoker spends about $179.40 per month on cigarettes. Suppose the smoker invests that amount each month in a savings account at 4.8% interest compounded monthly. What would the account be worth after 40 years? ***Source: Tobacco Free Kids.*** $259,900.62

50. Retirement Planning A 45-year-old man puts $2500 in a retirement account at the end of each quarter until he reaches the age of 60, then makes no further deposits. If the account pays 6% interest compounded quarterly, how much will be in the account when the man retires at age 65? $323,967.96

51. Retirement Planning At the end of each quarter, a 50-year-old woman puts $3000 in a retirement account that pays 5% interest compounded quarterly. When she reaches 60, she withdraws the entire amount and places it in a mutual fund that pays 6.9% interest compounded monthly. From then on she deposits $300 in the mutual fund at the end of each month. How much is in the account when she reaches age 65? $239,315.17

Individual Retirement Accounts **Suppose a 40-year-old person deposits $4000 per year in an Individual Retirement Account until age 65. Find the total in the account with the following assumptions of interest rates. (Assume quarterly compounding, with payments of $1000 made at the end of each quarter period.) Find the total amount of interest earned.**

52. 6% **53.** 8% **54.** 4% **55.** 10%

52. $228,803.04; $128,803.04 **53.** $312,232.31; $212,232.31
54. $170,481.38; $70,481.38 **55.** $432,548.65; $332,548.65

56. Savings Hector Amaya needs \$10,000 in 8 years.

(a) What amount should he deposit at the end of each quarter at 8% compounded quarterly so that he will have his \$10,000? \$226.11

(b) Find Hector's quarterly deposit if the money is deposited at 6% compounded quarterly. \$245.78

57. Buying Equipment Harv, the owner of Harv's Meats, knows that he must buy a new deboner machine in 4 years. The machine costs \$12,000. In order to accumulate enough money to pay for the machine, Harv decides to deposit a sum of money at the end of each 6 months in an account paying 6% compounded semiannually. How much should each payment be? \$1349.48

58. Buying a Car Michelle Christian wants to have a \$20,000 down payment when she buys a new car in 6 years. How much money must she deposit at the end of each quarter in an account paying 3.2% compounded quarterly so that she will have the down payment she desires? \$759.22

59. Savings Taylor Larson is paid on the first day of the month and \$80 is automatically deducted from her pay and deposited in a savings account. If the account pays 2.5% interest compounded monthly, how much will be in the account after 3 years and 9 months? \$3777.89

60. Savings A father opened a savings account for his daughter on the day she was born, depositing \$1000. Each year on her birthday he deposits another \$1000, making the last deposit on her 21st birthday. If the account pays 5.25% interest compounded annually, how much is in the account at the end of the day on his daughter's 21st birthday? How much interest has been earned? \$39,664.40; \$17,664.40

61. Savings Jessica Elbern deposits \$2435 at the beginning of each semiannual period for 8 years in an account paying 6% compounded semiannually. She then leaves that money alone, with no further deposits, for an additional 5 years. Find the final amount on deposit after the entire 13-year period. \$67,940.98

62. Savings Nic Daubenmire deposits \$10,000 at the beginning of each year for 12 years in an account paying 5% compounded annually. He then puts the total amount on deposit in another account paying 6% compounded semiannually for another 9 years. Find the final amount on deposit after the entire 21-year period. \$284,527.35

In Exercises 63 and 64, use a graphing calculator to find the value of *i* that produces the given value of *S*. (See Example 5(b).)

63. Retirement To save for retirement, Karla Harby put \$300 each month into an ordinary annuity for 20 years. Interest was compounded monthly. At the end of the 20 years, the annuity was worth \$147,126. What annual interest rate did she receive? 6.5%

64. Rate of Return Kelli Hughes made payments of \$250 per month at the end of each month to purchase a piece of property. At the end of 30 years, she completely owned the property, which she sold for \$330,000. What annual interest rate would she need to earn on an annuity for a comparable rate of return? 7.397%

+ indicates more challenging problem.

65. Lottery In a 1992 Virginia lottery, the jackpot was \$27 million. An Australian investment firm tried to buy all possible combinations of numbers, which would have cost \$7 million. In fact, the firm ran out of time and was unable to buy all combinations but ended up with the only winning ticket anyway. The firm received the jackpot in 20 equal annual payments of \$1.35 million. Assume these payments meet the conditions of an ordinary annuity. ***Source: The Washington Post***.

(a) Suppose the firm can invest money at 8% interest compounded annually. How many years would it take until the investors would be further ahead than if they had simply invested the \$7 million at the same rate? (*Hint:* Experiment with different values of n, the number of years, or use a graphing calculator to plot the value of both investments as a function of the number of years.) 7 yr

(b) How many years would it take in part (a) at an interest rate of 12%? 9 yr

66. Buying Real Estate Erin D'Aquanni sells some land in Nevada. She will be paid a lump sum of \$60,000 in 7 years. Until then, the buyer pays 8% simple interest quarterly.

(a) Find the amount of each quarterly interest payment on the \$60,000. \$1200

(b) The buyer sets up a sinking fund so that enough money will be present to pay off the \$60,000. The buyer will make semiannual payments into the sinking fund; the account pays 6% compounded semiannually. Find the amount of each payment into the fund. \$3511.59

67. Buying Rare Stamps Trevor Stieg bought a rare stamp for his collection. He agreed to pay a lump sum of \$4000 after 5 years. Until then, he pays 6% simple interest semiannually on the \$4000.

(a) Find the amount of each semiannual interest payment. \$120

(b) Trevor sets up a sinking fund so that enough money will be present to pay off the \$4000. He will make annual payments into the fund. The account pays 8% compounded annually. Find the amount of each payment. \$681.83

+ **68. Down Payment** A conventional loan, such as for a car or a house, is similar to an annuity but usually includes a down payment. Show that if a down payment of D dollars is made at the beginning of the loan period, the future value of all the payments, including the down payment, is

$$S = D(1+i)^n + R\left[\frac{(1+i)^n - 1}{i}\right].$$

YOUR TURN ANSWERS

1. 39,364
2. \$39,719.98
3. \$184.41
4. \$21,227.66

5.3 Present Value of an Annuity; Amortization

APPLY IT

Teaching Tip: This section extends the ideas of the previous section to show how to calculate payments on a loan and create an amortization table.

What monthly payment will pay off a $17,000 car loan in 36 monthly payments at 6% annual interest?

The answer to this question is given in Example 2 in this section. We shall see that it involves finding the present value of an annuity.

Suppose that at the end of each year, for the next 10 years, $500 is deposited in a savings account paying 7% interest compounded annually. This is an example of an ordinary annuity. The **present value of an annuity** is the amount that would have to be deposited in one lump sum today (at the same compound interest rate) in order to produce exactly the same balance at the end of 10 years. We can find a formula for the present value of an annuity as follows.

Suppose deposits of R dollars are made at the end of each period for n periods at interest rate i per period. Then the amount in the account after n periods is the future value of this annuity:

$$S = R \cdot s_{\overline{n}|i} = R\left[\frac{(1+i)^n - 1}{i}\right].$$

On the other hand, if P dollars are deposited today at the same compound interest rate i, then at the end of n periods, the amount in the account is $P(1+i)^n$. If P is the present value of the annuity, this amount must be the same as the amount S in the formula above; that is,

$$P(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right].$$

FOR REVIEW

Recall from Section R.6 that for any nonzero number a, $a^0 = 1$. Also, by the product rule for exponents, $a^x \cdot a^y = a^{x+y}$. In particular, if a is any nonzero number, $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$.

To solve this equation for P, multiply both sides by $(1+i)^{-n}$.

$$P = R(1+i)^{-n}\left[\frac{(1+i)^n - 1}{i}\right]$$

Use the distributive property; also recall that $(1+i)^{-n}(1+i)^n = 1$.

$$P = R\left[\frac{(1+i)^{-n}(1+i)^n - (1+i)^{-n}}{i}\right] = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

The amount P is the *present value of the annuity*. The quantity in brackets is abbreviated as $a_{\overline{n}|i}$, so

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}.$$

(The symbol $a_{\overline{n}|i}$ is read "*a*-angle-*n* at *i*." Compare this quantity with $s_{\overline{n}|i}$ in the previous section.) The formula for the present value of an annuity is summarized below.

Present Value of an Ordinary Annuity

The present value P of an annuity of n payments of R dollars each at the end of consecutive interest periods with interest compounded at a rate of interest i per period is

$$\boldsymbol{P = R\left[\frac{1-(1+i)^{-n}}{i}\right]} \quad \text{or} \quad \boldsymbol{P = Ra_{\overline{n}|i}}.$$

CAUTION Don't confuse the formula for the present value of an annuity with the one for the future value of an annuity. Notice the difference: The numerator of the fraction in the present value formula is $1 - (1 + i)^{-n}$, but in the future value formula, it is $(1 + i)^n - 1$.

EXAMPLE 1 Present Value of an Annuity

Tim Wilson and Carol Britz are both graduates of the Brisbane Institute of Technology (BIT). They both agree to contribute to the endowment fund of BIT. Tim says that he will give \$500 at the end of each year for 9 years. Carol prefers to give a lump sum today. What lump sum can she give that will equal the present value of Tim's annual gifts, if the endowment fund earns 7.5% compounded annually?

YOUR TURN 1 Find the present value of an annuity of \$120 at the end of each month put into an account yielding 4.8% compounded monthly for 5 years.

SOLUTION Here, $R = 500$, $n = 9$, and $i = 0.075$, and we have

$$P = R \cdot a_{\overline{9}|0.075} = 500\left[\frac{1 - (1.075)^{-9}}{0.075}\right] \approx 3189.44.$$

Therefore, Carol must donate a lump sum of \$3189.44 today. **TRY YOUR TURN 1**

One of the most important uses of annuities is in determining the equal monthly payments needed to pay off a loan, as illustrated in the next example.

EXAMPLE 2 Car Payments

A car costs \$19,000. After a down payment of \$2000, the balance will be paid off in 36 equal monthly payments with interest of 6% per year on the unpaid balance. Find the amount of each payment.

APPLY IT

SOLUTION A single lump sum payment of \$17,000 today would pay off the loan. So, \$17,000 is the present value of an annuity of 36 monthly payments with interest of $6\%/12 = 0.5\%$ per month. Thus, $P = 17{,}000$, $n = 36$, $i = 0.005$, and we must find the monthly payment R in the formula

$$P = R\left[\frac{1 - (1 + i)^{-n}}{i}\right]$$

$$17{,}000 = R\left[\frac{1 - (1.005)^{-36}}{0.005}\right]$$

$$R \approx 517.17.$$

YOUR TURN 2 Find the car payment in Example 2 if there are 48 equal monthly payments and the interest rate is 5.4%.

A monthly payment of \$517.17 will be needed. **TRY YOUR TURN 2**

Each payment in Example 2 includes interest on the unpaid balance, with the remainder going to reduce the loan. For example, the first payment of \$517.17 includes interest of $0.005(\$17{,}000) = \$85$ and is divided as follows.

$$\underset{\text{payment}}{\overset{\text{monthly}}{\$517.17}} - \underset{\text{due}}{\overset{\text{interest}}{\$85}} = \underset{\text{the balance}}{\overset{\text{to reduce}}{\$432.17}}$$

At the end of this section, amortization schedules show that this procedure does reduce the loan to \$0 after all payments are made (the final payment may be slightly different).

TECHNOLOGY NOTE

As in the first two sections, the financial feature of the TI-84 Plus C calculator can simplify the calculations.

In Example 1, we are looking for the present value. In the `TVM Solver`, we enter the number of payments `N` = 9, the interest rate `I%` = 7.5, the payment `PMT` = −500, and the number of payments per year `P/Y` = 1. (We leave the present value `PV` and future value `FV` zero.) Move the cursor to the `PV` line and press `SOLVE`. See Figure 13. The present value, rounded to the nearest cent, is $3189.44.

```
N=9
I%=7.5
▪PV=3189.443514
 PMT=⁻500
 FV=0
 P/Y=1
 C/Y=1
 PMT: END  BEGIN
```

FIGURE 13

```
N=36
I%=6
PV=17000
▪PMT=⁻517.1729367
 FV=0
 P/Y=12
 C/Y=12
 PMT: END  BEGIN
```

FIGURE 14

In Example 2, we are looking for the amount of each car payment. Again we use the `TVM Solver`, but this time we will enter the present value (the amount we are borrowing for the car) and we leave the payment zero. Since we are borrowing the money, the present value is an inflow and is entered as a positive value. We enter `N` = 36, `I%` = 6, `PV` = 17,000, `PMT` = `FV` = 0, and `P/Y` = 12. Move the cursor to the `PMT` line and press `SOLVE`. See Figure 14. The monthly payment (rounded) is $517.17.

Amortization

A loan is **amortized** if both the principal and interest are paid by a sequence of equal periodic payments. In Example 2, a loan of $17,000 at 6% interest compounded monthly could be amortized by paying $517.17 per month for 36 months.

The periodic payment needed to amortize a loan may be found, as in Example 2, by solving the present value equation for R.

Amortization Payments

A loan of P dollars at interest rate i per period may be amortized in n equal periodic payments of R dollars made at the end of each period, where

$$\boldsymbol{R = \frac{P}{\left[\dfrac{1-(1+i)^{-n}}{i}\right]} = \frac{Pi}{1-(1+i)^{-n}} \quad \text{or} \quad R = \frac{P}{a_{\overline{n}|i}}}.$$

EXAMPLE 3 Home Mortgage

The Perez family buys a house for $275,000, with a down payment of $55,000. They take out a 30-year mortgage for $220,000 at an annual interest rate of 6%.

(a) Find the amount of the monthly payment needed to amortize this loan.

SOLUTION Here $P = 220{,}000$ and the monthly interest rate is $i = 0.06/12 = 0.005$.* The number of monthly payments is $n = 12(30) = 360$. Therefore,

$$R = \frac{220{,}000}{a_{\overline{360}|0.005}} = \frac{220{,}000}{\left[\dfrac{1-(1.005)^{-360}}{0.005}\right]} = 1319.01.$$

Monthly payments of $1319.01 are required to amortize the loan.

*Mortgage rates are quoted in terms of annual interest, but it is always understood that the monthly rate is 1/12 of the annual rate and that interest is compounded monthly.

(b) Find the total amount of interest paid when the loan is amortized over 30 years.

SOLUTION The Perez family makes 360 payments of \$1319.01 each, for a total of \$474,843.60. Since the amount of the loan was \$220,000, the total interest paid is

$$\$474{,}843.60 - \$220{,}000 = \$254{,}843.60.$$

This large amount of interest is typical of what happens with a long mortgage. A 15-year mortgage would have higher payments but would involve significantly less interest.

(c) Find the part of the first payment that is interest and the part that is applied to reducing the debt.

SOLUTION During the first month, the entire \$220,000 is owed. Interest on this amount for 1 month is found by the formula for simple interest, with $r =$ annual interest rate and $t =$ time in years.

$$I = Prt = 220{,}000(0.06)\frac{1}{12} = \$1100$$

At the end of the month, a payment of \$1319.01 is made; since \$1100 of this is interest, a total of

$$\$1319.01 - \$1100 = \$219.01$$

is applied to the reduction of the original debt. **TRY YOUR TURN 3**

YOUR TURN 3 Find the monthly payment and total amount of interest paid in Example 3 if the mortgage is for 15 years and the interest rate is 7%.

It can be shown that the unpaid balance after x payments is given by the function

$$y = R\left[\frac{1 - (1 + i)^{-(n-x)}}{i}\right],$$

although this formula will only give an approximation if R is rounded to the nearest penny. For example, the unrounded value of R in Example 3 is 1319.011155. When this value is put into the above formula, the unpaid balance is found to be

$$y = 1319.011155\left[\frac{1 - (1.005)^{-359}}{0.005}\right] = 219{,}780.99,$$

while rounding R to 1319.01 in the above formula gives an approximate balance of \$219,780.80. A graph of this function is shown in Figure 15.

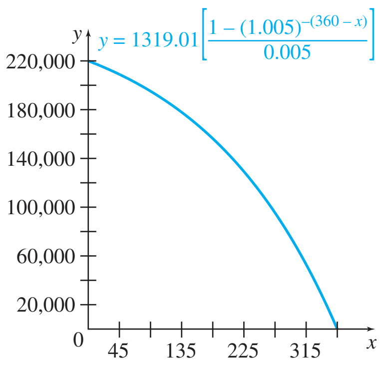


FIGURE 15

We can find the unpaid balance after any number of payments, x, by finding the y-value that corresponds to x. For example, the remaining balance after 5 years or 60 payments is shown at the bottom of the graphing calculator screen in Figure 16(a). You may be surprised that the remaining balance on a \$220,000 loan is as large as \$204,719.41. This is because most of the early payments on a loan go toward interest, as we saw in Example 3(c).

By adding the graph of $y = (1/2)220{,}000 = 110{,}000$ to the figure, we can find when half the loan has been repaid. From Figure 16(b) we see that 252 payments are required. Note that only 108 payments remain at that point, which again emphasizes the fact that the earlier payments do little to reduce the loan.

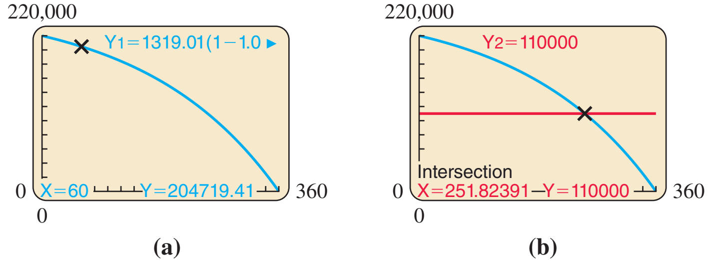


FIGURE 16

Amortization Schedules In the preceding example, 360 payments are made to amortize a \$220,000 loan. The loan balance after the first payment is reduced by only \$219.01, which is much less than $(1/360)(220{,}000) \approx \$611.11$. Therefore, even though equal *payments* are made to amortize a loan, the loan *balance* does not decrease in equal steps. This fact is very important if a loan is paid off early.

EXAMPLE 4 Early Payment

Kassy Morgan borrows \$1000 for 1 year at 12% annual interest compounded monthly. Verify that her monthly loan payment is \$88.8488, which is rounded to \$88.85. After making three payments, she decides to pay off the remaining balance all at once. How much must she pay?

SOLUTION Since nine payments remain to be paid, they can be thought of as an annuity consisting of nine payments of \$88.85 at 1% interest per period. The present value of this annuity is

$$88.8488\left[\frac{1-(1.01)^{-9}}{0.01}\right] \approx 761.08.$$

So Kassy's remaining balance, computed by this method, is \$761.08.

An alternative method of figuring the balance is to consider the payments already made as an annuity of three payments. At the beginning, the present value of this annuity was

$$88.85\left[\frac{1-(1.01)^{-3}}{0.01}\right] \approx 261.31.$$

So she still owes the difference $\$1000 - \$261.31 = \$738.69$. Furthermore, she owes the interest on this amount for 3 months, for a total of

$$(738.69)(1.01)^3 \approx \$761.07.$$

This balance due differs from the one obtained by the first method by 1 cent because the monthly payment and the other calculations were rounded to the nearest penny. If we had used the more accurate value of $R = 88.8488$ and not rounded any intermediate answers, both methods would have given the same value of \$761.08. TRY YOUR TURN 4

YOUR TURN 4 Find the remaining balance in Example 4 if the balance was to be paid off after four months. Use the unrounded value for R of \$88.8488.

Although most people would not quibble about a difference of 1 cent in the balance due in Example 4, the difference in other cases (larger amounts or longer terms) might be more than that. A bank or business must keep its books accurately to the nearest penny, so it must determine the balance due in such cases unambiguously and exactly. This is done by means of an **amortization schedule**, which lists how much of each payment is interest and how much goes to reduce the balance, as well as how much is owed after *each* payment.

EXAMPLE 5 Amortization Schedule

In Example 4, Kassy Morgan borrowed \$1000 for 1 year at 12% annual interest compounded monthly. Her monthly payment was determined to be \$88.85. Construct an amortization schedule for the loan and then determine the exact amount Kassy owes after three monthly payments.

SOLUTION An amortization schedule for the loan is shown on the next page. The annual interest rate is 12% compounded monthly, so the interest rate per month is $i = 0.12/12 = 0.01$. Each payment, except the final payment, is the same, but the amount of money paid toward the interest decreases each month.

The first monthly payment is \$88.85. To determine how much of the payment goes toward interest, multiply the principal balance (\$1000) by the monthly interest rate: $0.01(1000) = \$10$. The remainder of the payment, $\$88.85 - \$10 = \$78.85$, is applied to repayment of the principal. So after the first payment, the remaining balance is $\$1000 - \$78.85 = \$921.15$, as shown in the payment 1 line of the schedule.

The second monthly payment is again \$88.85. However, the monthly interest is owed on the new, smaller principal balance of \$921.15. The monthly interest is now $0.01(921.15) = \$9.21$. The remainder of the payment, $\$88.85 - \$9.21 = \$79.64$, is applied to the principal. After the second payment, the remaining balance is $\$921.15 - \$79.64 = \$841.51$, as shown in the payment 2 line.

The remaining lines of the schedule are found in a similar fashion. We continue to calculate the interest based on the remaining principal balance until the loan is paid off. Each month the amount of interest declines and the amount applied to the principal balance increases. After 12 payments, the loan is fully paid off. Notice that all the payments are the same except the last one. It is often necessary to adjust the amount of the final payment to account for rounding off earlier and to ensure that the final balance is exactly \$0.

The schedule shows that after three payments, she still owes \$761.08, an amount that agrees with the first method in Example 4.

Amortization Table

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	\$1000.00
1	\$88.85	\$10.00	\$78.85	\$921.15
2	\$88.85	\$9.21	\$79.64	\$841.51
3	\$88.85	\$8.42	\$80.43	\$761.08
4	\$88.85	\$7.61	\$81.24	\$679.84
5	\$88.85	\$6.80	\$82.05	\$597.79
6	\$88.85	\$5.98	\$82.87	\$514.92
7	\$88.85	\$5.15	\$83.70	\$431.22
8	\$88.85	\$4.31	\$84.54	\$346.68
9	\$88.85	\$3.47	\$85.38	\$261.30
10	\$88.85	\$2.61	\$86.24	\$175.06
11	\$88.85	\$1.75	\$87.10	\$87.96
12	\$88.84	\$0.88	\$87.96	\$0.00

TECHNOLOGY NOTE

A graphing calculator program to produce an amortization schedule is available in the *Graphing Calculator and Excel Spreadsheet Manual* available with this book. The TI-84 Plus C includes a built-in program to find the amortization payment. Spreadsheets are another useful tool for creating amortization tables. Microsoft Excel has a built-in feature for calculating monthly payments. Figure 17 shows an Excel amortization table for Example 5. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual*, available with this book.

	A	B	C	D	E	F
1	Pmt#	Payment	Interest	Principal	End Prncpl	
2	0				1000	
3	1	88.85	10.00	78.85	921.15	
4	2	88.85	9.21	79.64	841.51	
5	3	88.85	8.42	80.43	761.08	
6	4	88.85	7.61	81.24	679.84	
7	5	88.85	6.80	82.05	597.79	
8	6	88.85	5.98	82.87	514.92	
9	7	88.85	5.15	83.70	431.22	
10	8	88.85	4.31	84.54	346.68	
11	9	88.85	3.47	85.38	261.30	
12	10	88.85	2.61	86.24	175.06	
13	11	88.85	1.75	87.10	87.96	
14	12	88.85	0.88	87.97	-0.01	

FIGURE 17

EXAMPLE 6 Paying Off a Loan Early

Suppose that in Example 2, the car owner decides that she can afford to make payments of \$700 rather than \$517.17. How much earlier would she pay off the loan? How much interest would she save?

SOLUTION Putting $R = 700$, $P = 17{,}000$, and $i = 0.005$ into the formula for the present value of an annuity gives

$$17{,}000 = 700\left[\frac{1 - 1.005^{-n}}{0.005}\right].$$

Multiply both sides by 0.005 and divide by 700 to get

$$\frac{85}{700} = 1 - 1.005^{-n},$$

or

$$1.005^{-n} = 1 - \frac{85}{700}.$$

Solve this using either logarithms or a graphing calculator, as in Example 10 in Section 5.1, to get $n = 25.956$. This means that 25 payments of \$700, plus a final, smaller payment, would be sufficient to pay off the loan. Create an amortization table to verify that the final payment would be \$669.47 (the sum of the principal after the penultimate payment plus the interest on that principal for the final month). The loan would then be paid off after 26 months, or 10 months early.

The original loan required 36 payments of \$517.17, or $36(517.17) = \$18{,}618.12$, although the amount is actually \$18,618.24 because the final payment was \$517.29, as an amortization table would show. With larger payments, the car owner paid $25(700) + 669.47 = \$18{,}169.47$. Therefore, the car owner saved $\$18{,}618.24 - \$18{,}169.47 = \$448.77$ in interest by making larger payments each month.

5.3 WARM-UP EXERCISES

W1. Suppose \$150 is deposited at the end of each month for 5 years in an account paying 3% compounded monthly. Determine the future value and the amount of interest earned. *(Sec. 5.2)* \$9697.01; \$697.01

W2. How much money must be deposited at the end of each month into an account paying 4.8% compounded monthly so that it will accumulate to \$10,000 in 5 years? Determine the total amount of interest earned. *(Sec. 5.2)* \$147.80; \$1132

5.3 EXERCISES

1. Explain the difference between the present value of an annuity and the future value of an annuity. For a given annuity, which is larger? Why?
2. What does it mean to amortize a loan?

Find the present value of each ordinary annuity.

3. Payments of \$890 each year for 16 years at 6% compounded annually \$8994.25
4. Payments of \$1400 each year for 8 years at 7% compounded annually \$8359.82
5. Payments of \$10,000 semiannually for 15 years at 5% compounded semiannually \$209,302.93
6. Payments of \$50,000 quarterly for 10 years at 4% compounded quarterly \$1,641,734.31
7. Payments of \$15,806 quarterly for 3 years at 6.8% compounded quarterly \$170,275.47
8. Payments of \$18,579 every 6 months for 8 years at 5.4% compounded semiannually \$238,816.23

Find the lump sum deposited today that will yield the same total amount as payments of $10,000 at the end of each year for 15 years at each of the given interest rates.

9. 4% compounded annually $111,183.87

10. 6% compounded annually $97,122.49

Find (a) the payment necessary to amortize each loan; (b) the total payments and the total amount of interest paid based on the calculated monthly payments, and (c) the total payments and total amount of interest paid based upon an amortization table.

11. $2500; 6% compounded quarterly; 6 quarterly payments
(a) $438.81 **(b)** $2632.86; $132.86 **(c)** $2632.88; $132.88

12. $41,000; 8% compounded semiannually; 10 semiannual payments
(a) $5054.93 **(b)** $50,549.30; $9549.30 **(c)** $50,549.28; $9549.28

13. $90,000; 6% compounded annually; 12 annual payments
(a) $10,734.93 **(b)** $128,819.16; $38,819.16 **(c)** $128,819.20; $38,819.20

14. $140,000; 8% compounded quarterly; 15 quarterly payments
(a) $10,895.57 **(b)** $163,433.55; $23,433.55 **(c)** $163,433.48; $23,433.48

15. $7400; 6.2% compounded semiannually; 18 semiannual payments
(a) $542.60 **(b)** $9766.80; $2366.80 **(c)** $9766.88; $2366.88

16. $5500; 10% compounded monthly; 24 monthly payments
(a) $253.80 **(b)** $6091.20; $591.20 **(c)** $6091.12; $591.12

Suppose that in the loans described in Exercises 13–16, the borrower paid off the loan after the time indicated below. Calculate the amount needed to pay off the loan, using either of the two methods described in Example 4.

17. After 3 years in Exercise 13 $73,015.71

18. After 5 quarters in Exercise 14 $97,870.35

19. After 3 years in Exercise 15 $5368.98

20. After 7 months in Exercise 16 $4007.35

Use the amortization table in Example 5 to answer the questions in Exercises 21–24.

21. How much of the fourth payment is interest? $7.61

22. How much of the eleventh payment is used to reduce the debt? $87.10

23. How much interest is paid in the first 4 months of the loan? $35.24

24. How much interest is paid in the last 4 months of the loan? $8.71

25. What sum deposited today at 5% compounded annually for 8 years will provide the same amount as $1000 deposited at the end of each year for 8 years at 6% compounded annually? $6699

26. What lump sum deposited today at 8% compounded quarterly for 10 years will yield the same final amount as deposits of $4000 at the end of each 6-month period for 10 years at 6% compounded semiannually? About $48,677.34

Find the monthly house payments necessary to amortize each loan. Then calculate the total payments and the total amount of interest paid.

27. $199,000 at 7.01% for 25 years $1407.76; $422,328; $223,328

28. $175,000 at 6.24% for 30 years $1076.37; $387,493.20; $212,493.20

29. $253,000 at 6.45% for 30 years $1590.82; $572,695.20; $319,695.20

30. $310,000 at 5.96% for 25 years $1989.76; $596,928; $286,928

Suppose that in the loans described in Exercises 13–16, the borrower made a larger payment, as indicated below. Calculate (a) the time needed to pay off the loan, (b) the total amount of the payments, and (c) the amount of interest saved, compared with part (c) of Exercises 13–16.

31. $16,000 in Exercise 13 (a) 8 years (b) $113,086.84 (c) $15,732.36

32. $18,000 in Exercise 14 (a) 9 quarters (b) $153,729.65 (c) $9703.83

33. $850 in Exercise 15 (a) 11 semiannual periods (b) $8760.50 (c) $1006.38

34. $400 in Exercise 16 (a) 15 months (b) $5866.13 (c) $224.99

APPLICATIONS

35. $1856.44; $114,168.20. The payments are $537.48 more than for the 30-yr loan, but the total interest paid is $140,675.40 less.

Business and Economics

35. House Payments Calculate the monthly payment and total amount of interest paid in Example 3 with a 15-year loan, and then compare with the results of Example 3.

36. Installment Buying TV Town sells a big screen smart HDTV for $600 down and monthly payments of $30 for the next 3 years. If the interest rate is 1.25% per month on the unpaid balance, find

(a) the cost of the TV. $1465.42

(b) the total amount of interest paid. $214.58

37. Car Payments David Kurzawa buys a car costing $14,000. He agrees to make payments at the end of each monthly period for 4 years. He pays 7% interest, compounded monthly.

(a) What is the amount of each payment? $335.25

(b) Find the total amount of interest David will pay. $2092

38. Credit Card Debt Tom Shaffer charged $8430 on his credit card to relocate for his first job. When he realized that the interest rate for the unpaid balance was 27% compounded monthly, he decided not to charge any more on that account. He wants to have this account paid off by the end of 3 years, so he arranges to have automatic payments sent at the end of each month.

(a) What monthly payment must he make to have the account paid off by the end of 3 years? $344.16

(b) How much total interest will he have paid? $3959.76

39. New Car Financing In 2014, some dealers offered two different incentives on the Ford Mustang. The first offer was for 0% financing for 60 months plus $1500 bonus cash. The second offer was for $3500 cash back. ***Source: Ford*.**

(a) If the buyer chose the first offer and, after receiving the $1500 bonus cash, still needed to finance $20,000 for 60 months, determine the monthly payment. Find the total amount the buyer paid for this option.
$333.33; $20,000 (or $19,999.80 using the rounded payments)

(b) Suppose the buyer chose the second offer and needed to finance only $18,000. At the time, it was possible to get a new car loan at 2.69% for 60 months, compounded monthly, from an Internet bank. Determine the monthly payment and find the total amount the buyer paid for this option. $320.96; $19,257.60

(c) Discuss which deal was best and why.

40. New Car Financing In 2014, some dealers offered two different financing incentives on the Honda Civic. The first option was 0.9% financing for loans from 24 to 36 months, while the second option was 1.9% financing for loans from 37 to 60 months. Suppose that a buyer needed to finance $15,000. ***Source: Honda***.

(a) Determine the payment if the buyer chose the 0.9% financing for 36 months. Find the total amount that the buyer paid for this option. $422.47; $15,208.92

(b) Determine the payment if the buyer chose the 1.9% financing for 60 months. Find the total amount that the buyer paid for this option. $262.26; $15,735.60

(c) Some buyers look for the lowest payment, while others look for the lowest total cost. Discuss which deal was best and why.

41. Lottery Winnings In most states, the winnings of million-dollar lottery jackpots are divided into equal payments given annually for 20 years. (In Colorado, the results are distributed over 25 years.) This means that the present value of the jackpot is worth less than the stated prize, with the actual value determined by the interest rate at which the money could be invested. ***Source: The New York Times Magazine***.

(a) Find the present value of a $1 million lottery jackpot distributed in equal annual payments over 20 years, using an interest rate of 5%. $623,110.52

(b) Find the present value of a $1 million lottery jackpot distributed in equal annual payments over 20 years, using an interest rate of 9%. $456,427.28

(c) Calculate the answer for part (a) using the 25-year distribution time in Colorado. $563,757.78

(d) Calculate the answer for part (b) using the 25-year distribution time in Colorado. $392,903.18

Student Loans Student borrowers now have more options to choose from when selecting repayment plans. The standard plan repays the loan in up to 10 years with equal monthly payments. The extended plan allows up to 25 years to repay the loan. Suppose that a student borrows $55,000 at 4.66% compounded monthly. *Source: U.S. Department of Education.*

42. Find the monthly payment and total interest paid under the standard plan over 10 years. $574.26; $13,911.20

43. Find the monthly payment and total interest paid under the extended plan over 25 years. $310.72; $38,216

Installment Buying In Exercises 44–46, prepare an amortization schedule showing the first four payments for each loan.

44. An insurance firm pays $4000 for a new printer for its computer. It amortizes the loan for the printer in 4 annual payments at 8% compounded annually. *

45. Large semitrailer trucks cost $110,000 each. Ace Trucking buys such a truck and agrees to pay for it by a loan that will be amortized with 9 semiannual payments at 8% compounded semiannually. *

46. One retailer charges $1048 for a laptop computer. A firm of tax accountants buys 8 of these laptops. They make a down payment of $1200 and agree to amortize the balance with monthly payments at 6% compounded monthly for 4 years. *

*indicates answer is in the Additional Instructor Answers at end of the book.

47. Investment In 1995, Oseola McCarty donated $150,000 to the University of Southern Mississippi to establish a scholarship fund. What is unusual about her is that the entire amount came from what she was able to save each month from her work as a washer woman, a job she began in 1916 at the age of 8, when she dropped out of school. ***Sources: The New York Times***.

(a) How much would Ms. McCarty have to put into her savings account at the end of every 3 months to accumulate $150,000 over 79 years? Assume she received an interest rate of 5.25% compounded quarterly. $32.49

(b) Answer part (a) using a 2% and a 7% interest rate. $195.52; $10.97

48. Loan Payments When Heather Bruce opened her law office, she bought $14,000 worth of law books and $7200 worth of office furniture. She paid $1200 down and agreed to amortize the balance with semiannual payments for 5 years, at 8% compounded semiannually.

(a) Find the amount of each payment. $2465.82

(b) Refer to the text and Figure 16. When her loan had been reduced below $5000, Heather received a large tax refund and decided to pay off the loan. How many payments were left at this time? 2

49. House Payments Jason Hoffa buys a house for $285,000. He pays $60,000 down and takes out a mortgage at 6.5% on the balance. Find his monthly payment and the total amount of interest he will pay if the length of the mortgage is

(a) 15 years; $1959.99; $127,798.20 **(b)** 20 years; $1677.54; $177,609.60
(c) 25 years. $1519.22; $230,766.00

(d) Refer to the text and Figure 16. When will half the 20-year loan in part (b) be paid off? After 157 payments

50. House Payments The Chavara family buys a house for $225,000. They pay $50,000 down and take out a 30-year mortgage on the balance. Find their monthly payment and the total amount of interest they will pay if the interest rate is

(a) 6%; $1049.21; $202,715.60 **(b)** 6.5%; $1106.12; $223,203.20

(c) 7%. $1164.28; $244,140.80

(d) Refer to the text and Figure 16. When will half the 7% loan in part (c) be paid off? After 261 payments

51. Refinancing a Mortgage Fifteen years ago, the Budai family bought a home and financed $150,000 with a 30-year mortgage at 8.2%.

(a) Find their monthly payment, the total amount of their payments, and the total amount of interest they will pay over the life of this loan. $1121.63; $403,786.80; $253,786.80

(b) The Budais made payments for 15 years. Estimate the unpaid balance using the formula $115,962.66; $201,893.40

$$y = R\left[\frac{1 - (1 + i)^{-(n-x)}}{i}\right],$$

and then calculate the total of their remaining payments.

(c) Suppose interest rates have dropped since the Budai family took out their original loan. One local bank now offers a 30-year mortgage at 6.5%. The bank fees for refinancing are $3400. If the Budais pay this fee up front and refinance the balance of their loan, find their monthly payment. Including the refinancing fee, what is the total amount of their payments? Discuss whether or not the family should refinance with this option. $732.96; $267,265.60

(d) A different bank offers the same 6.5% rate but on a 15-year mortgage. Their fee for financing is $4500. If the Budais pay this fee up front and refinance the balance of their loan, find their monthly payment. Including the refinancing fee, what is the total amount of their payments? Discuss whether or not the family should refinance with this option. $1010.16; $186,328.80

52. Inheritance Neldy Rubio has inherited $25,000 from her grandfather's estate. She deposits the money in an account offering 6% interest compounded annually. She wants to make equal annual withdrawals from the account so that the money (principal and interest) lasts exactly 8 years.

(a) Find the amount of each withdrawal. $4025.90

(b) Find the amount of each withdrawal if the money must last 12 years. $2981.93

53. Charitable Trust The trustees of a college have accepted a gift of $150,000. The donor has directed the trustees to deposit the money in an account paying 6% per year, compounded semiannually. The trustees may make equal withdrawals at the end of each 6-month period; the money must last 5 years.

(a) Find the amount of each withdrawal. $17,584.58

(b) Find the amount of each withdrawal if the money must last 6 years. $15,069.31

Amortization **Prepare an amortization schedule for each loan.**

54. A loan of $37,948 with interest at 6.5% compounded annually, to be paid with equal annual payments over 10 years. *

55. A loan of $4836 at 7.25% interest compounded semiannually, to be repaid in 5 years in equal semiannual payments. *

56. Perpetuity A *perpetuity* is an annuity in which the payments go on forever. We can derive a formula for the present value of a perpetuity by taking the formula for the present value of an annuity and looking at what happens when n gets larger and larger. Explain why the present value of a perpetuity is given by

$$P = \frac{R}{i}.$$

57. Perpetuity Using the result of Exercise 56, find the present value of perpetuities for each of the following.

(a) Payments of $1000 a year with 4% interest compounded annually $25,000

(b) Payments of $600 every 3 months with 6% interest compounded quarterly $40,000

YOUR TURN ANSWERS

1. $6389.86	**2.** $394.59
3. $1977.42, $135,935.60	**4.** $679.84

5 CHAPTER REVIEW

SUMMARY

In this chapter we introduced the mathematics of finance. We first extended simple interest calculations to compound interest, which is interest earned on interest previously earned. We then developed the mathematics associated with the following financial concepts.

- In an annuity, money continues to be deposited at regular intervals, and compound interest is earned on that money as well.
- In an ordinary annuity, payments are made at the end of each time period, and the compounding period is the same as the time between payments, which simplifies the calculations.
- An annuity due is slightly different, in that the payments are made at the beginning of each time period.
- A sinking fund is like an ordinary annuity; a fund is set up to receive periodic payments. The payments plus the compound interest will produce a desired sum by a certain date.
- The present value of an annuity is the amount that would have to be deposited today to produce the same amount as the annuity at the end of a specified time.
- An amortization schedule shows how a loan is paid back after a specified time. It shows the payments broken down into interest and principal.

We have presented a lot of new formulas in this chapter. By answering the following questions, you can decide which formula to use for a particular problem.

1. Is simple or compound interest involved?

   Simple interest is normally used for investments or loans of a year or less; compound interest is normally used in all other cases.

2. If simple interest is being used, what is being sought: interest amount, future value, present value, or interest rate?

3. If compound interest is being used, does it involve a lump sum (single payment) or an annuity (sequence of payments)?

   (a) For a lump sum, what is being sought: present value, future value, number of periods at interest, or effective rate?

   (b) For an annuity,

   i. Is it an ordinary annuity (payment at the end of each period) or an annuity due (payment at the beginning of each period)?

   ii. What is being sought: present value, future value, or payment amount?

Once you have answered these questions, choose the appropriate formula and work the problem. As a final step, consider whether the answer you get makes sense. For instance, present value should always be less than future value. The amount of interest or the payments in an annuity should be fairly small compared to the total future value.

List of Variables

r is the annual interest rate.
i is the interest rate per period.
t is the number of years.
n is the number of periods.
m is the number of periods per year.
P is the principal or present value.
A is the future value of a lump sum.
S is the future value of an annuity.
R is the periodic payment in an annuity.

$$\boldsymbol{i = \frac{r}{m} \qquad n = tm}$$

	Simple Interest	Compound Interest	Continuous Compounding
Interest	$I = Prt$	$I = A - P$	$I = A - P$
Future Value	$A = P(1 + rt)$	$A = P(1 + i)^n$	$A = Pe^{rt}$
Present Value	$P = \dfrac{A}{1 + rt}$	$P = \dfrac{A}{(1 + i)^n} = A(1 + i)^{-n}$	$P = Ae^{-rt}$
Effective Rate		$r_E = \left(1 + \dfrac{r}{m}\right)^m - 1$	$r_E = e^r - 1$

Ordinary Annuity

Future Value $\quad S = R\left[\dfrac{(1 + i)^n - 1}{i}\right] = R \cdot s_{\overline{n|}i}$

Present Value $\quad P = R\left[\dfrac{1 - (1 + i)^{-n}}{i}\right] = R \cdot a_{\overline{n|}i}$

Annuity Due

Future Value $\quad S = R\left[\dfrac{(1 + i)^{n+1} - 1}{i}\right] - R$

Sinking Fund Payment

$$R = \frac{Si}{(1 + i)^n - 1} = \frac{S}{s_{\overline{n|}i}}$$

Amortization Payments

$$R = \frac{Pi}{1 - (1 + i)^{-n}} = \frac{P}{a_{\overline{n|}i}}$$

KEY TERMS

5.1
simple interest
principal
rate
time
future value
maturity value
compound interest
compound amount
nominal (stated) rate
effective rate
present value
rule of 70
rule of 72
continuous compounding

5.2
geometric sequence
terms
common ratio
annuity
ordinary annuity
payment period
future value of an annuity
term of an annuity
future value of an ordinary annuity
sinking fund
annuity due
future value of an annuity due

5.3
present value of an annuity
amortize a loan
amortization schedule

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. For a particular interest rate, compound interest will accumulate faster than simple interest. True
2. The sequence 1, 2, 4, 6, 8, . . . is a geometric sequence. False
3. If a geometric sequence has first term 3 and common ratio 2, then the sum of the first 5 terms is $S_5 = 93$. True
4. The value of a sinking fund should decrease over time. False
5. For payments made on a mortgage, the (noninterest) portion of the payment applied on the principal increases over time. True
6. On a 30-year conventional home mortgage, at recent interest rates, it is common to pay more money on the interest on the loan than the actual loan itself. True
7. One can use the amortization payments formula to calculate the monthly payment of a car loan. True
8. The effective rate formula can be used to calculate the present value of a loan. False
9. The following calculation gives the monthly payment on a $25,000 loan, compounded monthly at a rate of 5% for a period of six years:

$$25{,}000\left[\frac{(1 + 0.05/12)^{72} - 1}{0.05/12}\right].$$ False

10. The following calculation gives the present value of an annuity of $5,000 payments at the end of each year for 10 years. The fund earns 4.5% compounded annually.

$$5000\left[\frac{1 - (1.045)^{-10}}{0.045}\right]$$ True

15. For a given amount of money at a given interest rate for a given time period, does simple interest or compound interest produce more interest? Compound interest
16. What is meant by the present value of an amount A?

PRACTICE AND EXPLORATION

Find the simple interest for each loan.

11. $15,903 at 6% for 8 months $636.12
12. $4902 at 5.4% for 11 months $242.65
13. $42,368 at 5.22% for 7 months $1290.11
14. $3478 at 6.8% for 88 days (assume a 360-day year) $57.81

Find the compound amount in each loan.

17. $19,456.11 at 8% compounded semiannually for 7 years $33,691.69
18. $2800 at 7% compounded annually for 10 years $5508.02
19. $57,809.34 at 6% compounded quarterly for 5 years $77,860.80
20. $312.45 at 5.6% compounded semiannually for 16 years $756.07

Find the amount of interest earned by each deposit.

21. $12,699.36 at 5% compounded semiannually for 7 years $5244.50
22. $3954 at 8% compounded annually for 10 years $4582.39
23. $34,677.23 at 4.8% compounded monthly for 32 months $4725.22
24. $12,903.45 at 6.4% compounded quarterly for 29 quarters $7543.26

Find the present value of each amount.

25. $42,000 in 7 years, 6% compounded monthly $27,624.86
26. $17,650 in 4 years, 4% compounded quarterly $15,052.30
27. $1347.89 in 3.5 years, 6.77% compounded semiannually $1067.71
28. $2388.90 in 44 months, 5.93% compounded monthly $1923.09
29. Write the first five terms of the geometric sequence with $a = 2$ and $r = 3$. 2, 6, 18, 54, 162
30. Write the first four terms of the geometric sequence with $a = 4$ and $r = 1/2$. 4, 2, 1, 1/2
31. Find the sixth term of the geometric sequence with $a = -3$ and $r = 2$. -96
32. Find the fifth term of the geometric sequence with $a = -2$ and $r = -2$. -32
33. Find the sum of the first four terms of the geometric sequence with $a = -3$ and $r = 3$. -120
34. Find the sum of the first five terms of the geometric sequence with $a = 8000$ and $r = -1/2$. 5500

35. Find $s_{\overline{30}|0.02}$. 40.56808

36. Find $s_{\overline{20}|0.06}$. 36.78559

37. What is meant by the future value of an annuity?

Find the future value of each annuity and the amount of interest earned.

38. \$500 deposited at the end of each 6-month period for 10 years; money earns 6% compounded semiannually. \$13,435.19; \$3435.19

39. \$1288 deposited at the end of each year for 14 years; money earns 4% compounded annually. \$23,559.98; \$5527.98

40. \$4000 deposited at the end of each quarter for 7 years; money earns 5% compounded quarterly. \$133,117.54; \$21,117.54

41. \$233 deposited at the end of each month for 4 years; money earns 4.8% compounded monthly. \$12,302.78; \$1118.78

42. \$672 deposited at the beginning of each quarter for 7 years; money earns 4.4% compounded quarterly. \$22,136.73; \$3320.73

43. \$11,900 deposited at the beginning of each month for 13 months; money earns 6% compounded monthly. \$160,224.29; \$5524.29

44. What is the purpose of a sinking fund?

Find the amount of each payment that must be made into a sinking fund to accumulate each amount.

45. \$6500; money earns 5% compounded annually for 6 years. \$955.62

46. \$57,000; money earns 4% compounded semiannually for $8\frac{1}{2}$ years. \$2848.29

47. \$233,188; money earns 5.2% compounded quarterly for $7\frac{3}{4}$ years. \$6156.15

48. \$1,056,788; money earns 7.2% compounded monthly for $4\frac{1}{2}$ years. \$16,628.83

Find the present value of each ordinary annuity.

49. Deposits of \$850 annually for 4 years at 6% compounded annually \$2945.34

50. Deposits of \$1500 quarterly for 7 years at 5% compounded quarterly \$35,235.78

51. Payments of \$4210 semiannually for 8 years at 4.2% compounded semiannually \$56,711.93

52. Payments of \$877.34 monthly for 17 months at 6.4% compounded monthly \$14,222.42

53. Give two examples of the types of loans that are commonly amortized. A home loan and an auto loan

Find the amount of the payment necessary to amortize each loan. Calculate the total interest paid.

54. \$80,000; 5% compounded annually; 9 annual payments \$11,255.21; \$21,296.89

55. \$3200; 8% compounded quarterly; 12 quarterly payments \$302.59; \$431.08

56. \$32,000; 6.4% compounded quarterly; 17 quarterly payments \$2164.87; \$4802.79

57. \$51,607; 8% compounded monthly; 32 monthly payments \$1796.20; \$5871.40

Find the monthly house payments for each mortgage. Calculate the total payments and interest.

58. \$256,890 at 5.96% for 25 years \$1648.87; \$494,661; \$237,771

59. \$177,110 at 6.68% for 30 years \$1140.50; \$410,580; \$233,470

A portion of an amortization table is given below for a \$127,000 loan at 8.5% interest compounded monthly for 25 years.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
1	\$1022.64	\$899.58	\$123.06	\$126,876.94
2	\$1022.64	\$898.71	\$123.93	\$126,753.01
3	\$1022.64	\$897.83	\$124.81	\$126,628.20
4	\$1022.64	\$896.95	\$125.69	\$126,502.51
5	\$1022.64	\$896.06	\$126.58	\$126,375.93
6	\$1022.64	\$895.16	\$127.48	\$126,248.45
7	\$1022.64	\$894.26	\$128.38	\$126,120.07
8	\$1022.64	\$893.35	\$129.29	\$125,990.78
9	\$1022.64	\$892.43	\$130.21	\$125,860.57
10	\$1022.64	\$891.51	\$131.13	\$125,729.44
11	\$1022.64	\$890.58	\$132.06	\$125,597.38
12	\$1022.64	\$889.65	\$132.99	\$125,464.39

Use the table to answer the following questions.

60. How much of the fifth payment is interest? \$896.06

61. How much of the twelfth payment is used to reduce the debt? \$132.99

62. How much interest is paid in the first 3 months of the loan? \$2696.12

63. How much has the debt been reduced at the end of the first year? \$1535.61

APPLICATIONS

Business and Economics

64. **Personal Finance** Carla Truss owes \$5800 to her mother. She has agreed to repay the money in 10 months at an interest rate of 5.3%. How much will she owe in 10 months? How much interest will she pay? \$6056.17; \$256.17

65. **Business Financing** Candice Cotton needs to borrow \$9820 to buy new equipment for her business. The bank charges her 6.7% simple interest for a 7-month loan. How much interest will she be charged? What amount must she pay in 7 months? \$10,203.80; \$383.80

66. **Business Financing** An accountant loans \$28,000 at simple interest to her business. The loan is at 6.5% and earns \$1365 interest. Find the time of the loan in months. 9 mo

67. **Business Investment** A developer deposits \$84,720 for 7 months and earns \$4055.46 in simple interest. Find the interest rate. 8.21%

68. **Personal Finance** In 3 years Christine Ellington must pay a pledge of \$7500 to her college's building fund. What lump sum can she deposit today, at 5% compounded semiannually, so that she will have enough to pay the pledge? \$6467.23

69. **Personal Finance** Ali Williams, a graduate student, is considering investing \$500 now, when he is 23, or waiting until he is 40 to invest \$500. How much more money will he have at the age of 65 if he invests now, given that he can earn 5% interest compounded quarterly? \$2298.58

70. Pensions Pension experts recommend that you start drawing at least 40% of your full pension as early as possible. Suppose you have built up a pension of $12,000-annual payments by working 10 years for a company. When you leave to accept a better job, the company gives you the option of collecting half of the full pension when you reach age 55 or the full pension at age 65. Assume an interest rate of 8% compounded annually. By age 75, how much will each plan produce? Which plan would produce the larger amount? ***Source: Smart Money***. *

71. Business Investment A firm of attorneys deposits $5000 of profit-sharing money at the end of each semiannual period for $7\frac{1}{2}$ years. Find the final amount in the account if the deposits earn 10% compounded semiannually. Find the amount of interest earned. $107,892.82; $32,892.82

72. Business Financing A small resort must add a swimming pool to compete with a new resort built nearby. The pool will cost $28,000. The resort borrows the money and agrees to repay it with equal payments at the end of each quarter for $6\frac{1}{2}$ years at an interest rate of 8% compounded quarterly. Find the amount of each payment. $1391.58

73. Business Financing The owner of Eastside Hallmark borrows $48,000 to expand the business. The money will be repaid in equal payments at the end of each year for 7 years. Interest is 6.5%. Find the amount of each payment and the total amount of interest paid. $8751.91; $13,263.37

74. Personal Finance To buy a new computer, David Berg borrows $3250 from a friend at 4.2% interest compounded annually for 4 years. Find the compound amount he must pay back at the end of the 4 years. $3831.37

75. Effective Rate On April 2, 2015, First Internet Bank of Indiana paid 2.10% interest, compounded monthly, on a 5-year CD, while Discover Bank paid 2.08% compounded daily. What are the effective rates for the two CDs, and which bank paid a higher effective rate? ***Source: Bankrate.com***.
2.12% and 2.10%; First Internet Bank of Indiana

76. Home Financing When the Lee family bought their home, they borrowed $315,700 at 7.5% compounded monthly for 25 years. If they make all 300 payments, repaying the loan on schedule, how much interest will they pay? (Assume the last payment is the same as the previous ones.) $384,200

77. Financing In 2014, an advertisement for the Erie Federal Credit Union offered two home equity loans. The first offer was a 2.99% annual interest rate for loans from 0 to 60 months. The second offer was a 3.49% annual interest rate for loans from 61 to 84 months. ***Source: Erie Federal Credit Union***.

(a) Determine the monthly payment if a customer borrowed $15,000 for 60 months. Find the total amount that the customer paid for this option. $269.46; $16,167.60

(b) Determine the monthly payment if a customer borrowed $15,000 for 84 months. Find the total amount that the customer paid for this option. $201.53; $16,928.52

(c) Discuss which deal was best and why.

78. New Car In Spring 2014, some dealers offered a cash-back allowance of $4000 or 0% financing for 72 months on a Chevrolet Silverado. ***Source: cars.com***.

(a) Determine the payments on a Chevrolet Silverado if a buyer chooses the 0% financing option and needs to finance $30,000 for 72 months. Find the total amount of the payments.
$416.67, $30,000 (or $30,000.24 using the rounded payments)

(b) Determine the payments on a Chevrolet Silverado if a buyer chooses the cash-back option and now needs to finance only $26,000. At the time, it was possible to get a new car loan at 3.18% for 48 months, compounded monthly. Find the total amount of the payments. $577.56; $27,722.88

(c) Discuss which deal is best and why.

(d) Find the interest rate at the bank that would make the total amount of payments for the two options equal. 7.20%

79. Buying and Selling a House The Bahary family bought a house for $191,000. They paid $40,000 down and took out a 30-year mortgage for the balance at 6.5%.

(a) Find their monthly payment. $954.42

(b) How much of the first payment is interest? $817.92

After 180 payments, the family sells its house for $238,000. They must pay closing costs of $3700 plus 2.5% of the sale price.

(c) Estimate the current mortgage balance at the time of the sale using one of the methods from Example 4 in Section 3.
Method 1: $109,563.99; Method 2: $109,565.13

(d) Find the total closing costs. $9650

(e) Find the amount of money they receive from the sale after paying off the mortgage.
Method 1: $118,786.01; Method 2: $118,784.87

The following exercise is from an actuarial examination. *Source: The Society of Actuaries.*

+ **80. Death Benefit** The proceeds of a $10,000 death benefit are left on deposit with an insurance company for 7 years at an annual effective interest rate of 5%. The balance at the end of 7 years is paid to the beneficiary in 120 equal monthly payments of X, with the first payment made immediately. During the payout period, interest is credited at an annual effective interest rate of 3%. Calculate X. Choose one of the following. (d)

(a) 117 **(b)** 118 **(c)** 129

(d) 135 **(e)** 158

81. Investment *The New York Times* posed a scenario with two individuals, Sue and Joe, who each have $1200 a month to spend on housing and investing. Each takes out a mortgage for $140,000. Sue gets a 30-year mortgage at a rate of 6.625%. Joe gets a 15-year mortgage at a rate of 6.25%. Whatever money is left after the mortgage payment is invested in a mutual fund with a return of 10% annually. ***Source: The New York Times***.

(a) What annual interest rate, when compounded monthly, gives an effective annual rate of 10%? 9.569%

(b) What is Sue's monthly payment? $896.44

(c) If Sue invests the remainder of her $1200 each month, after the payment in part (b), in a mutual fund with the interest rate in part (a), how much money will she have in the fund at the end of 30 years? $626,200.88

(d) What is Joe's monthly payment? $1200.39

* indicates answer is in the Additional Instructor Answers at end of the book.
+ indicates more challenging problem.

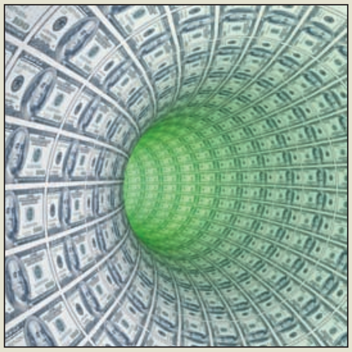
(e) You found in part (d) that Joe has nothing left to invest until his mortgage is paid off. If he then invests the entire \$1200 monthly in a mutual fund with the interest rate in part (a), how much money will he have at the end of 30 years (that is, after 15 years of paying the mortgage and 15 years of investing)? \$478,134.14

(f) Who is ahead at the end of the 30 years and by how much? Sue is ahead by \$148,066.74

(g) Discuss to what extent the difference found in part (f) is due to the different interest rates or to the different amounts of time.

EXTENDED APPLICATION

TIME, MONEY, AND POLYNOMIALS



A *time line* is often helpful for evaluating complex investments. For example, suppose you buy a \$1000 CD at time t_0. After one year \$2500 is added to the CD at t_1. By time t_2, after another year, your money has grown to \$3851 with interest. What rate of interest, called *yield to maturity* (YTM), did your money earn? A time line for this situation is shown in Figure 18.

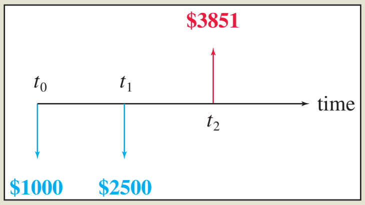


FIGURE 18

Assuming interest is compounded annually at a rate i, and using the compound interest formula, gives the following description of the YTM.

$$1000(1+i)^2 + 2500(1+i) = 3851$$

To determine the yield to maturity, we must solve this equation for i. Since the quantity $1+i$ is repeated, let $x = 1+i$ and first solve the second-degree (quadratic) polynomial equation for x.

$$1000x^2 + 2500x - 3851 = 0$$

We can use the quadratic formula with $a = 1000$, $b = 2500$, and $c = -3851$.

$$x = \frac{-2500 \pm \sqrt{2500^2 - 4(1000)(-3851)}}{2(1000)}$$

We get $x = 1.0767$ and $x = -3.5767$. Since $x = 1 + i$, the two values for i are $0.0767 = 7.67\%$ and $-4.5767 = -457.67\%$. We reject the negative value because the final accumulation is greater than the sum of the deposits. In some applications, however, negative rates may be meaningful. By checking in the first equation, we see that the yield to maturity for the CD is 7.67%.

Now let us consider a more complex but realistic problem. Suppose David Lopez has contributed for 4 years to a retirement fund. He contributed \$6000 at the beginning of the first year. At the beginning of the next 3 years, he contributed \$5840, \$4000, and \$5200, respectively. At the end of the fourth year, he had \$29,912.38 in his fund. The interest rate earned by the fund varied between 21% and −3%, so Lopez would like to know the YTM $= i$ for his hard-earned retirement dollars. From a time line (see Figure 19), we set up the following equation in $1 + i$ for Lopez's savings program.

$$6000(1+i)^4 + 5840(1+i)^3 + 4000(1+i)^2 + 5200(1+i) = 29{,}912.38$$

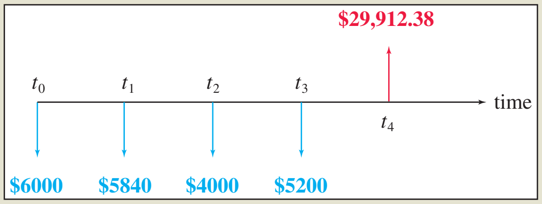


FIGURE 19

Let $x = 1 + i$. We need to solve the fourth-degree polynomial equation

$$f(x) = 6000x^4 + 5840x^3 + 4000x^2 + 5200x - 29{,}912.38 = 0.$$

There is no simple way to solve a fourth-degree polynomial equation, so we will use a graphing calculator.

We expect that $0 < i < 1$, so that $1 < x < 2$. Let us calculate $f(1)$ and $f(2)$. If there is a change of sign, we will know that there is a solution to $f(x) = 0$ between 1 and 2. We find that

$$f(1) = -8872.38 \quad \text{and} \quad f(2) = 139{,}207.62.$$

Using a graphing calculator, we find that there is one positive solution to this equation, $x = 1.14$, so $i = \text{YTM} = 0.14 = 14\%$.

***Source:* COMAP**.

EXERCISES

1. Lorri Morgan received \$50 on her 16th birthday, and \$70 on her 17th birthday, both of which she immediately invested in the bank, with interest compounded annually. On her 18th birthday, she had \$127.40 in her account. Draw a time line, set up a polynomial equation, and calculate the YTM.

2. At the beginning of the year, Yvette Virgil invested \$10,000 at 5% for the first year. At the beginning of the second year, she added \$12,000 to the account. The total account earned 4.5% for the second year.

   (a) Draw a time line for this investment.

   (b) How much was in the fund at the end of the second year?

   (c) Set up and solve a polynomial equation and determine the YTM. What do you notice about the YTM?

3. On January 2 each year for 3 years, Lauren O'Brien deposited bonuses of \$1025, \$2200, and \$1850, respectively, in an account. She received no bonus the following year, so she made no deposit. At the end of the fourth year, there was \$5864.17 in the account.

   (a) Draw a time line for these investments.

   (b) Write a polynomial equation in x $(x = 1 + i)$ and use a graphing calculator to find the YTM for these investments.

   (c) Go to the website WolframAlpha.com, and ask it to solve the polynomial from part (b). Compare this method of solving the equation with using a graphing calculator.

4. Don Beville invested yearly in a fund for his children's college education. At the beginning of the first year, he invested \$1000; at the beginning of the second year, \$2000; at the third through the sixth, \$2500 each year; and at the beginning of the seventh, he invested \$5000. At the beginning of the eighth year, there was \$21,259 in the fund.

   (a) Draw a time line for this investment program.

   (b) Write a seventh-degree polynomial equation in $1 + i$ that gives the YTM for this investment program.

   (c) Use a graphing calculator to show that the YTM is less than 5.07% and greater than 5.05%.

   (d) Use a graphing calculator to calculate the solution for $1 + i$ and find the YTM.

   (e) Go to the website WolframAlpha.com, and ask it to solve the polynomial from part (b). Compare this method of solving the equation with using a graphing calculator.

5. People often lose money on investments. Karen McFadyen invested \$50 at the beginning of each of 2 years in a mutual fund, and at the end of 2 years her investment was worth \$90.

   (a) Draw a time line and set up a polynomial equation in $1 + i$. Solve for i.

   (b) Examine each negative solution (rate of return on the investment) to see if it has a reasonable interpretation in the context of the problem. To do this, use the compound interest formula on each value of i to trace each \$50 payment to maturity.

DIRECTIONS FOR GROUP PROJECT

Assume that you are in charge of a group of financial analysts and that you have been asked by the broker at your firm to develop a time line for each of the people listed in the exercises above. Prepare a report for each client that presents the YTM for each investment strategy. Make sure that you describe the methods used to determine the YTM in a manner that the average client should understand.