Time-Varying Dispersion Integer-Valued GARCH Models

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Abstract

We propose a general class of INteger-valued Generalized AutoRegressive Conditionally Heteroskedastic (INGARCH) processes by allowing time-varying mean and dispersion parameters, which we call time-varying dispersion INGARCH (tv-DINGARCH) models. More specifically, we consider mixed Poisson INGARCH models and allow for a dynamic modeling of the dispersion parameter (as well as the mean), similarly to the spirit of the ordinary GARCH models. We derive conditions to obtain first and second order stationarity, and ergodicity as well. Estimation of the parameters is addressed and their associated asymptotic properties established as well. A restricted bootstrap procedure is proposed for testing constant dispersion against time-varying dispersion. Monte Carlo simulation studies are presented for checking point estimation, standard errors, and the performance of the restricted bootstrap approach. The inclusion of covariates is also addressed and applied to the daily number of deaths due to COVID-19 in Ireland. Insightful results were obtained in the data analysis, including a superior performance of the tv-DINGARCH processes over the ordinary INGARCH models.

Keywords: Autocorrelation; Count time series; Overdispersion; Time-varying dispersion parameter; Volatility.

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1 Introduction

Modelling count time series data is challenging and very exciting research topic with applications in many different areas such as epidemiology, sociology, economics, and health science. A wellestablished methodology for dealing with count time series data is the INteger-valued Generalized AutoRegressive Conditional Heterokedastic (INGARCH) models, which have been initially studied and explored by Heinen (2003), Ferland et al. (2006), Fokianos et al. (2009), and Fokianos (2011). The INGARCH nomenclature emerges from the fact that for a Poisson distribution (assumption considered in the aforementioned papers), the mean equals its variance, therefore modelling of mean implies modelling of variance like the classic continuous GARCH models introduced by Bollerslev (1986). Consequently, they are considered in the literature as an integer counterpart of the GARCH models although this terminology should be used cautionsly because ordinary GARCH processes do not consider a dynamics for the conditional mean; for instance, see Fokianos et al. (2009).

A Poisson INGARCH(p,q) (with $p,q \in \mathbb{N}_0$) model $\{Y_t\}_{t \in \mathbb{N}}$ is defined by

$$Y_t | \mathcal{F}_{t-1} \sim \text{Poisson}(\lambda_t), \quad \lambda_t \equiv E(Y_t | \mathcal{F}_{t-1}) = \beta_0 + \sum_{i=1}^p \beta_i Y_{t-i} + \sum_{j=1}^q \alpha_j \lambda_{t-j}, \tag{1}$$

where $\mathcal{F}_{t-1} = \sigma\{Y_{t-1}, \ldots, Y_0, \lambda_0\}, \beta_0 > 0, \beta_i \ge 0$ and $\alpha_j \ge 0$, for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Since $E(Y_t | \mathcal{F}_{t-1}) = \operatorname{Var}(Y_t | \mathcal{F}_{t-1}) = \lambda_t$, the model specifies dynamic processes for both the conditional mean and variance, as previously stated.

The Poisson INGARCH model has been extensively considered in the literature. However, due to limitations for fitting adequately real count time series data (e.g. not capturing all the source of overdispersion although the model is overdispersed, inflation or deflation of zeros), other variants have been proposed such as the negative binomial INGARCH process by Zhu (2011) and Christou and Fokianos (2014) or the more general mixed Poisson INGARCH models by Christou and Fokianos (2015), Silva and Barreto-Souza (2019), among others. Zero-inflated versions of the Poisson and negative binomial INGARCH models were proposed by Zhu (2012a), while processes dealing with both overdispersion and underdispersion were proposed by Zhu (2012b,c) and Xu et al. (2012).

Other INGARCH processes have been proposed. An alternative process to the linear model for the conditional mean, as in Equation (1), was introduced by Fokianos and Tjøstheim (2011) through the log-linear INGARCH processes. These models cope both negative and positive autocorrelations and facilitate the inclusion of covariates in an easier way than in the linear INGARCH model (1). A different approach was recently proposed by Weiß et al. (2020), where a softplus link function is assumed instead of a logarithmic one.

The main goal in this paper is to propose a novel general class of INGARCH processes based on the mixed Poisson distributions and by *allowing time-varying mean and dispersion parameters*, which we call time-varying dispersion INGARCH (tv-DINGARCH) models. This flexibility leads to the development of models for integer-valued time series, where the assumption of constant dispersion might not be satisfied. The advantage of the tv-DINGARCH approach over the ordinary INGARCH models will be illustrated in this work.

For example, in a count regression context, Barreto-Souza and Simas (2016) demonstrated through a data application on attendance behavior of high school juniors that such an assumption can be violated. Generalized linear models allowing regression structures for both mean and dispersion/precision parameters were discussed by Efron (1986) and Smyth (1989), among others. In this work, we develop this concept further with emphasis to the INGARCH methodology and demonstrate its practical importance and impact. Besides exploring the traditional INGARCH models,

the proposed class contains the case that admits constant mean and time-dependent variance, similarly to the spirit of the ordinary GARCH models. Another important feature is the volatility modelling, which is a well-explored topic for the continuous time series but less explored in the discrete case. Although the traditional INGARCH processes consider a time-dependent conditional variance, this in turn it is driven by the dynamics of the mean process. The main disadvantage of this approach is that imposes severe restrictions as illustrated in the real data application in this paper. The tv-DINGARCH models relax such an assumption by considering a time-dependent dispersion parameter, therefore also controlling the conditional variance and providing a different source of volatility.

The paper is organized as follows. In Section 2, we define the class of tv-DINGARCH models and then derive its stochastic properties. We establish conditions ensuring stationarity and ergodicity of the count processes. Section 3 is devoted to estimation of the parameters and the derivation of their associated asymptotic properties as well. Aiming at testing constant dispersion in practice, a restricted bootstrap procedure is proposed in Section 4. Monte Carlo simulation studies are presented for checking point estimation, standard errors, and the performance of the restricted bootstrap approach. This work is motivated in part by the challenges to analyzing the COVID-19 daily deaths in Ireland. Then, a log-linear tv-DINGARCH process allowing for covariates is explored in Section 5 and considered for this real data application. The proposed tv-DINGARCH approach produced interesting and insightful results that were missed by the ordinary INGARCH models in terms of fitting and out-of-sample prediction. Some technical results and proofs are provided in the Appendix. This paper contains Supplementary Material, which can be obtained from the authors upon request.

2 The tv-DINGARCH Models

In this section we propose a class of linear time-varying dispersion INGARCH models. The corresponding log-linear version will be discussed and illustrated in Section 5. We begin by introducing some useful notations.

We say that a random variable Y follows a mixed Poisson (MP) distribution if satisfies the stochastic representation $Y|Z = z \sim \text{Poisson}(\lambda z)$, with Z following some non-negative distribution with E(Z) = 1 (standardization) and $\text{Var}(Z) = \phi$, for $\lambda, \phi > 0$. We denote $Y \sim \text{MP}(\lambda, \phi)$. In this case, $E(Y) = \lambda$ and $\text{Var}(Y) = \lambda + \phi \lambda^2$. A random variable Z following a Gamma distribution, with respective shape and scale parameters a > 0 and b > 0, is denoted by $Z \sim \text{Gamma}(a, b)$, where E(Z) = a/b and $\text{Var}(Z) = a/b^2$. If the mixing variable $Z \sim \text{Gamma}(\phi^{-1}, \phi^{-1})$, then we obtain that Y follows a negative binomial (NB) distribution, i.e. $Y \sim \text{NB}(\lambda, \phi)$. When Z follows an inverse-Gaussian distribution with mean 1 and variance ϕ , we obtain that Y follows a Poisson inverse-Gaussian (PIG) distribution, which we denoted by $Y \sim \text{PIG}(\lambda, \phi)$.

We now introduce some notation to be used in Subsection 2.1. For a *l*-dimensional vector $\mathbf{x} = (x_1, \ldots, x_l)^{\top}$, let $\|\mathbf{x}\|_p = (\sum_{i=1}^l |x_i|^p)^{1/p}$, for $p \in [1, \infty)$, and $\|\mathbf{x}\|_p = \max_{1 \le i \le l} |x_i|$ for $p = \infty$. The induced *p*-norm for a $m \times l$ matrix \mathbf{C} is then defined by $\|\mathbf{C}\|_p = \max_{\mathbf{x} \neq \mathbf{0}} \{\|\mathbf{C}\mathbf{x}\|_p / \|\mathbf{x}\|_p : \mathbf{x} \in \mathbb{R}^l\}$, for $p \in [1, \infty]$.

Definition 2.1. (tv-DINGARCH processes) A tv-DINGARCH(p_1, p_2, q_1, q_2) process $\{Y_t\}_{t \in \mathbb{N}}$ is defined by $Y_t | \mathcal{F}_{t-1} \sim MP(\lambda_t, \phi_t)$, with

$$\lambda_t = f(Y_{t-1}, \dots, Y_{t-p_1}; \lambda_{t-1}, \dots, \lambda_{t-q_1}), \quad \phi_t = g(Y_{t-1}, \dots, Y_{t-p_2}; \phi_{t-1}, \dots, \phi_{t-q_2}), \tag{2}$$

where $\mathcal{F}_{t-1} = \sigma\{Y_{t-1}, \ldots, Y_0, \phi_0, \lambda_0\}$, and (λ_0, ϕ_0) denoting some starting value, $f : \mathbb{N}^{p_1} \times (0, \infty)^{q_1} \to (0, \infty)$, and $g : \mathbb{N}^{p_2} \times (0, \infty)^{q_2} \to (0, \infty)$.

Along with this paper, we will consider that the conditional distribution in Definition 2.1 will be negative binomial or inverse-Gaussian, with emphasis on the former. The conditional probability function of Y_t given \mathcal{F}_{t-1} under the negative binomial assumption is

$$P(Y_t = y | \mathcal{F}_{t-1}) = \frac{\Gamma(y + \phi_t^{-1})}{y! \Gamma(\phi_t^{-1})} \left(\frac{\lambda_t \phi_t}{\lambda_t \phi_t + 1}\right)^y \left(\frac{1}{\lambda_t \phi_t + 1}\right)^{1/\phi_t}, \quad y \in \mathbb{N}_0.$$

The model in Definition 2.1 is too general and some additional assumptions on the function fand g are necessary to obtain desirable properties such stationarity and ergodicity. A natural model to be first explored is the first-order ($p_1 = p_2 = q_1 = q_2 = 1$) linear tv-DINGARCH processes, which will be defined in what follows. Due to the linear form assumed for the functions f and g, we will see that explicit conditions ensuring stationarity and ergodicity can be obtained. Another model to be explored in Section 5 is the first-order log-linear tv-DINGARCH process, which allows for the inclusion of covariates. We would like to emphasize that these particular parametric forms (linear and log-linear) are common choices considered in the literature.

Definition 2.2. (Linear tv-DINGARCH processes) A linear tv-DINGARCH(1, 1, 1, 1) process $\{Y_t\}_{t\geq 1}$ is given as in Definition 2.1 with $f(\cdot)$ and $g(\cdot)$ being linear parametric functions of the forms

$$\lambda_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 \lambda_{t-1}, \quad \phi_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 \phi_{t-1}, \tag{3}$$

where $\beta_0, \alpha_0 > 0$ and $\beta_1, \beta_2, \alpha_1, \alpha_2 \ge 0$.

Remark 2.1. Some particular cases are obtained from the linear tv-DINGARCH class defined above. For $\alpha_1 = \alpha_2 = 0$, we obtain the ordinary linear mixed Poisson INGARCH (Christou and Fokianos, 2015; Silva and Barreto-Souza, 2019) models as particular cases. Additionally, by taking $\alpha_0 \rightarrow 0^+$, we also obtain the Poisson INGARCH model as a limiting member of our proposed class. Another interesting and novel model arises when $\beta_1 = \beta_2 = 0$. Under this setting, the mean of the INGARCH process is constant and the variance depends on the time like in the ordinary GARCH models (Bollerslev, 1986). This feature is not possible to be accomodated by the standard INGARCH model (1). In this case, we refer our model as Pure INGARCH (P-INGARCH) process. We use term "pure" to connect the fact that our model mimicks the traditional continuous GARCH models (constant mean and time-varying variance).

Simulated trajectories of the linear tv-DINGARCH models for some parameter settings are presented in the Supplementary Material.

2.1 Stationarity and Ergodicity

From now on we assume that the conditional distribution in Definition 2.1 is $NB(\lambda_t, \phi_t)$ or $PIG(\lambda_t, \phi_t)$, with λ_t and ϕ_t as in (3), for $t \ge 1$, unless otherwise mentioned. We now explore the stochastic properties of the tv-DINGARCH(1, 1, 1, 1) models. This is assumption is simple but powerful enough for many practical situations in terms of fitting. Further, the linearity is a common assumption in the literature as discussed before, which is justified due to successful empirical applications.

Let $\{N_t(0,s]; s \ge 0\}_{t\ge 1}$ be a sequence of independent and identically distributed (i.i.d.) Poisson processes with rate 1 independent of $\{Z_t(0,s]; s \ge 0\}_{t\ge 1}$, which is assumed to be a i.i.d.

sequence of standard gamma Lévy processes with $Z_t(0,s] \sim \text{Gamma}(s,1)$. Then, $\{Y_t\}_{t\geq 1} \sim \text{NB tv-DINGARCH}(1,1,1,1)$ admits the following stochastic representation:

$$Y_t \stackrel{d}{=} N_t \bigg(0, f(Y_{t-1}, \lambda_{t-1}) Z_t^* \Big(0, g(Y_{t-1}, \phi_{t-1}) \Big] \bigg], \tag{4}$$

where we have defined $Z_t^* = 0$ for s = 0, $\{Z_t^*(0, s]; s > 0\} \equiv \{s Z_t(0, s^{-1}]; s > 0\}$, $f(Y_{t-1}, \lambda_{t-1}) = \beta_0 + \beta_1 Y_{t-1} + \beta_2 \lambda_{t-1}$, and $g(Y_{t-1}, \phi_{t-1}) = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 \phi_{t-1}$, for $t \ge 1$. The model in (4) is a time-changed Poisson process driven by a gamma process, so yielding a negative binomial process with time-varying dispersion parameter. To the best of our knowledge, this is the first time that such a model has been proposed and studied. A similar representation can be derived for the PIG model by considering a time-changed Poisson process driven by an inverse-Gaussian Lévy process, but here we focus on the NB case. With this representation at hand, conditions for the existence and stationarity of the process (4) can be established, for example, by following the strategy by Christou and Fokianos (2014), which relies on establishing weak dependence by Doukhan and Wintenberger (2008). We here obtain such desirable properties based on another approach, the e-chains theory (Meyn and Tweedie, 1993), as follows.

The dynamic latent processes $\{\lambda_t\}_{t\geq 1}$ and $\{\phi_t\}_{t\geq 1}$ given in (3) can be rewritten in a matrix form. By defining $\Delta_t = (\lambda_t, \phi_t)^{\top}$, for $t \geq 1$, $\boldsymbol{\tau} = (\beta_0, \alpha_0)^{\top} \mathbf{A} = \begin{pmatrix} \beta_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}$, we have that $\Delta_t = \boldsymbol{\tau} + \mathbf{B}(Y_{t-1}, Y_{t-1})^{\top} + \mathbf{A}\Delta_{t-1}$. Under this matrix representation, we can adapt the strategy in Liu (2012) to establish the existence and stationarity of our process. In that work, the author provided stochastic properties of a bivariate Poisson INGARCH model. The key point is to show that $\{\Delta_t\}_{t\in\mathbb{N}}$ is an e-chain (Meyn and Tweedie, 1993). We are now ready to enunciate and prove one of the main results of this section, which will be important in our numerical experiments in what follows.

Theorem 2.2. Let $\{Y_t\}_{t\in\mathbb{N}}$ be a tv-DINGARCH process as in Definition 2.1. If there exists $p \in [1, \infty]$ such that $\|\mathbf{A}\|_p + 2^{1-1/p} \|\mathbf{B}\|_p < 1$, then the trivariate process $\{\Omega_t\}_{t\in\mathbb{N}} = \{(Y_t, \lambda_t, \phi_t)\}_{t\in\mathbb{N}}$ has a unique stationary and ergodic solution.

Proof. As mentioned above, the key ingredient to establish the desired result is to prove that $\{\Delta_t\}_{t\in\mathbb{N}}$ is an e-chain, that is, for any continuous function w with compact support on $(0,\infty)^2$ and for every $\epsilon > 0$, there exists $\eta > 0$ such that, for $\mathbf{x}, \mathbf{z} \in (0,\infty)^2$, $\|\mathbf{x} - \mathbf{z}\| < \eta$ implies that $|E(w(\Delta_k)|\Delta_0 = \mathbf{x}) - E(w(\Delta_k)|\Delta_0 = \mathbf{z})| < \epsilon \ \forall k \ge 1$, where $\|\cdot\|$ is some norm.

Let g be a continuous function with compact support on $(0, \infty)^2$ and assumed to be bounded |g| < 1 without loss of generality. Consider k = 1, $\mathbf{x}, \mathbf{z} \in (0, \infty)^2$, and $\epsilon > 0$ arbitrary. Denote by $f_{\mathbf{\Delta}}^{mp}(\cdot)$ the probability function of a mixed Poisson distribution with mean λ and variance $\lambda + \phi \lambda^2$, where $\mathbf{\Delta} = (\lambda, \phi)^{\top}$. It follows that

$$|E(w(\mathbf{\Delta}_{1})|\mathbf{\Delta}_{0} = \mathbf{x}) - E(w(\mathbf{\Delta}_{1})|\mathbf{\Delta}_{0} = \mathbf{z})| \leq \sum_{y=0}^{\infty} |w(\boldsymbol{\tau} + \mathbf{B}(y, y)^{\top} + \mathbf{A}\mathbf{x})f_{\mathbf{x}}^{mp}(y) - w(\boldsymbol{\tau} + \mathbf{B}(y, y)^{\top} + \mathbf{A}\mathbf{z})f_{\mathbf{z}}^{mp}(y)| \leq \sum_{y=0}^{\infty} f_{\mathbf{x}}^{mp}(y)|w(\boldsymbol{\tau} + \mathbf{B}(y, y)^{\top} + \mathbf{A}\mathbf{x}) - w(\boldsymbol{\tau} + \mathbf{B}(y, y)^{\top} + \mathbf{A}\mathbf{z})| + \sum_{y=0}^{\infty} |w(\boldsymbol{\tau} + \mathbf{B}(y, y)^{\top} + \mathbf{A}\mathbf{z})| |f_{\mathbf{x}}^{mp}(y) - f_{\mathbf{z}}^{mp}(y)|.$$

$$(5)$$

We now find an upper bound for the term (6). Denote by $f_{\lambda}^{pois}(\cdot)$ the probability function of a Poisson distribution with mean λ . From the mixed Poisson stochastic representation, we have that $f_{\mathbf{x}}^{mp}(y) = E(f_{x_1Z_1}^{pois}(y))$, where Z_1 is the associated latent random variables with distribution depending on x_2 . Similar representation holds for $f_{\mathbf{z}}^{mp}(y)$ in terms of an associated latent factor Z_2 with distribution depending on z_2 . Using this and the recalling the fact that |g| < 1, we obtain that

$$\sum_{y=0}^{\infty} |w(\boldsymbol{\tau} + \mathbf{B}(y, y)^{\top} + \mathbf{A}\mathbf{z})| |f_{\mathbf{x}}^{mp}(y) - f_{\mathbf{z}}^{mp}(y)| \le \sum_{y=0}^{\infty} |f_{\mathbf{x}}^{mp}(y) - f_{\mathbf{z}}^{mp}(y)| = \sum_{y=0}^{\infty} |E(f_{x_{1}Z_{1}}^{pois}(y)) - E(f_{z_{1}Z_{2}}^{pois}(y))| \le E\left(\sum_{y=0}^{\infty} |f_{x_{1}Z_{1}}^{pois}(y) - f_{x_{2}Z_{2}}^{pois}(y)|\right).$$
(7)

By using inequality (A.1) from Wang et al. (2014) (see also Liu (2012)), we have that $\sum_{y=0}^{\infty} |f_{x_1Z_1}^{pois}(y) - f_{x_2Z_2}^{pois}(y)| \le 2(1 - \exp\{-|x_1Z_1 - x_2Z_2|\})$. Hence, (7) is bounded above by

$$E\left(\sum_{y=0}^{\infty} |f_{x_1Z_1}^{pois}(y) - f_{x_2Z_2}^{pois}(y)|\right) \le 2E\left(1 - e^{-|x_1Z_1 - x_2Z_2|}\right) \le 2E\left(1 - e^{-|x_1 - x_2|(Z_1 + Z_2)}\right) \le 2E\left(|x_1 - x_2|(Z_1 + Z_2)) = 2|x_1 - x_2| \le 2||\mathbf{x} - \mathbf{z}||_p,$$

for $p \in [1, \infty]$, where we have used in the second inequality the fact that $|ab - cd| \leq |a - c|(b + d)$ for a, b, c, d > 0. In the third inequality, we used that $1 - e^{-x} \leq x$ for all x > 0. Finally, the forth inequality follows, for instance, by Liu (2012) (page 108).

The upper bound for the term given in (5) follows exactly as discussed in Liu (2012): choose $\epsilon' > 0$ and $\eta > 0$ small enough such as $\epsilon' + \frac{8\eta}{1 - \|\mathbf{A}\|_p} < \epsilon$ and $\|\mathbf{x} - \mathbf{z}\|_p < \eta$ implying $|w(\mathbf{x}) - w(\mathbf{z})| < \epsilon'$, with $p \in [1, \infty]$. In this part, we are using the fact that $\|\mathbf{A}\|_p < 1$ for some $p \in [1, \infty]$, which follows from the assumption that there exists p such that $\|\mathbf{A}\|_p + 2^{1-1/p} \|\mathbf{B}\|_p < 1$.

By combining the above results, we obtain that $|E(w(\Delta_1)|\Delta_0 = \mathbf{x}) - E(w(\Delta_1)|\Delta_0 = \mathbf{z})| < \epsilon' + 2||\mathbf{x} - \mathbf{z}||_p$. For general $k \geq 2$, the result follows by using mathematical induction, exactly as done in Chapter 4 from Liu (2012), and therefore it is omitted. With the e-chain property established for the bivariate process $\{\Delta_t\}$ and under the assumption of existence of $p \in [1, \infty]$ such that $||\mathbf{A}||_p + 2^{1-1/p} ||\mathbf{B}||_p < 1$, following the same steps that proofs of Proposition 4.2.1(b) and Proposition 4.3.1 from Liu (2012), we obtain the desired result.

Remark 2.3. The importance of Theorem 2.2 for data analysis is that these conditions are necessary to have consistency and asymptotic normality of the conditional maximum likelihood estimators, as will be addressed in next section.

Remark 2.4. The results given in Theorem 2.2 can be extended to the higher-order linear tv-DINGARCH(p_1, p_2, q_1, q_2) process with $\Delta_t = (\lambda_t, \phi_t)^\top = \tau + \sum_{j=1}^q \mathbf{B}_j(Y_{t-j}, Y_{t-j})^\top + \sum_{i=1}^m \mathbf{A}_i \Delta_{t-i}$, where $\{\mathbf{A}_i\}_{i=1}^m$ and $\{\mathbf{B}_j\}_{j=1}^q$ are diagonal matrices of parameters, $m = \max(p_1, p_2)$, and $q = \max(q_1, q_2)$. Then, following the same steps than in proof of Theorem 2.2, it can be established that $\{\Omega_t\}_{t\in\mathbb{N}} = \{(Y_t, \lambda_t, \phi_t)\}_{t\in\mathbb{N}}$ has a unique stationary and ergodic solution if there exists $p \in [1, \infty]$ such that $\sum_{i=1}^m \|\mathbf{A}_i\|_p + 2^{1-1/p} \sum_{j=1}^q \|\mathbf{B}_j\|_p < 1$.

3 Estimation and Asymptotic Theory

In this section we propose conditional maximum likelihood estimators, establish their asymptotic properties, and provide some numerical experiments for the linear tv-DINGARCH(1, 1, 1, 1)process in Definition 2.2 under the assumption of negative binomial conditional distributions, which is the focus of the present paper.

3.1 Estimators and Asymptotics

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_6)^\top = (\beta_0, \beta_1, \beta_2, \alpha_0, \alpha_1, \alpha_2)^\top$ be the parameter vector, and y_1, \dots, y_n be a realization of a NB tv-DINGARCH process $\{Y_t\}_{t=1}^n$. The conditional log-likelihood function of Y_2, \dots, Y_n given $Y_1 = y_1$ is given by $\ell(\boldsymbol{\theta}) = \sum_{t=2}^n \ell_t(\boldsymbol{\theta})$, where

$$\ell_t(\theta) = y_t(\log \lambda_t + \log \phi_t) - (y_t + \phi_t^{-1}) \log(\lambda_t \phi_t + 1) + \log \Gamma(y_t + \phi_t^{-1}) - \log \Gamma(\phi_t^{-1}) - \log y_t!,$$

for t = 2, ..., n, where we have omitted the dependence of λ_t and ϕ_t on $\boldsymbol{\theta}$ for simplicity of notation. In practice, it is necessary to set some initial value for λ_1 and ϕ_1 , which are fixed in this paper. More specifically, we get such initial values based on the two first empirical moments of the count time series. This strategy has worked well as demonstrated in the simulated results to be discussed in the sequence. The conditional maximum likelihood estimator (CMLE) of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}$, is given by $\hat{\boldsymbol{\theta}} = \operatorname{argmax}_{\boldsymbol{\theta} \in \Theta} \ell(\boldsymbol{\theta})$, where $\Theta = (0, \infty) \times [0, \infty)^2 \times (0, \infty) \times [0, \infty)^2$ denotes the parameter space.

The score function associated to the the conditional log-likelihood function is $\mathbf{U}(\boldsymbol{\theta}) = \partial \ell(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} = \sum_{t=2}^{n} \mathbf{U}_{t}(\boldsymbol{\theta})$, with

$$\mathbf{U}_{t}(\boldsymbol{\theta}) = \left(S_{1t}\frac{\partial\lambda_{t}}{\partial\beta_{0}}, S_{1t}\frac{\partial\lambda_{t}}{\partial\beta_{1}}, S_{1t}\frac{\partial\lambda_{t}}{\partial\beta_{2}}, S_{2t}\frac{\partial\phi_{t}}{\partial\alpha_{0}}, S_{2t}\frac{\partial\phi_{t}}{\partial\alpha_{1}}, S_{2t}\frac{\partial\phi_{t}}{\partial\alpha_{2}}\right)^{\top},$$

and

$$S_{1t} = \frac{y_t - \lambda_t}{\lambda_t (\lambda_t \phi_t + 1)}, \quad S_{2t} = \phi_t^{-1} \frac{y_t - \lambda_t}{\lambda_t \phi_t + 1} + \phi_t^{-2} \{ \log(\lambda_t \phi_t + 1) - \Psi(y_t + \phi_t^{-1}) + \Psi(\phi_t^{-1}) \},$$

for t = 2, ..., n, where $\Psi(x) = d \log \Gamma(x)/dx$, for x > 0, is the digamma function. Explicit expressions for the derivatives involving λ_t and ϕ_t are presented in the Supplementary Material. To establish the asymptotic normality of the CMLEs, the following proposition is necessary.

Proposition 3.1. Let $\mathbf{U}_t(\boldsymbol{\theta})$ be the t-th term of the score function, for $t \geq 2$. Then, $E(\mathbf{U}_t(\boldsymbol{\theta})|\mathcal{F}_{t-1}) = 0$ for all $t \geq 2$, where the conditional expectation is taken regarding the model with true parameter vector $\boldsymbol{\theta}$.

Proof. Since $E(Y_t|\mathcal{F}_{t-1}) = \lambda_t$ by definition, we immediately obtain that $E(S_{1t}(\boldsymbol{\theta})|\mathcal{F}_{t-1}) = 0$. We now compute the conditional expectation of S_{2t} given \mathcal{F}_{t-1} , which involves $E(\Psi(Y_t + \phi_t^{-1})|\mathcal{F}_{t-1})$. For a > 0 and |c| < 1, we have that

$$\sum_{y=0}^{\infty} \frac{\Gamma'(y+a)}{y!} c^y = \frac{d}{da} \sum_{y=0}^{\infty} \frac{\Gamma(y+a)}{y!} c^y = \frac{d}{da} \frac{\Gamma(a)}{(1-c)^a} = \frac{\Gamma(a)}{(1-c)^a} \left\{ \Psi(a) - \log(1-c) \right\}$$

where $\Gamma'(a) = d\Gamma(a)/da$. Using the above result, it follows that

$$E\left(\Psi(Y_t + \phi_t^{-1}) | \mathcal{F}_{t-1}\right) = \frac{(\lambda_t \phi_t + 1)^{-1/\phi_t}}{\Gamma(\phi_t^{-1})} \sum_{y=0}^{\infty} \frac{\Gamma'(y + \phi_t^{-1})}{y!} \left(\frac{\lambda_t \phi_t}{\lambda_t \phi_t + 1}\right)^y = \Psi(\phi_t^{-1}) + \log(\lambda_t \phi_t + 1),$$

and then we obtain that $E(S_{2t}(\boldsymbol{\theta})|\mathcal{F}_{t-1}) = 0$. These conditional expectations give us that, for all $t \geq 1, E(\mathbf{U}_t(\boldsymbol{\theta})|\mathcal{F}_{t-1}) = 0.$

Let us now discuss the asymptotic distribution of the CMLE for the NB tv-DINGARCH model. A more detailed technical result will be established in Theorem 3.2 for one particular case when $\beta_1 = \beta_2 = 0$. By using Proposition 3.1 and assuming that $\mathbf{U}(\boldsymbol{\theta})$ is square integrable, we can apply the Central Limit Theorem for martingale difference (for instance, see Corollary 3.1 from Hall and Heyde (1980)) to obtain that

$$n^{-1/2}\mathbf{U}(\boldsymbol{\theta}) \stackrel{d}{\longrightarrow} N(\mathbf{0}, \boldsymbol{\Omega}_1(\boldsymbol{\theta})), \quad \boldsymbol{\Omega}_1(\boldsymbol{\theta}) \equiv \text{plim}_{n \to \infty} \mathbf{J}_1(\boldsymbol{\theta}),$$
 (8)

as $n \to \infty$, with

. –

$$\mathbf{J}_{1}(\boldsymbol{\theta}) \equiv \frac{1}{n} \sum_{t=2}^{n} \operatorname{Var}(\mathbf{U}_{t}(\boldsymbol{\theta}) | \mathcal{F}_{t-1}) = \frac{1}{n} \sum_{t=2}^{n} \begin{pmatrix} b_{t} \frac{\partial \lambda_{t}}{\partial \boldsymbol{\theta}_{*}} \frac{\partial \lambda_{t}}{\partial \boldsymbol{\theta}_{*}}^{\top} & m_{t} \frac{\partial \lambda_{t}}{\partial \boldsymbol{\theta}_{*}} \frac{\partial \phi_{t}}{\partial \boldsymbol{\theta}^{*}}^{\top} \\ m_{t} \frac{\partial \lambda_{t}}{\partial \boldsymbol{\theta}_{*}}^{\top} \frac{\partial \phi_{t}}{\partial \boldsymbol{\theta}^{*}} & l_{t} \frac{\partial \phi_{t}}{\partial \boldsymbol{\theta}^{*}} \frac{\partial \phi_{t}}{\partial \boldsymbol{\theta}^{*}}^{\top} \end{pmatrix},$$

$$\begin{aligned} \boldsymbol{\theta}_{*} &= (\theta_{1}, \theta_{2}, \theta_{3})^{\top}, \, \boldsymbol{\theta}^{*} = (\theta_{4}, \theta_{5}, \theta_{6})^{\top}, \, \Psi'(x) = d\Psi(x)/dx, \\ b_{t} &= \frac{1}{\lambda_{t}(\lambda_{t}\phi_{t}+1)}, \quad m_{t} = \frac{\phi_{t} - \Psi'(\phi_{t}^{-1}+1) + \Psi(\phi_{t}^{-1})}{\phi_{t}^{2}(\lambda_{t}\phi_{t}+1)}, \\ l_{t} &= \frac{\lambda_{t}}{\phi_{t}^{2}(\lambda_{t}\phi_{t}+1)} + \frac{1}{\phi_{t}^{4}} \Big\{ \left(\Psi(\phi_{t}^{-1}) + \log(\lambda_{t}\phi_{t}+1) \right) \left(1 - \Psi(\phi_{t}^{-1}) - \log(\lambda_{t}\phi_{t}+1) \right) \\ &+ \Psi'(\phi_{t}^{-1}) - E \left(\Psi'(Y_{t} + \phi_{t}^{-1}) | \mathcal{F}_{t-1} \right) \Big\} - \frac{2}{\phi_{t}^{3}(\lambda_{t}\phi_{t}+1)} \Big\{ \log(\lambda_{t}\phi_{t}+1) + \lambda_{t} \left(\Psi'(\phi_{t}^{-1}+1) - \log(\lambda_{t}\phi_{t}+1) - \Psi'(\phi_{t}^{-1}) \right) \Big\}, \quad t = 2, \dots, n, \end{aligned}$$

where we have assumed that $\Omega_1(\theta)$ exists and it is a positive definite matrix.

We sketch the proof assuming that conditions of Theorem 2.2 are satisfied. Then, we apply the Law of Large Numbers for stationary and ergodic sequences to obtain that

$$n^{-1} \frac{\partial \mathbf{U}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \xrightarrow{p} \mathbf{\Omega}_2(\boldsymbol{\theta}), \quad \mathbf{\Omega}_2(\boldsymbol{\theta}) \equiv \operatorname{plim}_{n \to \infty} \mathbf{J}_2(\boldsymbol{\theta}),$$
(9)

as $n \to \infty$, where

$$\mathbf{J}_{2}(\boldsymbol{\theta}) \equiv \frac{1}{n} \sum_{t=2}^{n} E(-\nabla \mathbf{U}_{t}(\boldsymbol{\theta}) | \mathcal{F}_{t-1}) = \frac{1}{n} \sum_{t=2}^{n} \begin{pmatrix} b_{t} \frac{\partial \lambda_{t}}{\partial \boldsymbol{\theta}_{*}} \frac{\partial \lambda_{t}}{\partial \boldsymbol{\theta}_{*}}^{\top} & \mathbf{0} \\ \mathbf{0} & d_{t} \frac{\partial \phi_{t}}{\partial \boldsymbol{\theta}^{*}} \frac{\partial \phi_{t}}{\partial \boldsymbol{\theta}^{*}}^{\top} \end{pmatrix},$$

with

$$d_t = -\frac{\Psi'(\phi_t^{-1}) + E\left(\Psi'(Y_t + \phi_t^{-1}) | \mathcal{F}_{t-1}\right)}{\phi_t^4} - \frac{\lambda_t}{\phi_t^2(\lambda_t \phi_t + 1)}, \quad t = 2, \dots, n,$$

and the other terms as defined above.

By combining the above results and use a Taylor's expansion to obtain that

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \stackrel{d}{\longrightarrow} N(\boldsymbol{0}, \boldsymbol{\Sigma}(\boldsymbol{\theta})) \text{ as } n \to \infty, \text{ with } \boldsymbol{\Sigma}(\boldsymbol{\theta}) = \boldsymbol{\Omega}_2^{-1}(\boldsymbol{\theta})\boldsymbol{\Omega}_1(\boldsymbol{\theta})\boldsymbol{\Omega}_2^{-1}(\boldsymbol{\theta}).$$
 (10)

We now discuss how to consistently estimate the standard errors of the conditional maximum likelihood estimates based on the asymptotic normality discussed above. We have that $\widehat{\Sigma} = \mathbf{J}_2^{-1}(\widehat{\boldsymbol{\theta}})\mathbf{J}_1(\widehat{\boldsymbol{\theta}})\mathbf{J}_2^{-1}(\widehat{\boldsymbol{\theta}})$ is a consistent estimator for Σ since $\mathbf{J}_1(\widehat{\boldsymbol{\theta}})$ and $\mathbf{J}_2(\widehat{\boldsymbol{\theta}})$ are consistent for $\Omega_1(\boldsymbol{\theta})$ and $\Omega_2(\boldsymbol{\theta})$, respectively. Another consistent estimator for Σ is given by $\widetilde{\Sigma} = \mathbf{S}_2^{-1}(\widehat{\boldsymbol{\theta}})\mathbf{S}_1(\widehat{\boldsymbol{\theta}})\mathbf{S}_2^{-1}(\widehat{\boldsymbol{\theta}})$, where $\mathbf{S}_1(\boldsymbol{\theta}) = n^{-1}\sum_{t=2}^n \mathbf{U}_t(\boldsymbol{\theta})\mathbf{U}_t(\boldsymbol{\theta})^{\top}$ and $\mathbf{S}_2(\boldsymbol{\theta}) = -n^{-1}\sum_{t=2}^n \nabla \mathbf{U}_t(\boldsymbol{\theta})$.

There are required additional technical conditions to ensure the consistency and asymptotic normality of the CML estimators. We provide such details for the negative binomial P-INGARCH process ($\beta_1 = \beta_2 = 0$) in the next theorem, where its proof is given in the Appendix; here we denote $\boldsymbol{\theta} = (\beta_0, \alpha_0, \alpha_1, \alpha_2)^{\top}$. Before to do that, we have to consider lower and upper bounds for the possible values that the parameters can assume and then we define

$$\mathcal{B} = \{ \boldsymbol{\theta} : \beta_{0,low} < \beta_0 < \beta_{0,up}, \ \alpha_{0,low} < \alpha_0 < \alpha_{0,up}, \ \alpha_{1,low} < \alpha_1 < \alpha_{1,up}, \ \gamma_{1,low} < \alpha_2 < \gamma_{1,up} \},$$
(11)

where it is assumed that the true value of θ is contained in \mathcal{B} .

Theorem 3.2. Let $\{Y_t\}_{t=1}^n$ be a trajectory of the negative binomial P-INGARCH process. Assume that $0 < \alpha_1 + \alpha_2 < 1$. Then, there exists an open set $\mathcal{A} \subset \mathcal{B}$ such that the conditional loglikelihood function $\ell(\cdot)$ has a global maximum point on \mathcal{A} , say $\hat{\theta}$, with probability tending to 1 as $n \to \infty$. Furthermore, $\hat{\theta}$ is consistent for θ and satisfy the asymptotic normality (10) with $\theta = (\beta_0, \alpha_0, \alpha_1, \alpha_2)^{\top}$.

Proof. Please see the Appendix.

In the next subsection, we investigate the performance of the CMLEs and their asymptotic distribution as well through Monte Carlo simulation studies.

3.2 Monte Carlo Simulation

In this subsection, we report Monte Carlo simulation studies with 1000 replications, using sample sizes of n = 200,500 for assessing the finite-sample behaviour of the CMLEs for the NB tv-DINGARCH(1, 1, 1, 1) model. A burn-in period of 500 iterations is applied first for all simulated trajectories to reduce the influence of initial values. Results for two parameter configurations are reported here and additional results are given in the Supplementary Material. Stationarity and uniqueness of the resulting tv-DINGARCH processes by all parameter settings is guaranteed by ensuring that Theorem 2.2 holds for p = 1, that is, $\|\mathbf{A}\|_1 + \|\mathbf{B}\|_1 = \max\{\beta_1, \alpha_1\} + \max\{\beta_2, \alpha_2\} < 1$. The true values of $\boldsymbol{\theta}$ under Settings I and II are respectively $\boldsymbol{\theta} = (\beta_0, \alpha_0, \beta_1, \beta_2, \alpha_1, \alpha_2)^{\top} =$ $(1, 0.5, 0.5, 0.3, 0.3, 0.2)^{\top}$ and $\boldsymbol{\theta} = (\beta_0, \alpha_0, \beta_1, \beta_2, \alpha_1, \alpha_2)^{\top} = (1, 1, 0.3, 0.4, 0.2, 0.3)^{\top}$. Empirical means, standard deviation (SD), and mean squared error (MSE) of the CMLEs for the NB tv-DINGARCH model are provided in Table 1, where we can observe that the asymptotic normality provides adequate estimation results.

Figure 1 illustrates the asymptotic normality of the CMLEs for the NB case. Non-parametric density estimator plots of the standardized parameter estimates obtained under the two sample sizes of Setting I are displayed alongside with the standard Gaussian density curve. The density curves of the parameter estimates due to n = 200 and n = 500 are mostly overlapping despite for β_2 , in which case the normality approximation improves slightly with the increase of the sample size. Additional density plots for other parameter configurations can be found in the Supplementary Material. These plots confirm further the asymptotic normality of the CMLEs.

Setting I		$\beta_0 = 1$	$\alpha_0 = 0.5$	$\beta_1 = 0.5$	$\beta_2 = 0.3$	$\alpha_1=0.3$	$\alpha_2 = 0.2$
	Mean	1.245	0.545	0.488	0.239	0.305	0.181
n = 200	SD	0.516	0.321	0.165	0.147	0.104	0.136
	MSE	0.326	0.105	0.027	0.025	0.011	0.019
	Mean	1.116	0.524	0.503	0.268	0.301	0.189
n = 500	SD	0.304	0.191	0.109	0.096	0.062	0.090
	MSE	0.106	0.037	0.012	0.010	0.004	0.008
Setting II		$\beta_0 = 1$	$\alpha_0 = 1$	$\beta_1 = 0.3$	$\beta_2 = 0.4$	$\alpha_1 = 0.2$	$\alpha_2 = 0.3$
	Mean	1.289	1.029	0.308	0.305	0.212	0.276
n = 200	SD	0.592	0.505	0.151	0.216	0.108	0.232
	MSE	0.434	0.255	0.023	0.056	0.012	0.054
	Mean	1.146	1.036	0.299	0.354	0.205	0.283
n = 500	SD	0.419	0.361	0.095	0.158	0.069	0.163
	MSE	0.197	0.132	0.009	0.027	0.005	0.027

Table 1: Empirical mean, standard deviation (SD), and mean square error (MSE) of the Monte Carlo estimates for the NB tv-DINGARCH model under Settings I and II. Results are based on 1000 Monte Carlo replications.

4 Testing Constant Dispersion

In applications, it is often of interest to test for constant dispersion, i.e. $\phi_t = \alpha_0$ for all t. This is a challenging problem because it imposes conditions on the corresponding parameters to belong the boundary of the parameter space. Our approach to this problem is based on the likelihood ratio test combined with the methodology of bootstrap to test constant dispersion.

Recall this section is dedicated to evaluating methods to test the hypothesis of constant volatility, i.e., $H_0 : \alpha_1 = \alpha_2 = 0$ against the alternative $H_1 : \alpha_1 = 0$ or $\alpha_2 = 0$. This is challenging because the null hypothesis belongs to the boundary of the parameter space for which case the classical inference results might not hold (Self and Liang, 1987; Andrews, 2001; Crainiceanu and Ruppert, 2004). Instead, we develop and compare two parametric bootstrap methods; the classical or unrestricted, and the restricted bootstrap recently developed by Cavaliere et al. (2016). The first method considers the usual parametric bootstrap (Efron and Tibshirani, 1993) replications based on the unrestricted CMLEs, while the latter uses the CMLEs under H_0 . Algorithm 1 describes the estimation of the test's p-value with B replications of the restricted or unrestricted bootstrap.

We use a Monte Carlo simulation study to investigate how these methods achieve the desirable significance levels. Trajectories of the NB tv-DINGARCH process with 200 time points are simulated under the null hypothesis using four different (varying) settings of the parameter vector $(\beta_0, \beta_1, \beta_2, \alpha_0)^{\top}$ as follows: (C1) $\beta_0 = 2, \alpha_0 = 1, \beta_1 = 0.4, \beta_2 = 0.3, (C2) \beta_0 = 2, \alpha_0 = 1, \beta_1 = 0, \beta_2 = 0, (C3) \beta_0 = 3, \alpha_0 = 0.5, \beta_1 = 0.3, \beta_2 = 0.4, and (C4) \beta_0 = 3, \alpha_0 = 0.5, \beta_1 = 0, \beta_2 = 0.$ For each configuration, 500 Monte Carlo replications are used to calculate the empirical significance levels by employing the competing methodologies. The number of replications used to estimate bootstrap *p*-values is B = 500. Table 2 displays the proportion of times that the restricted and unrestricted bootstrap procedures rejected the null hypothesis. Parameter configurations are set in a way that C2 and C4 are variations of C1 and C3 that do not include effects on the mean.



Figure 1: Non-parametric density plots of standardized parameter estimates due to Setting I under sample sizes n = 200 and n = 500. Solid line corresponds to the standard Gaussian density function.

Algorithm 1: Bootstrap likelihood ratio test of constant $(\mathcal{H}_0 : \alpha_1 = \alpha_2 = 0)$ versus timevarying dispersion $(\mathcal{H}_1 : \alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0)$ for a tv NB-INGARCH model. Alternatives to step 3 yield restricted (3A) or unrestricted bootstrap (3B) estimators of the test's p-value, p_B , where B is the number of replications.

Input: \boldsymbol{Y} observed count time series data B bootstrap replications α significance level 1. Obtain $\hat{\boldsymbol{\theta}}_{\mathcal{H}_0}$ and $\hat{\boldsymbol{\theta}}$, the model CMLEs under \mathcal{H}_0 and \mathcal{H}_1 ; 2. Compute the observed likelihood ratio $LR = -2(\ell(\hat{\boldsymbol{\theta}}_{\mathcal{H}_0}) - \ell(\hat{\boldsymbol{\theta}}))$; for $b \leftarrow 1 : B$ do $\begin{vmatrix} 3A \rangle \boldsymbol{Y}_b \sim \text{tv NB-INGARCH}(\hat{\boldsymbol{\theta}}_{\mathcal{H}_0}); \ // \text{ if restricted bootstrap} \\ 3B \rangle \boldsymbol{Y}_b \sim \text{tv NB-INGARCH}(\hat{\boldsymbol{\theta}}); \ // \text{ if unrestricted bootstrap} \\ 4. Obtain <math>\hat{\boldsymbol{\theta}}_{\mathcal{H}_0}^b$ and $\hat{\boldsymbol{\theta}}^b$ fitting tv NB-INGARCH models to \boldsymbol{Y}_b under the null and alternative hypothesis; 5. Let $LR^b = -2(\ell(\hat{\boldsymbol{\theta}}_{\mathcal{H}_0}^b) - \ell(\hat{\boldsymbol{\theta}}^b))$, the replicated LR statistic; end 6. If $p_B = \sum_{b=1}^{B} I\{LR^b > LR\}/B < \alpha$ reject \mathcal{H}_0 ;

Configuration	Significance level	Restricted Bootstrap	Unrestricted Bootstrap
C1: $\beta_0 = 2, \alpha_0 = 1$ $\beta_1 = 0.4, \beta_2 = 0.3$	$\begin{array}{c} 0.05\\ 0.10\end{array}$	$\begin{array}{c} 0.046 \\ 0.088 \end{array}$	$0.000 \\ 0.002$
C2 : $\beta_0 = 2, \alpha_0 = 1$ $\beta_1 = 0, \beta_2 = 0$	$\begin{array}{c} 0.05 \\ 0.10 \end{array}$	$0.052 \\ 0.098$	$\begin{array}{c} 0.012\\ 0.036\end{array}$
$\overline{\mathbf{C3:} \ \beta_0 = 3, \alpha_0 = 0.5} \\ \beta_1 = 0.3, \beta_2 = 0.4$	$\begin{array}{c} 0.05 \\ 0.10 \end{array}$	$0.064 \\ 0.104$	$0.002 \\ 0.002$
C4: $\beta_0 = 3, \alpha_0 = 0.5$ $\beta_1 = 0, \beta_2 = 0$	$\begin{array}{c} 0.05\\ 0.10\end{array}$	$0.042 \\ 0.094$	$\begin{array}{c} 0.022\\ 0.058\end{array}$

Table 2: Nominal significance levels produced by the restricted and unrestricted bootstrap hypothesist tests for $H_0: \alpha_1 = \alpha_2 = 0$ for various parameter configurations.

Additional evidence is provided in the Supplementary Material.

Notably, the restricted parametric bootstrap agrees with the set significance levels, something that occurs under the four parameter configurations investigated in this study, while the unrestricted parametric bootstrap does not provide satisfactory results by underestimating the significance levels. These results show the importance of considering a restricted bootstrap for testing constant dispersion in the tv-DINGARCH models.

Given that the restricted parametric bootstrap yields excellent results in terms of Type I error, we now focus our attention on the test power resulting from this method. In this study, Configurations C1 - C4 of the intercepts and mean effects are combined with a grid of α_1 and α_2 values. More specifically, we take α_1 , $\alpha_2 \in [0, 1]$ with 0.1 spacing such that they lie in the stationary region. This yields 34 and 39 combinations to evaluate under C1 and C2, respectively. Under each of these, 200 trajectories of length 200 are simulated from the NB tv-DINGARCH(1,1,1,1) model and 500 restricted bootstrap replications estimate the p-value. The estimated test power is the proportion of times that the bootstrap *p*-value is less than the set significance level.

Results in Figure 2 show that the test power tends to increase with α_1 and α_2 as expected since in this case we moving away from the null hypothesis, but there is a much higher effect of the first parameter than the latter. For instance, it is estimated to be 0.910 and 0.560 when $\alpha_1 = 0.1, \alpha_2 = 0$ respectively for C1 and C2. Under $\alpha_1 = 0, \alpha_2 = 0.1$, these are 0.030 and 0.045, respectively. Additionally, recalling that C2 is a variation of C1, which takes the same intercepts but set the mean effects to zero, we can conclude that a higher power is expected when β_1 and β_2 are non-zero. In the Supplementary Material, the surfaces due to C3 and C4 are displayed, supporting the same conclusions.

5 Log-Linear tv-DINGARCH Models with Covariates

In this section, we present the novel first-order tv-DINGARCH model allowing for both negative and positive autocorrelation and inclusion of covariates. These features are important and commonly required when dealing with real data sets. We here focus our attention to the statistical modeling rather than theoretical aspects, which we will be addressing in future work.



Figure 2: Restricted bootstrap power for testing the hypothesis $H_0: \alpha_1 = \alpha_2 = 0$. Intercept and mean effect parameters are those of Configuration 1 (left-side) and Configuration 2 (right-side). Values of α_1 and α_2 are taken such that $\alpha_1, \alpha_2 \in [0, 1]$ with 0.1 spacing and ensuring they lie in the stationary region. Test power is estimated from 200 Monte Carlo replications of the restricted parametric bootstrap with B = 500.

Definition 5.1. (Log-linear tv-DINGARCH processes with covariates) A first-order log-linear tv-DINGARCH process $\{Y_t\}_{t\geq 1}$ allowing for covariates/exogenous time series is given as in Definition 2.1 with $\lambda_t \equiv \exp(\mu_t)$ and $\phi_t \equiv \exp(\nu_t)$, where

$$\mu_{t} = \beta_{0} + \beta_{1} \log(Y_{t-1} + 1) + \beta_{2} \mu_{t-1} + \boldsymbol{\delta}^{\top} \mathbf{X}_{t}, \quad \nu_{t} = \alpha_{0} + \alpha_{1} \log(Y_{t-1} + 1) + \alpha_{2} \nu_{t-1} + \boldsymbol{\gamma}^{\top} \mathbf{W}_{t}, \quad (12)$$

where $\beta_i, \alpha_i \in \mathbb{R}$ for i = 0, 1, 2, with \mathbf{X}_t and \mathbf{W}_t covariates with associated real-valued coefficients $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$.

In our definition above, we followed Fokianos and Tjøstheim (2011) and chose $\log(Y_{t-1} + 1)$ in (12) to keep the transformed time series, μ_t , and ϕ_t in the same scale. Since the coefficients now can assume negative values, a negative autocorrelation is also plausible, as discussed by Fokianos and Tjøstheim (2011). As a note, we suspect that the stationary linear model can only produce non-negative autocorrelation since the coefficients are assumed to be positive, but this needs to be further investigated. By asuming a negative binomial distribution for $Y_t | \mathcal{F}_{t-1}$, we obtain that the conditional log-likelihood function associated to a observed trajectory y_1, \ldots, y_n assumes the form $\ell^*(\boldsymbol{\theta}) = \sum_{t=2}^n \ell_t^*(\boldsymbol{\theta})$, where

$$\ell_t^*(\boldsymbol{\theta}) = y_t(\mu_t + \nu_t) - (y_t + e^{-\nu_t})\log(e^{\mu_t + \nu_t} + 1) + \log\Gamma(y_t + e^{-\nu_t}) - \log\Gamma(e^{-\nu_t}) - \log y_t!,$$

with μ_t and ν_t given in (12), for t = 2, ..., n, and $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_2, \boldsymbol{\delta}^\top, \alpha_0, \alpha_1, \alpha_2, \boldsymbol{\gamma}^\top)^\top$ is the parameter vector. By maximizing the above log-likelihood function, we get the conditional maximum likelihood estimators.

The ordinary log-linear INGARCH model is obtained as a particular case from the the log-linear tv-DINGARCH process by taking $\alpha_1 = \beta_2 = 0$ and $\gamma = 0$. This can be tested using some classic statistic like the likelihood ratio. Since the parameters involved do not belong to the boundary of

the parameter space, the classic theory holds in contrast with the hypothesis testing addressed for the linear tv-DINGARCH process in Section 4.

The usefulness of the log-linear tv-DINGARCH models with exogenous time series is empirically demonstrated in the next subsection, where COVID-19 data from Ireland is investigated.

5.1 Ireland COVID-19 Data Analysis

The log-linear negative binomial (NB) tv-DINGARCH model is applied to modeling the number of daily COVID-19 deaths recorded in Ireland between March 12^{th} , 2020 and May 14^{th} , 2021. This count time series, say $\{Y_t\}$, consists of 429 observations and is publicly available at the website https://ourworldindata.org/, from where we also collect data on the number of hospitalized patients due to the disease. The logarithm of the former, denoted as log-hosp, is chosen as a potential covariate for explaining $\{Y_t\}$. We explore including the log-hosp effect in the mean and dispersion of a log-linear NB tv-DINGARCH model, yielding the regression models

$$\begin{cases} \mu_t = \beta_0 + \beta_1 \log(Y_{t-1} + 1) + \beta_2 \mu_{t-1} + \delta_1 \log - hosp_t \\ \nu_t = \alpha_0 + \alpha_1 \log(Y_{t-1} + 1) + \alpha_2 \nu_{t-1} + \gamma_1 \log - hosp_t, \end{cases}$$
(13)

where $Y_t | \mathcal{F}_{t-1} \sim \text{NB}(\lambda_t, \phi_t)$ with $\lambda_t \equiv \exp\{\mu_t\}$ and $\phi_t \equiv \exp\{\nu_t\}$, for $t = 1, \ldots, 429$. In what follows, we denote as \mathcal{M}_{tv} the log-linear tv-DINGARCH negative binomial model, that will be compared to the ordinary log-linear INGARCH negative binomial process, denoted by \mathcal{M}_{ord} .

Figure 3 shows $\{Y_t\}$ on the left, and the log-daily count of patients in hospital due to COVID-19 on the right. This period comprises the first and second waves of the disease in the country, separated by a period with low number of deaths and hospitalizations that are latter increased after the easing social distance measures.



Figure 3: Daily COVID-19 deaths (left) and log-count of hospitalized patients (right) in Ireland between March 12^{th} , 2020 and May 14^{th} , 2021.

The fit of the \mathcal{M}_{tv} and \mathcal{M}_{ord} models to the data is reported in Table 3, where the CMLEs of the model parameters are given, with their standard errors (SE) in parenthesis. Uncertainty quantification of parameter estimates is done with 500 replications of a parametric bootstrap for \mathcal{M}_{tv} . The \mathcal{M}_{ord} model fit is readily obtained with the tscount R package (Liboschik et al., 2017), where standard errors can be estimated asymptotically of via bootstrap with the exception of α_0 ,

Parameter	tv-DINGARCH	INGARCH	Paramotor	tv-DINGARCH	INGARCH
	Estimate (SE)	Estimate (SE)	1 arameter	Estimate (SE)	Estimate (SE)
β_0	-0.611(0.080)	-0.599(0.215)	α_0	2.682(0.823)	1.288(3.551)
β_1	$0.181 \ (0.026)$	$0.066\ (0.050)$	α_1	-0.492 (0.098)	-(-)
β_2	$0.723\ (0.029)$	$0.800\ (0.057)$	α_2	-0.430(0.478)	-(-)
δ_1	$0.144\ (0.020)$	$0.157 \ (0.052)$	γ_1	-0.493(0.181)	-(-)

Table 3: Conditional maximum likelihood estimates with their respective standard errors for the log-linear tv-DINGARCH and ordinary log-linear INGARCH models fitted to the daily COVID-19 deaths in Ireland.

in which case only the former method applies. In this case, we obtain the SE for the estimate of α_0 by setting the number of replications to 500 for comparison. We find out that there is great uncertainty in estimating the dispersion under \mathcal{M}_{ord} . The 95% bootstrap confidence interval for α_0 is (0.999, 4.325), and although bootstrap standard errors are also available for the remaining \mathcal{M}_{ord} model parameters, the asymptotic ones are considerably smaller so we report these instead.

Comparison between \mathcal{M}_{tv} and \mathcal{M}_{ord} shows that the mean related are in close resemblance, but those of \mathcal{M}_{tv} have smaller variability. As to be expected, the log-count of patients in hospital is significant to explain the mean number of daily deaths and can be interpreted by taking $\exp(\delta_1)$. This gives that one unit increment in the log-count of hospitalized patients increases deaths in 16% in average. Under the tv-DNGARCH structure, this covariate is also significant to the explain the dispersion, as shown by γ_1 estimated to be -0.493 with a small standard error. Although interpretation of this parameter is not as straightforward, we can infer by the negative sign of γ_1 that once $\log-hosp_t$ is known, $Y_t|\mathcal{F}_{t-1}$ can be modelled with reduced variability. The left hand side of Figure 4 illustrates the proximity amongst the fitted mean processes alongside the observed data. The different line types give $\{\hat{\mu}_t\}$ from the alternative models, but these mostly overlap. The fitted dispersion process $\{\hat{\phi}_t\}$ obtained with the tv-DINGARCH structure (13) is displayed on the right, showing high volatility periods around July 2020 and in the end of the study period. In what follows, we contrast the fit of the \mathcal{M}_{tv} and \mathcal{M}_{ord} models with respect to their goodness-of-fit to the observed data and in terms of one-step-ahead forecasting.



Figure 4: Left: Fitted means obtained from \mathcal{M}_{tv} (solid line) and \mathcal{M}_{ord} (dashed line) are displayed alongside the observed values of $\{Y_t\}$. Right: Fitted dispersion process $\{\hat{\phi}_t\}$ (solid line) from the tv-DINGARCH structure in (13) with 95% bootstrap confidence interval (dashed lines).

The AIC and BIC information criteria under the negative binomial assumption are calculated for the alternative models, returning (2397.672, 2430.164) for \mathcal{M}_{tv} and (2462.138, 2482.445) for \mathcal{M}_{ord} , both indicating the proposed model. The hypothesis of a non-constant dispersion is further supported by conducting the likelihood ratio (LR) test, with statistic given by LR = $2\{\ell^*(\hat{\theta}) - \ell^*(\hat{\theta}_0)\}$, where $\ell^*(\cdot)$ is defined in Section 5, $\hat{\theta}$ is the unrestricted CMLE and $\hat{\theta}_0$ is the restricted CMLE under $\alpha_1 = \alpha_2 = 0$ and $\gamma = 0$. The LR test gives high evidence (p-value < 10⁻⁵) in favour of \mathcal{M}_{tv} over \mathcal{M}_{ord} .

A final goodness-of-fit assessment is done with Probability Integral Transform (PIT) plots (Czado et al., 2009), an approach that enables the comparison of count data models via their predictive distributions to the observed data. A model providing a good fit to the data in this aspect will render a PIT plot resembling a uniform distribution, where major deviations typically indicate problems of overdispersion or underdispersion by the model's predictive distribution. These are reported in Figure 5 for the alternative models, and while that of \mathcal{M}_{tv} approaches the uniform as desired, the upside-down U shaped PIT of \mathcal{M}_{ord} points to a not adequate fit.



Figure 5: Probability Integral Transform (PIT) assessment of the alternative \mathcal{M}_{ord} (on the left), and \mathcal{M}_{tv} (to the right) models.

In addition to the data adherence, another feature that is often of interest to practitioners in choosing between models is their forecasting power. We consider forecasting by exploring one-stepahead (OSA) prediction from \mathcal{M}_{tv} and \mathcal{M}_{ord} , doing this in a recursive manner that is explained in what follows. Start by defining the initial training data set from the beginning of the study until December 15th 2020, fit the competing models, and predict the next day count via the conditional mean and median of the distributions. Once this is obtained, add December 15th 2020 to the training set, refitting both models and gather the new OSA predictions. Proceeding until the end of the study period gives a total of 150 predictions from each model that we summarise via their root mean squared forecasting error (RMSFE). Pseudocode describing the steps to this prediction exercise is given in Algorithm 2.

Let n_0 denote the time point chosen to starting the prediction exercise, in our case $n_0 = 279$ (December 15th, 2020), the RMSFE of the forecasting step t is $\text{RMSFE}_t = \sqrt{\frac{1}{t-n_0} \sum_{s=n_0+1}^t (Y_s - \hat{Y}_s)^2}$, where Y_s and \hat{Y}_s are the observed and forecasted counts, respectively. To the out-of-sample exercise considered here, we consider the conditional mean and median as forecasters. Computation

Algorithm 2: Recursive algorithm for obtaining \hat{Y} , the one-step-ahead (OSA) predicted values of $Y_{[n_0+1:n]}$. The training data at iteration s, Y(s), is incremented and the model is refitted to obtain the OSA forecast \hat{Y}_{s+1} . Steps 5A) and 5B) provide alternative prediction methods via the mean or median.

Input: \boldsymbol{Y} observed count trajectory of length n \boldsymbol{X} $(n \times p)$ matrix of mean covariates \boldsymbol{W} $(n \times q)$ matrix of dispersion covariates n_0 starting point of prediction exercise 0. $s \leftarrow n_0$; while s < n do $\begin{vmatrix} 1. \boldsymbol{Y}(s) \leftarrow Y_{[1:s]}; \ // \text{ train data} \\ 2. \text{ Fit the tv-NBINGARCH model to } \boldsymbol{Y}(s); \end{vmatrix}$

3. From 2, gather the CMLEs $\widehat{\theta}(s)$ and the fitted $\widehat{\mu}(s)$ and $\widehat{\nu}(s)$ of step s;

4. Obtain the OSA log-mean $\hat{\mu}_{s+1} = \hat{\beta}_0^s + \hat{\beta}_1^s \log(Y_s + 1) + \hat{\beta}_2^s \hat{\mu}_s + \hat{\delta}_s^T \boldsymbol{X}[\boldsymbol{s},]$ and log-dispersion $\hat{\nu}_{s+1} = \hat{\alpha}_0^s + \hat{\alpha}_1^s \log(Y_s + 1) + \hat{\alpha}_2^s \hat{\mu}_s + \hat{\boldsymbol{\gamma}}_s^T \boldsymbol{W}[\boldsymbol{s},];$

 $\begin{array}{|c|c|c|c|c|} & 5\mathrm{A}) \ \widehat{Y}_{s+1} = \exp(\widehat{\mu}_{s+1}) \ // \ \mathrm{prediction \ via \ mean} \\ & 5\mathrm{B}) \ \widehat{Y}_{s+1} = \mathtt{qbinom}(0.5, \mathtt{size} = e^{-\widehat{\nu}_{s+1}}, \mathtt{mean} = e^{\widehat{\mu}_{s+1}}) \ // \ \mathtt{prediction \ via \ median} \\ & \mathrm{s} = \mathrm{s}{+1}; \ // \ \mathtt{increment \ step} \\ & \mathtt{end} \\ & \mathbf{Output:} \ \widehat{Y} \ \mathrm{vector \ of} \ (n-n_0) \ \mathrm{OSA \ predictions}; \end{array}$

of MSFE from the 150 total predictions from \mathcal{M}_{tv} and \mathcal{M}_{ord} renders Figure 6. On the left, \widehat{Y}_s is the conditional mean, which yields a reduced prediction error in comparison to the median, that is displayed on the right. Especially for the tv-DINGARCH, the MSFE is considerably lower under the mean, whereas the difference amongst point estimates of \widehat{Y}_s is more subtle under \mathcal{M}_{ord} . In either case, allowing for a time-varying dispersion has shown to provide great advantage in terms of forecasting, once the MSFEs due to \mathcal{M}_{tv} are much lower than those of \mathcal{M}_{ord} for the entire period.



Figure 6: Root mean squared forecasting error (RMSFE) of predictions obtained with the fit of ordinary (dashed lines) and time-varying dispersion (solid lines) INGARCH models to the daily count of COVID-19 deaths in Ireland in 2021. On the left, predicted values are the conditional mean of the distributions, whereas the median is taken as the point estimate to produce the plot on the right.

This empirical illustration to the number of deaths due to COVID-19 in Ireland showed that the time-varying dispersion INGARCH models are promising and an important extension of the ordinary INGARCH processes in terms of goodness-of-fit and forecasting.

6 Conclusions

We proposed a class of time-varying dispersion INGARCH (tv-DINGARCH) models and explore stochastic properties such as stationarity and ergodicity. Estimation of parameters was addressed through conditional maximum likelihood estimation (CMLE) and its associated asymptotic theory was established. Monte Carlo simulations were conducted to evaluate the performance of the CMLE. Moreover, we developed bootstrap methodologies to testing for constant dispersion and showed via simulated studies that the restricted bootstrap is preferred over the unrestricted parametric one. Log-linear tv-DINGARCH models were also proposed to allow the inclusion of covariates. We analyzed the daily number of deaths due to COVID-19 data in Ireland and found that the log-linear tv-DINGARCH approach delivers much better results regarding goodness-of-fit and prediction when compared to the ordinary log-linear INGARCH models.

We have also observed this superior performance of our methodology over the INGARCH processes in other count time series data. R codes for fitting tv-DINGARCH models will be available at github very soon. Although we focus on a dispersion parameterization, the very same approach and strategy to obtain the theoretical results here can be derived under a precision parameterization. Our idea is to make the R codes available to handle both parameterizations.

Appendix

Technical results and proofs

Lemma 6.1. For $0 < \alpha_1 + \alpha_2 < 1$ and $\beta_1 = \beta_2 = 0$, the negative binomial P-INGARCH process $\{Y_t\}_{t\in\mathbb{N}}$ has finite moments of order $k \in \mathbb{N}$, for an arbitrary k.

Proof. Under the conditions $0 < \alpha_1 + \alpha_2 < 1$ and $\beta_1 = \beta_2 = 0$, we obtain from Theorem 2.2 that $\{Y_t\}_{t\in\mathbb{N}}$ has a unique stationary and ergodic solution. Moreover, this process is a Markov chain. Therefore, we will now show that the Tweedie's criterion is satisfied for a test function in order to show the finiteness of arbitrary moments stated in the lemma (Meyn and Tweedie, 1993).

Consider the test function $V(x) = 1 + x^k$, for x > 0 and arbitrary $k \in \mathbb{N}$. It follows that

$$E(V(\phi_t)|\phi_{t-1} = \phi) = 1 + E\left((\alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 \phi)^k | \phi_{t-1} = \phi\right)$$

= $1 + (\alpha_2 \phi)^k + \sum_{j=0}^{k-1} \binom{k-1}{j} (\alpha_2 \phi)^j E\left((\alpha_0 + \alpha_1 Y_{t-1})^{k-j} | \phi_{t-1} = \phi\right)$
= $V(\phi) \frac{1 + (\alpha_2 \phi)^k + h(\phi)}{1 + \phi^k},$ (14)

where $h(\phi) \equiv \sum_{j=0}^{k-1} {k-1 \choose j} (\alpha_2 \phi)^j E\left((\alpha_0 + \alpha_1 Y_{t-1})^{k-j} | \phi_{t-1} = \phi\right)$. From Equation (A.2) from Christou and Fokianos (2014), after a reparameterization and rearranging terms, we obtain that the *r*-th moment of a NB(λ, ϕ) distributed random variable, say *Y*, for $r \geq 2$, satisfies the recursive equation

$$E(Y^{r}) = \lambda \left\{ \sum_{j=0}^{r-2} \binom{r-1}{j} \left[E(Y^{j}) + \phi E(Y^{j+1}) \right] + E(Y^{r-1}) \right\},$$
(15)

with $E(Y) = \lambda$. From this result, we immediately obtain that $h(\phi) = \mathcal{O}(\phi^{k-1})$.

Since $\lim_{\phi\to\infty} \frac{1+(\alpha_2\phi)^k}{1+\phi^k} = \alpha_2^k < 1$ and $\lim_{\phi\to\infty} \frac{h(\phi)}{1+\phi^k} = 0$, there exist real constants $\kappa_1 \in (0,1), \kappa_2 > 0$, and a > 0 such that (14) is bounded above as follows:

$$E(V(\phi_t)|\phi_{t-1} = \phi) \le (1 - \kappa_1)V(\phi) + \kappa_2 I\{0 < \phi < a\}.$$

In other words, the Tweedie's criterion is satisfied and we conclude that the k-th moment of ϕ_t is finite for arbitrary $k \in \mathbb{N}$. Now, using the fact that $Y_t | \mathcal{F}_{t-1} \sim \text{NB}(\lambda, \phi_t)$ and (15), we have that the k-th conditional moment of Y_t given \mathcal{F}_{t-1} is a polynomial on ϕ_t . Therefore, the unconditional k-th moment of Y_t is finite for all $k \geq 1$.

Lemma 6.2. Let \mathcal{B} as defined in (11). Under the negative binomial P-INGARCH model with $0 < \alpha_1 + \alpha_2 < 1$, there exists a sequence of random variables $\{M_n\}_{n \in \mathbb{N}}$ such that

$$\max_{i,j,k=1,2,3,4} \sup_{\boldsymbol{\theta} \in \mathcal{B}} \left| \frac{1}{n} \frac{\partial^3 \ell(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| \le M_n,$$

where $M_n \xrightarrow{p} \bar{\mu}$ as $n \to \infty$, with $\bar{\mu} < \infty$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)^\top = (\beta_0, \alpha_0, \alpha_1, \alpha_2)^\top$.

Proof. We will show that the result holds for the third derivative of $\ell(\cdot)$ with respect to β_0 and also α_1 . The remaining cases follow the very same steps than these two and therefore are omitted.

By computing the third derivative $\partial^3 \ell_t(\boldsymbol{\theta}) / \partial \beta_0^3$ and using the triangle inequality, we obtain after some algebra that

$$\left|\frac{\partial^3 \ell_t(\boldsymbol{\theta})}{\partial \beta_0^3}\right| \le \frac{y_t}{\beta_{0,low}^3} \left(\frac{5}{\beta_{0,low}\alpha_{0,low}} + 2\right) + \frac{1}{\beta_{0,low}^3 \alpha_{0,low}} \equiv M_{1,t},$$

where $M_{1,t}$ does not depend on $\boldsymbol{\theta}$. Under the assumption $0 < \alpha_1 + \alpha_2 < 1$ and using Lemma 6.1, we apply the Law of Large Numbers for stationary and ergodic sequences to obtain that $n^{-1} \sum_{t=2}^{n} Y_t \xrightarrow{p} \beta_0$ and therefore $n^{-1} \sum_{t=2}^{n} M_{1,t}$ converges in probability to a finite constant, say $\bar{\mu}_1$, as $n \to \infty$.

The case for α_1 is more involving. We have that $\frac{\partial^3 \ell_t(\boldsymbol{\theta})}{\partial \alpha_1^3} = \frac{\partial^2 S_{2,t}}{\partial \phi_t^2} \frac{\partial \phi_t}{\partial \alpha_1}$, with $\frac{\partial \phi_t}{\partial \alpha_1} = \sum_{j=1}^{t-1} \alpha_2^{j-1} y_{t-j}$

and

$$\frac{\partial^2 S_{2,t}}{\partial \phi_t^2} = \frac{y_t - 2\beta_0}{\phi_t^4 (\beta_0 \phi_t + 1)^2} (3\beta_0 \phi_t^2 + 2\phi_t) + \beta_0 \frac{y_t - \beta_0}{\phi_t^2 (\beta_0 \phi_t + 1)^3} (3\beta_0 \phi_t + 1) - \frac{2\beta_0}{\phi_t^3 (\beta_0 \phi_t + 1)}$$
(16)

$$+\frac{6}{\phi_t^4}\log(\beta_0\phi_t+1) + 6\frac{\Psi(\phi_t^{-1}) - \Psi(y_t+\phi_t^{-1})}{\phi_t^4} + 6\frac{\Psi'(\phi_t^{-1}) - \Psi'(y_t+\phi_t^{-1})}{\phi_t^5}$$
(17)

$$+\frac{\Psi''(\phi_t^{-1}) - \Psi''(y_t + \phi_t^{-1})}{\phi_t^6}.$$
(18)

The modulus of the terms in (16) are bounded above (as done for the third derivative involving β_0) by a linear function of y_t with positive coefficients do not depend on $\boldsymbol{\theta}$. For the first term in (17), we have that $0 < \phi_t^{-4} \log(\beta_0 \phi_t + 1) < \alpha_{0,low}^{-3} \beta_{0,up}$ since $\log(x+1) < x$ for all x > -1 and $\phi_t > \alpha_{0,low}$ for all t. Now, we study the terms involving the polygamma functions.

Using inequality (14) by Guo and Qi (2013), we obtain that $|\Psi(\phi_t^{-1}) - \Psi(y_t + \phi_t^{-1})| \leq \log(y_t\phi_t + 1) + \phi_t \frac{1+2y_t\phi_t}{2(1+y_t\phi_t)} \leq y_t\phi_t + \phi_t + 1$. Hence, $\phi_t^{-4}|\Psi(\phi_t^{-1}) - \Psi(y_t + \phi_t^{-1})| \leq \frac{y_t}{\alpha_{0,low}} + \frac{1+\alpha_{0,low}^{-1}}{\alpha_{0,low}}$. Now, we consider inequality (15) by Guo and Qi (2013) to obtain that $|\Psi'(\phi_t^{-1}) - \Psi'(y_t + \phi_t^{-1})| \leq \frac{y_t\phi_t^2}{y_t\phi_t + 1} + \phi_t^2 \left(1 + \frac{1}{(y_t\phi_t + 1)^2}\right) \leq \phi_t(2\phi_t + 1)$. It follows that $\phi_t^{-5}|\Psi'(\phi_t^{-1}) - \Psi'(y_t + \phi_t^{-1})| \leq \alpha_{0,low}^{-3}(1+\alpha_{0,low}^{-3})$. Similarly, we have that the modulus of the term in (18), $\phi_t^{-6}|\Psi''(\phi_t^{-1}) - \Psi''(y_t + \phi_t^{-1})| = \phi_t^{-1}(y_t^{-1}) + \phi_t^{-1}(y_t^{-1}) = 0$.

From the above results, we conclude that there exist constants $c_1 > 0$ and $c_2 > 0$ do not depending on $\boldsymbol{\theta}$ such that $\left| \frac{\partial^2 S_{2,t}}{\partial \phi_t^2} \right| \leq c_1 y_t + c_2$. Hence, we obtain that

$$\left|\frac{\partial^3 \ell_t(\boldsymbol{\theta})}{\partial \alpha_1^3}\right| \le (c_1 y_t + c_2) \sum_{j=1}^{t-1} \gamma_{1,up}^{j-1} y_{t-j} \equiv M_{2,t}.$$

Now, by using our Lemma 6.1 and arguing exactly as done by Fokianos et al. (2009) (proof of Lemma 3.4 given in their Supplementary Material), we obtain that $n^{-1} \sum_{t=2}^{n} M_{2,t} \xrightarrow{p} \bar{\mu}_2$ when $n \to \infty$, where $\bar{\mu}_2$ is a finite constant.

Proof. (Theorem 3.2) We will show that the Conditions (A1), (A2), and (A3) from Lemma 3.1 by Jensen and Rahbek (2004) are satisfied, which give us the desired results stated in the theorem. As argued before for the general case, we use our Theorem 3.1 and the Central Limit Theorem for martingale difference to establish the weak convergence involving the score function in (8). The existence and finiteness of the matrix $\Omega_1(\theta)$ follows from Lemma 6.1 that provides the finiteness of the moments of all orders for the NB P-INARCH process under the assumption that $0 < \alpha_1 + \alpha_2 < 1$, which is in force. Therefore, Condition (A1) holds.

Moreover, under the assumption $0 < \alpha_1 + \alpha_2 < 1$, we obtain from Theorem 2.2 that $\{Y_t\}_{t \in \mathbb{N}}$ is a stationary and ergodic sequence. Then, the Law of Large Numbers for stationary and ergodic sequences give us that (9) is valid, with the existence and finiteness of the matrix $\Omega_2(\theta)$ being ensured again by Lemma 6.1. That is, Condition (A2) from Lemma 3.1 by Jensen and Rahbek (2004) is satisfied. Finally, the Condition (A3) has been established in Lemma 6.2.

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