An elliptic curve test for Mersenne primes

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Let $\ell \geq 3$ be a prime, and let $p = 2^\ell - 1$ be the corresponding Mersenne number. The Lucas-Lehmer test for the primality of p goes as follows. Define the sequence of integers x_k by the recursion

$$x_0 = 4, \quad x_k = x_{k-1}^2 - 2.$$

Then p is a prime if and only if each x_k is relatively prime to p, for $0 \leq k \leq \ell - 3$, and $gcd(x_{\ell-2}, p) > 1$. We show, in the first section, that this test is based on the successive squaring of a point on the one dimensional algebraic torus T over $\mathbb{Q}$, associated to the real quadratic field $k = \mathbb{Q}(\sqrt{3})$. This suggests that other tests could be developed, using different algebraic groups. As an illustration, we will give a second test involving the sucessive squaring of a point on an elliptic curve.

If we define the sequence of rational numbers x_k by the recursion

$$x_0 = -2, \quad x_k = \frac{(x_{k-1}^2 + 12)^2}{4 \cdot x_{k-1} \cdot (x_{k-1}^2 - 12)}$$

then we show that p is prime if and only if $x_k \cdot (x_k^2 - 12)$ is relatively prime to p, for $0 \leq k \leq \ell - 2$, and $gcd(x_{\ell-1}, p) > 1$. This test involves the successive squaring of a point on the elliptic curve E over $\mathbb{Q}$ defined by the equation

$$y^2 = x^3 - 12x.$$

We provide the details in the second section.

The two tests are remarkably similar. For example, both take place on groups with good reduction away from 2 and 3. Can one be derived from the other?

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1. Lucas-Lehmer

If $\ell \geq 3$ is a prime, and $p = 2^\ell - 1$ is the corresponding Mersenne number, then

$$p \equiv 7 \pmod{24}. \tag{1.1}$$

We will exploit this congruence throughout the paper.

In this section, we will consider the Lucas-Lehmer test for the primality of p. Lucas's original paper is [Lu], and Lehmer's addition is given in [Le]. A good modern treatment, similar to the one given here, can be found in [R].

Let $A = \mathbb{Z} + \mathbb{Z}\sqrt{3}$ be the ring of integers, of discriminant 12, inside the real quadratic field $k = \mathbb{Q}(\sqrt{3})$. Let σ be the non-trivial automorphism of k, for which $\sigma(\sqrt{3}) = -\sqrt{3}$. The ring A has class number 1 and fundamental unit

$$\epsilon = 2 + \sqrt{3}.$$

The unit ϵ is totally positive and satisfies $\epsilon \cdot \epsilon^\sigma = 1$. It provides an integral point on the algebraic torus T mentioned in the introduction.

Let q be a prime number, and let $T(q)$ be the subgroup of $(A/q)^*$ consisting of elements of norm 1 to $(\mathbb{Z}/q)^*$. By reduction $\pmod{q}$, we may consider ϵ as an element of the finite group $T(q)$.

Proposition 1.2. *If $q \equiv 7 \pmod{24}$ then $T(q)$ is cyclic of order $q+1$, and ϵ is not a square in $T(q)$.*

Proof. Since $q \equiv 7 \pmod{12}$ we have $\left(\frac{3}{q}\right) = -1$ by quadratic reciprocity. Hence q remains prime in A and A/q is a field with q^2 elements. Since the norm $(A/q)^* \to (\mathbb{Z}/q)^*$ is surjective, $T(q)$ is cyclic of order $q+1$.

The element ϵ is not a square provided

$$\epsilon^{\frac{q+1}{2}} \equiv -1 \pmod{q},$$

by Euler's criterion. But

$$\epsilon = \beta/\beta^\sigma$$

in k, with $\beta = 3 + \sqrt{3}$ satisfying $\beta\beta^\sigma = 6$. Writing this identity as

$$\epsilon = \beta^2/6,$$

and reducing $\pmod{q}$ then gives:

$$\begin{aligned} \epsilon^{\frac{q+1}{2}} &= \beta^{q+1}/6^{\frac{q+1}{2}} \\ &\equiv 6/6^{\frac{q+1}{2}} \qquad \text{as } \beta^\sigma \equiv \beta^q \\ &\equiv \left(\frac{6}{q}\right) = -1. \end{aligned}$$

The last identity follows from the congruence $q \equiv 7 \pmod{24}$ and quadratic reciprocity. This completes the proof.

Now define the (Lucas) sequence of integers x_k by the formula

$$x_k = \mathrm{Tr}(\epsilon^{2^k}).$$

The first few terms are

$$x_0 = 4, \quad x_1 = 14, \quad x_2 = 194, \quad x_3 = 37634.$$

The integers x_k can be computed via the recursion

$$x_k = x_{k-1}^2 - 2.$$

Proposition 1.3. *If the Mersenne number $p = 2^\ell - 1$ is prime, then $x_k \not\equiv 0 \pmod{p}$ for $0 \le k \le \ell - 3$ and $x_{\ell-2} \equiv 0 \pmod{p}$.*

Conversely, let $p = 2^\ell - 1$ be a Mersenne number. If x_k is a unit $\pmod{p}$ for $0 \le k \le \ell - 3$ and $gcd(x_{\ell-2}, p) > 1$, then p is prime.

Proof. If p is prime, then by (1.1) and Proposition 1.2, the group $T(p)$ is cyclic of order $p+1 = 2^\ell$. Since ϵ is not a square in $T(p)$, it is a generator. Hence $\epsilon^{2^{\ell-2}}$ has order 4 in $T(p)$, and satisfies the polynomial $x^2 + 1 \equiv 0 \pmod{p}$. In particular, $x_{\ell-2} = \mathrm{Tr}(\epsilon^{2^{\ell-2}}) \equiv 0 \pmod{p}$. No smaller power of ϵ has order 4, so x_k is a unit $\pmod{p}$ for $0 \le k \le \ell - 3$.

For the converse, assume that q is a prime factor of $p = 2^\ell - 1$ which divides $x_{\ell-2}$. Then $\epsilon^{2^{\ell-2}}$ has order 4 $\pmod{p}$, so ϵ has order $2^\ell = p + 1$ in the group $T(q)$. Since $T(q)$ has order $q \pm 1$, depending on the behavior of q in A, this forces $q = p$. Hence p is prime.

Corollary 1.4. *Assume that $p = 2^\ell - 1$ is prime. Then the order $B = \mathbb{Z} + p\mathbb{Z}\sqrt{3}$ of index p in A has class number 2 and fundamental unit $\eta = \epsilon^{2^{\ell-1}}$.*

Proof. Let $\hat{A} = A \otimes \hat{\mathbb{Z}}$ and $\hat{B} = B \otimes \hat{\mathbb{Z}}$ be the profinite completions of these rings. In general, we have an exact sequence [L-P-P]

$$1 \to A^*/B^* \to \hat{A}^*/\hat{B}^* \to \mathrm{Pic}(B) \to \mathrm{Pic}(A) \to 1.$$

In this case, $\mathrm{Pic}(A) = 1$ and

$$\hat{A}^*/\hat{B}^* = (A/p)^*/(\mathbb{Z}/p)^*.$$

Since ϵ has order $2^{\ell-1}$ in $(A/p)^*/(\mathbb{Z}/p)^*$, the quotient $\mathrm{Pic}(B)$ has order 2. Also $\eta = \epsilon^{2^{\ell-1}}$ is the smallest power of ϵ which lies in B^*.

Since the fundamental unit of B is so large, the continued fraction of the quadratic irrationality $p\sqrt{3}$ is quite complicated, when p is prime. Can this be converted into a primality test?

2. Elliptic curves

Let E be the elliptic curve over $\mathbb{Q}$ defined by the equation

$$y^2 = x^3 - 12x = x(x^2 - 12).$$

Then E has discriminant $\Delta = 2^{12} \cdot 3^3$ and conductor $N = 2^5 \cdot 3^2 = 288$. In Cremona's tables [C, pg 123], E is the curve $288 - A2$.

The Mordell-Weil group

$$E(\mathbb{Q}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$$

is generated by the points

$$P = (-2, 4) \quad \text{of infinite order,}$$

$$Q = (0, 0) \quad \text{of order 2.}$$

The curve E has good reduction at all primes $q > 3$. It has complex multiplication by the ring of Gaussian integers, defined over $\mathbb{Q}(i)$. An automorphism of order 4 is given by:

$$\varphi(x, y) = (-x, iy).$$

In particular, E has supersingular reduction at all primes $q > 3$ with $q \equiv 3 \pmod 4$, and at these primes the group $E(q)$ of points over $\mathbb{Z}/q$ has order $q + 1$ [S2, pg 184].

Proposition 2.1. *If $q \equiv 7 \pmod{24}$ then $E(q)$ is cyclic of order $q+1$, and $P = (-2, 4)$ is not divisible by 2 in $E(q)$.*

Proof. The group $E(q)$ is the kernel of the isogeny $F-1$ on E in characteristic q [S1, pg 131], where

$$F(x,y)=(x^q,y^q).$$

Hence $E(q)$ is cyclic if $F-1$ is not divisible by any prime ℓ in the ring $\mathrm{End}(E)$. Otherwise, $E(q)$ contains the group $(\mathbb{Z}/\ell)^2$ killed by multiplication by ℓ.

Since $F^2=-q$ in $\mathrm{End}(E)$, the only rational prime ℓ which *can* divide $F-1$ is $\ell=2$. Indeed, the quotient must be an algebraic integer. But 2 divides $F-1$ if and only if $\left(\frac{12}{q}\right)=+1$, when all 2-torsion is rational over $\mathbb{Z}/q$. Since $q\equiv 7 \pmod{12}$, $\left(\frac{12}{q}\right)=-1$, and $E(q)$ is cyclic.

A point (x,y) lies in $2E(q)$ provided both x and x^2-12 are squares in $(\mathbb{Z}/q)^*$ [S1, pg 280-282]. Since $q\equiv 7 \pmod{24}$, -2 is not a square and $P=(-2,4)$ is not divisible by 2.

Now define a sequence of rational numbers x_k by the formula

$$x_k=x(2^k\cdot P).$$

The first few terms are

$$x_0=-2,\quad x_1=4,\quad x_2=\frac{49}{4},\quad x_3=\frac{6723649}{1731856}.$$

The rational numbers x_k can be computed (cf. [S1, pg 59]) via the recursion

$$x_k=\frac{(x_{k-1}^2+12)^2}{4\cdot x_{k-1}\cdot(x_{k-1}^2-12)}.$$

Proposition 2.2. *If the Mersenne number $p = 2^\ell - 1$ is prime, then the rational numbers $x_k(x_k^2 - 12)$ are p-adic units for $0 \le k \le \ell - 2$ and $x_{\ell-1} \equiv 0 \pmod{p}$.*

Conversely, let $p = 2^\ell - 1$ be a Mersenne number. If $x_k(x_k^2 - 12)$ is relatively prime to p for $0 \le k \le \ell - 2$ and $\gcd(x_{\ell-1}, p) > 1$, then p is prime.

Proof. If p is prime, then by (1.1) and Proposition 2.1, the group $E(p)$ is cyclic of order $p + 1 = 2^\ell$. Since P is not divisible by 2 in $E(p)$, it is a generator. Hence

$$2^{\ell-1} \cdot P \equiv Q \pmod{p}$$

with $Q = (0, 0)$ the unique point of order 2. In particular, $x_{\ell-1} = x(2^{\ell-1}P) \equiv 0 \pmod{p}$. No smaller multiple of P has order 2, so $x_k(x_k^2 - 12)$ is a p-adic unit for $0 \le k \le \ell - 2$.

For the converse, assume that the rational number $x_k(x_k^2-12)$ is relatively prime to p for $0 \le k \le \ell - 2$ and that q is a prime factor of $p = 2^\ell - 1$ which divides $x_{\ell-1}$. Then $2^{\ell-1}P$ has order 2 in $E(q)$, so P has order $2^\ell = p + 1$ in $E(q)$. But the order of $E(q)$ has the form $q + 1 - a_q$ with $|a_q| \le 2\sqrt{q}$ [S1, pg 136]. Hence

$$p + 1 \le q + 1 + 2\sqrt{q}.$$

This forces $q = p$, so p is prime.

3. Bibliography

[C] Cremona, J.E. Algorithms for modular elliptic curves. Cambridge University Press, 1992.

[Le] Lehmer, D.H. On Lucas's test for the primality of Mersenne's numbers. J. London Math. Soc. 10 (1935), 162–165.

[L-P-P] Lenstra, H.W., Pila, J., and Pomerance, C. A hyperelliptic smoothness test. II. Proc. London Math. Soc. (3) 84 (2002), 105–146.

[Lu] Lucas, E. Noveaux théorèmes d'Arithmétique supérieure. C.R. Acad. Sci. Paris 83 (1876), 1286–1288.

[R] Rosen, M. A proof of the Lucas-Lehmer test. American Math. Monthly 95 (1988), 855–856.

[S1] Silverman, J.H. The arithmetic of elliptic curves. Springer GTM 106, 1986.

[S2] Silverman, J.H. Advanced topics in the arithmetic of elliptic curves. Springer GTM 151, 1994.