

1.1 THE REAL NUMBERS

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- The Rational Numbers
- The Number Line
- The Real Numbers
- Absolute Value

In arithmetic we use only positive numbers and zero, but in algebra we use negative numbers also. The numbers that we use in algebra are called the real numbers. We start the discussion of the real numbers with some simpler sets of numbers.

The Integers

The most fundamental collection or **set** of numbers is the set of **counting numbers** or **natural numbers**. Of course, these are the numbers that we use for counting. The set of natural numbers is written in symbols as follows.

The Natural Numbers

$$\{1, 2, 3, \dots\}$$

Braces, $\{ \}$, are used to indicate a set of numbers. The three dots after 1, 2, and 3, which are read “and so on,” mean that the pattern continues without end. There are infinitely many natural numbers.

The natural numbers, together with the number 0, are called the **whole numbers**. The set of whole numbers is written as follows.

The Whole Numbers

$$\{0, 1, 2, 3, \dots\}$$

Although the whole numbers have many uses, they are not adequate for indicating losses or debts. A debt of \$20 can be expressed by the negative number -20 (negative twenty). See Fig. 1.1. When a thermometer reads 10 degrees below zero on a Fahrenheit scale, we say that the temperature is -10°F . See Fig. 1.2. The whole numbers together with the negatives of the counting numbers form the set of **integers**.

The Integers

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

The Rational Numbers

A **rational number** is any number that can be expressed as a ratio (or quotient) of two integers. The set of rational numbers includes both the positive and negative fractions. We cannot list the rational numbers as easily as we listed the numbers in the other sets we have been discussing. So we write the set of rational numbers in symbols using **set-builder notation** as follows.

The Rational Numbers

$$\left\{ \frac{a}{b} \mid a \text{ and } b \text{ are integers, with } b \neq 0 \right\}$$

↑ The set of ↑ such that ↑ conditions

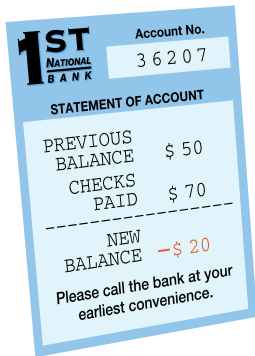
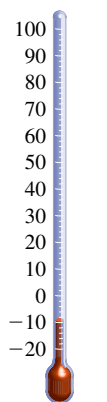


FIGURE 1.1



Degrees
Fahrenheit

FIGURE 1.2

helpful hint

Rational numbers are used for ratios. For example, if 2 out of 5 students surveyed attend summer school, then the ratio of students who attend summer school to the total number surveyed is $2/5$. Note that the ratio $2/5$ does not tell how many were surveyed or how many attend summer school.

We read this notation as “the set of numbers of the form $\frac{a}{b}$ such that a and b are integers, with $b \neq 0$.” Note how we use the letters a and b to represent numbers here. A letter used to represent some numbers is called a **variable**.

Examples of rational numbers are

$$\frac{3}{1}, \frac{5}{4}, -\frac{7}{10}, \frac{0}{6}, \frac{5}{1}, -\frac{77}{3}, \text{ and } \frac{-3}{-6}.$$

Note that we usually use simpler forms for some of these rational numbers. For instance, $\frac{3}{1} = 3$ and $\frac{0}{6} = 0$. The integers are rational numbers because any integer can be written with a denominator of 1.

If you divide the denominator into the numerator, then you can convert a rational number to decimal form. As a decimal, every rational number either repeats indefinitely (for example, $\frac{1}{3} = 0.333 \dots$) or terminates (for example, $\frac{1}{8} = 0.125$).

The Number Line

The number line is a diagram that helps us to visualize numbers and their relationships to each other. A number line is like the scale on the thermometer in Fig. 1.2. To construct a number line, we draw a straight line and label any convenient point with the number 0. Now we choose any convenient length and use it to locate other points. Points to the right of 0 correspond to the positive numbers, and points to the left of 0 correspond to the negative numbers. The number line is shown in Fig. 1.3.

**FIGURE 1.3**

The numbers corresponding to the points on the line are called the **coordinates** of the points. The distance between two consecutive integers is called a **unit** and is the same for any two consecutive integers. The point with coordinate 0 is called the **origin**. The numbers on the number line increase in size from left to right. *When we compare the size of any two numbers, the larger number lies to the right of the smaller on the number line.*

EXAMPLE 1**Comparing numbers on a number line**

Determine which number is the larger in each given pair of numbers.

- a) $-3, 2$ b) $0, -4$ c) $-2, -1$

Solution

- a) The larger number is 2, because 2 lies to the right of -3 on the number line. In fact, any positive number is larger than any negative number.
 b) The larger number is 0, because 0 lies to the right of -4 on the number line.
 c) The larger number is -1 , because -1 lies to the right of -2 on the number line. ■

The set of integers is illustrated or **graphed** in Fig. 1.4 by drawing a point for each integer. The three dots to the right and left below the number line and the blue arrows indicate that the numbers go on indefinitely in both directions.

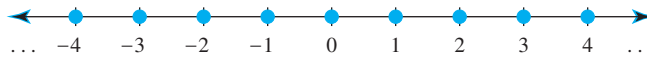


FIGURE 1.4

EXAMPLE 2 Graphing numbers on a number line

List the numbers described, and graph the numbers on a number line.

- The whole numbers less than 4
- The integers between 3 and 9
- The integers greater than -3

Solution

- a) The whole numbers less than 4 are 0, 1, 2, and 3. These numbers are shown in Fig. 1.5.



FIGURE 1.5

- b) The integers between 3 and 9 are 4, 5, 6, 7, and 8. Note that 3 and 9 are not considered to be *between* 3 and 9. The graph is shown in Fig. 1.6.

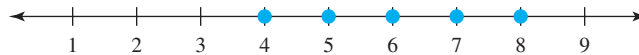


FIGURE 1.6

- c) The integers greater than -3 are -2 , -1 , 0 , 1 , and so on. To indicate the continuing pattern, we use three dots on the graph shown in Fig. 1.7.

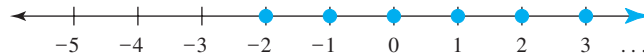


FIGURE 1.7

The Real Numbers

For every rational number there is a point on the number line. For example, the number $\frac{1}{2}$ corresponds to a point halfway between 0 and 1 on the number line, and $-\frac{5}{4}$ corresponds to a point one and one-quarter units to the left of 0, as shown in Fig. 1.8. Since there is a correspondence between numbers and points on the number line, the points are often referred to as numbers.

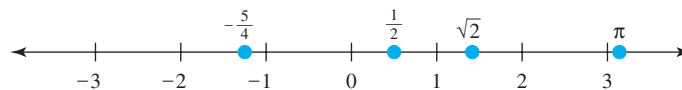


FIGURE 1.8

The set of numbers that corresponds to *all* points on a number line is called the set of **real numbers**. A graph of the real numbers is shown on a number line by shading all points as in Fig. 1.9. All rational numbers are real numbers, but there are

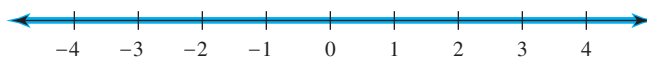


FIGURE 1.9

study tip

Start a personal library. This book as well as other books that you study from should be the basis for your library. You can also add books to your library at garage-sale prices when your bookstore sells its old texts. If you need to reference some material in the future, it is much easier to use a familiar book.

calculator

TOTAL PART %TOTAL \times^y

close-up

A calculator can give rational approximations for irrational numbers such as $\sqrt{2}$ and π .

$\sqrt{(2)}$	1.414213562
π	3.141592654

The calculator screens in this text may differ from the screen of the calculator model you use. If so, you may have to consult your manual to get the desired results.

points on the number line that do not correspond to rational numbers. Those real numbers that are not rational are called **irrational**. An irrational number cannot be written as a ratio of integers. It can be shown that numbers such as $\sqrt{2}$ (the square root of 2) and π (Greek letter pi) are irrational. The number $\sqrt{2}$ is a number that can be multiplied by itself to obtain 2 ($\sqrt{2} \cdot \sqrt{2} = 2$). The number π is the ratio of the circumference and diameter of any circle. Irrational numbers are not as easy to represent as rational numbers. That is why we use symbols such as $\sqrt{2}$, $\sqrt{3}$, and π for irrational numbers. When we perform computations with irrational numbers, we use rational approximations for them. For example, $\sqrt{2} \approx 1.414$ and $\pi \approx 3.14$. The symbol \approx means “is approximately equal to.” Note that not all square roots are irrational. For example, $\sqrt{9} = 3$, because $3 \cdot 3 = 9$. We will deal with irrational numbers in greater depth when we discuss roots in Chapter 9.

Figure 1.10 summarizes the sets of numbers that make up the real numbers, and shows the relationships between them.

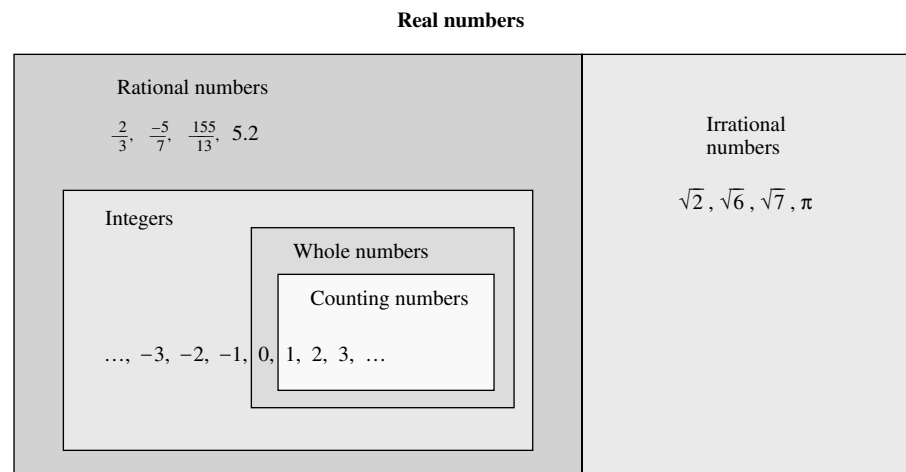


FIGURE 1.10

EXAMPLE 3

Types of numbers

Determine whether each statement is true or false.

- Every rational number is an integer.
- Every counting number is an integer.
- Every irrational number is a real number.

Solution

- False. For example, $\frac{1}{2}$ is a rational number that is not an integer.
- True, because the integers consist of the counting numbers, the negatives of the counting numbers, and zero.
- True, because the rational numbers together with the irrational numbers form the real numbers. ■

helpful hint

In mathematics, every statement is either true or false. If there is one exception to the statement, then the statement is false. If there is an exception and that is part of the statement, then the statement is true.

Absolute Value

The concept of absolute value will be used to define the basic operations with real numbers in Section 1.3. The **absolute value** of a number is the number's distance from 0 on the number line. For example, the numbers 5 and -5 are both five units away from 0 on the number line. So the absolute value of each of these numbers

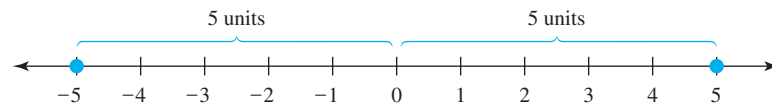


FIGURE 1.11

is 5. See Fig. 1.11. We write $|a|$ for “the absolute value of a .” So

$$|5| = 5 \quad \text{and} \quad |-5| = 5.$$

The notation $|a|$ represents distance, and distance is never negative. So $|a|$ is greater than or equal to zero for any real number a .

EXAMPLE 4 Finding absolute value

Evaluate.

a) $|3|$ b) $|-3|$ c) $|0|$ d) $\left|\frac{2}{3}\right|$ e) $|-0.39|$

Solution

a) $|3| = 3$ because 3 is three units away from 0.

b) $|-3| = 3$ because -3 is three units away from 0.

c) $|0| = 0$ because 0 is zero units away from 0.

d) $\left|\frac{2}{3}\right| = \frac{2}{3}$ e) $|-0.39| = 0.39$ ■

study tip

Exercise sets are designed to gradually increase in difficulty. So start from the beginning and work lots of exercises. If you get stuck, go back and study the corresponding examples. If you are still stuck, move ahead to a new type of exercise.

Two numbers that are located on opposite sides of zero and have the same absolute value are called **opposites** of each other. The numbers 5 and -5 are opposites of each other. We say that the opposite of 5 is -5 and the opposite of -5 is 5. The symbol “ $-$ ” is used to indicate “opposite” as well as “negative.” When the negative sign is used before a number, it should be read as “negative.” When it is used in front of parentheses or a variable, it should be read as “opposite.” For example, $-(5) = -5$ means “the opposite of 5 is negative 5,” and $-(-5) = 5$ means “the opposite of negative 5 is 5.” Zero does not have an opposite in the same sense as nonzero numbers. Zero is its own opposite. We read $-(0) = 0$ as the “the opposite of zero is zero.”

In general, $-a$ means “the opposite of a .” If a is positive, $-a$ is negative. If a is negative, $-a$ is positive. Opposites have the following property.

Opposite of an Opposite

For any real number a ,

$$-(-a) = a.$$

Remember that we have defined $|a|$ to be the distance between 0 and a on the number line. Using opposites, we can give a symbolic definition of absolute value.

Absolute Value

$$|a| = \begin{cases} a & \text{if } a \text{ is positive or zero} \\ -a & \text{if } a \text{ is negative} \end{cases}$$

EXAMPLE 5 Using the symbolic definition of absolute value

Evaluate.

- a) $|8|$
 b) $|0|$
 c) $|-8|$

Solution

- a) If a is positive, then $|a| = a$. Since $8 > 0$, $|8| = 8$.
 b) If a is 0, then $|a| = a$. So $|0| = 0$.
 c) If a is negative, then $|a| = -a$. So $|-8| = -(-8) = 8$. ■

WARM - UPS**True or false? Explain your answer.**

1. The natural numbers and the counting numbers are the same.
2. The number 8,134,562,877,565 is a counting number.
3. Zero is a counting number.
4. Zero is not a rational number.
5. The opposite of negative 3 is positive 3.
6. The absolute value of 4 is -4 .
7. $-(-9) = 9$
8. $-(-b) = b$ for any number b .
9. Negative six is greater than negative three.
10. Negative five is between four and six.

1.1 EXERCISES

Reading and Writing After reading this section write out the answers to these questions. Use complete sentences.

1. What are the integers?
2. What are the rational numbers?
3. What is the difference between a rational and an irrational number?
4. What is a number line?
5. How do you know that one number is larger than another?
6. What is the ratio of the circumference and diameter of any circle?

Determine which number is the larger in each given pair of numbers. See Example 1.

- | | |
|----------------|---------------|
| 7. $-3, 6$ | 8. $7, -10$ |
| 9. $0, -6$ | 10. $-8, 0$ |
| 11. $-3, -2$ | 12. $-5, -8$ |
| 13. $-12, -15$ | 14. $-13, -7$ |

List the numbers described and graph them on a number line. See Example 2.

15. The counting numbers smaller than 6
16. The natural numbers larger than 4
17. The whole numbers smaller than 5
18. The integers between -3 and 3
19. The whole numbers between -5 and 5
20. The integers smaller than -1
21. The counting numbers larger than -4
22. The natural numbers between -5 and 7
23. The integers larger than $\frac{1}{2}$
24. The whole numbers smaller than $\frac{7}{4}$

Determine whether each statement is true or false. Explain your answer. See Example 3.

25. Every integer is a rational number.
26. Every counting number is a whole number.
27. Zero is a counting number.
28. Every whole number is a counting number.
29. The ratio of the circumference and diameter of a circle is an irrational number.
30. Every rational number can be expressed as a ratio of integers.
31. Every whole number can be expressed as a ratio of integers.
32. Some of the rational numbers are integers.
33. Some of the integers are natural numbers.
34. There are infinitely many rational numbers.
35. Zero is an irrational number.
36. Every irrational number is a real number.

Determine the values of the following. See Examples 4 and 5.

- | | | |
|---------------|--------------------------------|---------------------------------|
| 37. $ -6 $ | 38. $ 4 $ | 39. $ 0 $ |
| 40. $ 2 $ | 41. $ 7 $ | 42. $ -7 $ |
| 43. $ -9 $ | 44. $ -2 $ | 45. $ -45 $ |
| 46. $ -30 $ | 47. $\left \frac{3}{4}\right $ | 48. $\left -\frac{1}{2}\right $ |
| 49. $ -5.09 $ | 50. $ 0.00987 $ | |

Select the smaller number in each given pair of numbers.

- | | |
|----------------------------------|--------------------------------|
| 51. $-16, 9$ | 52. $-12, -7$ |
| 53. $-\frac{5}{2}, -\frac{9}{4}$ | 54. $\frac{5}{8}, \frac{6}{7}$ |
| 55. $ -3 , 2$ | 56. $ -6 , 0$ |
| 57. $ -4 , 3$ | 58. $ 5 , -4$ |

Which number in each given pair has the larger absolute value?

59. $-5, -9$ 60. $-12, -8$ 61. $16, -9$ 62. $-12, 7$

True or false? Explain your answer.

63. If we add the absolute values of -3 and -5 , we get 8.
64. If we multiply the absolute values of -2 and 5 , we get 10.
65. The absolute value of any negative number is greater than 0.
66. The absolute value of any positive number is less than 0.
67. The absolute value of -9 is larger than the absolute value of 6.
68. The absolute value of 12 is larger than the absolute value of -11 .

GETTING MORE INVOLVED



69. **Writing.** Find a real-life question for which the answer is a rational number that is not an integer.



70. **Exploration.** a) Find a rational number between $\frac{1}{3}$ and $\frac{1}{4}$.
 b) Find a rational number between -3.205 and -3.114 .
 c) Find a rational number between $\frac{2}{3}$ and 0.6667 .
 d) Explain how to find a rational number between any two given rational numbers.



71. **Discussion.** Suppose that a is a negative real number. Determine whether each of the following is positive or negative, and explain your answer.
 a) $-a$ b) $|-a|$ c) $-|a|$ d) $-(-a)$ e) $-|-a|$