1.2 The Integers and Rational Numbers

The elements of the set of integers:

$$\mathbb{Z} = \{..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...\}$$

consist of three types of numbers:

I. The (positive) natural numbers $\{1, 2, 3, 4, 5, ...\}$,

II. The negative integers $\{-1, -2, -3, -4, -5, ...\}$, and

III. The number 0.

The ordering on the natural numbers extends to an ordering of the integers:

$$... < -5 < -4 < -3 < -2 < -1 < 0 < 1 < 2 < 3 < 4 < 5 < ...$$

but there is no well-ordered principle for the integers since many subsets of $\mathbb{Z}$ (including $\mathbb{Z}$ itself) have no smallest element.

Negative numbers may seem obvious today, but there was a long period of time when only positive numbers were used. The introduction of 0 is often cited as evidence of the scientific superiority of the eastern cultures during the middle ages. It is remarkable that we can easily (if tediously) extend the operations of addition and multiplication of natural numbers to include 0 and the negative integers, maintaining all the fundamental laws of arithmetic. Negative numbers are necessary today because, in our society based upon the right to the pursuit of happiness through credit card debt, we need to be able to **subtract**!

Definition of Addition. $a + b$ is defined on a case by case basis:

Case I. $b = n$ is positive. Then $a + n$ is defined by induction.

(i) $a + 1$ is the next number after a,

(ii) Each $a + (n + 1)$ is the next number after $a + n$

Case II. $b = -n$ is negative. Then $a + (-n)$ is also defined by induction.

(i′) $a + (-1)$ is the number immediately before a, and

(ii′) Each $a + (-(n + 1))$ is the number immediately before $a + (-n)$

Case III. $b = 0$. By definition,

$$a + 0 = a$$

The associative and commutative laws of addition can now be proved for this new definition of addition by the same proof-by-induction strategy we used in §1.1 (but it is tedious, involving lots of different cases, so we won't do it!) Since the first case of the definition of addition is identical to the definition of addition of natural numbers from §1.1, the two additions give the same result when applied to two natural numbers.

Before we tackle multiplication, we introduce:

The Negation Transformation: Negation is the function:

$$- : \mathbb{Z} \to \mathbb{Z}$$

defined by: $-(n) = -n$, $-(0) = 0$ and $-(-n) = n$. It is clear that

$$-(-a) = a \text{ for all integers } a$$

(double negatives cancel) and that taking negatives reverses order:

$$\text{if } \; a < b \; \text{ then } \; -b < -a$$

Negation has the following three important properties, too:

Proposition 1.2.1. *For all integers a,*

$$a + (-a) = 0$$

(and because of this, we say that $-a$ is an **additive inverse** *of a).*

Proof: This is clearly true for $a = 0$ since $0 + (-0) = 0 + 0 = 0$. Otherwise, either a or $-a$ must be a natural number, so it is enough by the commutativity of addition to prove the "additive inverse" sentence $n + (-n) = 0$ for all n, which we will now do by induction:

(i) $1 + (-1)$ is the number before 1, by addition definition (i′), which is 0.

(ii) For each n, once we know $n + (-n) = 0$, then since $-(n + 1)$ is the number before $-n$, we also know that:

$$(n + 1) + (-(n + 1)) = (n + 1) + (-n + (-1))$$

by addition definition (i′), and then

$$(n + 1) + (-n + (-1)) = (1 + (-1)) + (n + (-n))$$

by the commutative and associative laws of addition. But now:

$$1 + (-1) = 0, \quad n + (-n) = 0 \quad \text{and} \quad 0 + 0 = 0$$

allow us to conclude that $(n + 1) + (-(n + 1)) = 0$, hence the induction.

Proposition 1.2.2. *$-a$ is the* **only** *additive inverse of a.*

Proof: Suppose b is another additive inverse of a. Then:

$$-a + (a + b) = -a + 0 = -a$$

but using the associative law of addition, we also have:

$$-a + (a + b) = (-a + a) + b = 0 + b = b$$

so $-a = b$. Thus any other additive inverse of a is equal to $-a$, which is the same thing as saying that $-a$ is the only additive inverse of a.

Proposition 1.2.3. *Negation is a linear transformation, meaning:*

$$-(a+b) = -a + (-b)$$

Proof: By the laws of addition and Proposition 1.2.1:

$$(a+b) + (-a + (-b)) = (a + (-a)) + (b + (-b)) = 0$$

so $-a + (-b)$ is an additive inverse of $a + b$. From Proposition 1.2.2, we know that there is only one additive inverse of $a + b$, so $-a + (-b) = -(a + b)$.

Now we are finally ready for the:

Definition of Subtraction: For all integers a and b:

$$a - b = a + (-b)$$

(that is, subtraction of b is defined to be addition of the additive inverse of b)

Finally (for the integers), we use negatives to define multiplication:

Definition of Multiplication. ab is defined on a case-by-case basis.

Case I. $b = n$ is a positive. Then $a \times n$ is defined by induction:

(i) $a \times 1 = a$,

(ii) Each $a \times (n + 1) = a \times n + a$.

Case II. $b = -n$ is negative. Then $a \times (-n)$ is defined to be $-(a \times n)$.

Case III. $b = 0$. Then $a \times 0 = 0$

Remark: The definitions in Cases II and III are forced upon us, if we want multiplication to satisfy the distributive law! (see the exercises) Notice also that

$$a \times (-1) = -a$$

so the negation transformation is the same as multiplication by -1.

Again, we will not go through the tedious exercise of proving the rest of the basic laws of arithmetic, but it can be done with only induction and these definitions, if you are willing to work through all the cases.

Recap: The new number 0 is the **additive identity**, meaning that:

$$a + 0 = a$$

for all integers a. Negation takes an integer to its additive inverse, allowing us to define subtraction as addition of the additive inverse. Note that 1 is the **multiplicative identity**, meaning that $a \times 1 = a$ for all integers a, but integer multiplicative inverses only exist for the integers 1 and -1.

The rational numbers can be thought of geometrically as slopes of lines:

$$\mathbb{Q} = \{\text{(slopes of) lines that pass through } (0,0) \text{ and a point } (b,a)\}$$

where $a, b \in \mathbb{Z}$ and $b \neq 0$ (so the line isn't vertical.)

The line L passing through $(0,0)$ and (b,a) has equation:

$$by = ax$$

and the slope is also the y-coordinate of the intersection of L with the (vertical) line $x = 1$. In particular, different lines through the origin have different slopes.

Many different points with integer coordinates will lie on the same line L! If (b', a') is another point with integer coordinates, then by the equation above for the line L, we see that (b', a') is also on L exactly when:

$$ba' = ab'$$

Definition: An integer fraction is a symbol of the form:

$$\frac{a}{b}$$

where $a, b \in \mathbb{Z}$ and $b \neq 0$. Two integer fractions are **equivalent**, written:

$$\frac{a}{b} \sim \frac{a'}{b'}$$

if (b, a) and (b', a') are on the same line through the origin.

Note: The symbol "$\sim$" is called a **relation**, and it is easy to see that:

(i) $\sim$ is **reflexive**, meaning that $\frac{a}{b} \sim \frac{a}{b}$

(ii) $\sim$ is **symmetric** meaning that if $\frac{a}{b} \sim \frac{a'}{b'}$ then $\frac{a'}{b'} \sim \frac{a}{b}$

(iii) $\sim$ is **transitive** meaning that if $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{a'}{b'} \sim \frac{a''}{b''}$ then $\frac{a}{b} \sim \frac{a''}{b''}$

(A relation satisfying (i)-(iii) is called an **equivalence relation**.)

Definition: The **equivalence class**

$$\left[\frac{a}{b}\right]$$

is the set of all fractions that are equivalent to $\frac{a}{b}$. Notice that:

$$\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$$

whenever $\frac{a}{b} \sim \frac{a'}{b'}$, that is, whenever (b, a) and (b', a') are on the same line through the origin, or, as we noticed above, whenever $ba' = ab'$.

Thus we can reinterpret the rational numbers as:

$$\mathbb{Q} = \{\text{equivalence classes of integer fractions}\}$$

and this reinterpretation is very useful for seeing the arithmetic of $\mathbb{Q}$.

Before we do this, let's notice that the rational numbers are still ordered:

$$\left[\frac{a}{b}\right] < \left[\frac{c}{d}\right]$$

if the line through $(0,0)$ and (b,a) intersects the vertical line $x = 1$ at a point that is **below** the intersection of the line through $(0,0)$ and (d,c).

Unlike the integers, there is no such thing as the next rational number after a rational number $[\frac{a}{b}]$, so there is no way to use induction to define addition. Instead, we use the rule for adding fractions that we learned in gradeschool.

Definition of Addition: Given rational numbers $[\frac{a}{b}]$ and $[\frac{c}{d}]$, then:

$$\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$$

(using the arithmetic of integers to define the numerator and denominator).

But we need to check something!

Is addition is well-defined? Here's the problem. If

$$\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right] \text{ and } \left[\frac{c}{d}\right] = \left[\frac{c'}{d'}\right]$$

how do we know that:

$$\left[\frac{ad+bc}{bd}\right] = \left[\frac{a'd'+b'c'}{b'd'}\right] \ ?$$

This isn't obvious, and it needs to be checked, because if it weren't true, then this would be **bad** definition of addition, because it would only be an addition of integer fractions, and not of rational numbers, which are equivalence classes of integer fractions. This problem will arise whenever we try to make a definition involving equivalence classes of fractions (or other things). So beware!

Proof that addition is well-defined: Suppose:

$$\frac{a}{b} \sim \frac{a'}{b'} \text{ and } \frac{c}{d} \sim \frac{c'}{d'}$$

This means that $ba' = ab'$ and $dc' = cd'$. But then:

$$\begin{aligned}(bd)(a'd' + b'c') &= (ba')(dd') + (c'd)(bb') \\ &= (ab')(dd') + (cd')(bb') = (ad+bc)(b'd')\end{aligned}$$

(substituting, and applying the laws of arithmetic for integers). So:

$$\frac{ad+bc}{bd} \sim \frac{a'd'+b'c'}{b'd'}$$

as desired.

Now that the definition is OK, it is very useful!

Addition is associative. Proof:

$$\left(\left[\frac{a}{b}\right]+\left[\frac{c}{d}\right]\right)+\left[\frac{e}{f}\right] = \left[\frac{ad+bc}{bd}\right]+\left[\frac{e}{f}\right] = \left[\frac{((ad)f+(bc)f)+(bd)e}{(bd)f}\right]$$

and

$$\left[\frac{a}{b}\right]+\left(\left[\frac{c}{d}\right]+\left[\frac{e}{f}\right]\right) = \left[\frac{a}{b}\right]+\left[\frac{cf+de}{df}\right] = \left[\frac{a(df)+(b(cf)+b(de))}{b(df)}\right]$$

and these are the same because of the associative laws for integer arithmetic!

Similarly, you can prove that addition is commutative.

Definition of Multiplication:

$$\left[\frac{a}{b}\right]\left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$$

Proof that multiplication is well-defined: If $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$, then $ba' = ab'$ and $dc' = cd'$, so $(bd)(a'c') = (ba')(dc') = (ab')(cd') = (ac)(b'd')$. But this gives us:

$$\frac{ac}{bd} \sim \frac{a'c'}{b'd'}$$

which is just what we needed to check.

It is now easy to prove the associative and commutative laws for the multiplication of rational numbers. To see that multiplication distributes with addition, though, we need an extra:

Proposition 1.2.4. *Suppose $a = a'f$ and $b = b'f$, in which case we say that f is a* **common factor** *of both a and b. Then:*

$$\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$$

(i.e. we can cancel common factors of the numerator and denominator.)

Proof: We need to show that $\frac{a}{b} \sim \frac{a'}{b'}$. But:

$$ab' = (a'f)b'$$

by substituting, and likewise,

$$ba' = (b'f)a'$$

so indeed the two fractions are equivalent.

Multiplication distributes with addition. Proof:

$$\left(\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right]\right)\left[\frac{e}{f}\right] = \left[\frac{ad+bc}{bd}\right]\left[\frac{e}{f}\right] = \left[\frac{(ad)e+(bc)e}{(bd)f}\right] \text{ and}$$

$$\left[\frac{a}{b}\right]\left[\frac{e}{f}\right] + \left[\frac{c}{d}\right]\left[\frac{e}{f}\right] = \left[\frac{ae}{bf}\right] + \left[\frac{ce}{df}\right] = \left[\frac{(ae)(df)+(bf)(ce)}{(bf)(df)}\right]$$

There is a common factor of f in the numerator and denominator of the second fraction. Once we cancel it (Proposition 1.2.4), we see that the two rational numbers are the same!

Now for some extra goodies:

The additive identity is the rational number:

$$\left[\frac{0}{d}\right]$$

(and it doesn't matter what d is, as long as it isn't 0).

Proof: $\frac{0}{d} \sim \frac{0}{d}'$ since $0 \times d' = 0 = 0 \times d$, so all choices of denominator give the same rational number (namely the slope of the x-axis!). Next:

$$\left[\frac{a}{b}\right] + \left[\frac{0}{d}\right] = \left[\frac{ad}{bd}\right] = \left[\frac{a}{b}\right]$$

(using Proposition 1.2.4 again) proves that $[\frac{0}{d}]$ is the additive identity.

Notation: Mathematicians always denote the additive identity by:

$$0$$

so we will, too.

Every rational number has an additive inverse. Proof:

$$\left[\frac{a}{b}\right] + \left[\frac{-a}{b}\right] = \left[\frac{ab+(-a)b}{b^2}\right] = \left[\frac{a+(-a)}{b}\right] = \left[\frac{0}{b}\right] = 0$$

using Proposition 1.2.4. So $[\frac{-a}{b}]$ is an additive inverse to $[\frac{a}{b}]$.

As in Proposition 1.2.2, this is the **only** additive inverse!

Definition of Subtraction:

$$\left[\frac{a}{b}\right] - \left[\frac{c}{d}\right] = \left[\frac{a}{b}\right] + \left[\frac{-c}{d}\right]$$

As always, subtraction is addition of the additive inverse

The multiplicative identity is the rational number:

$$\left[\frac{d}{d}\right]$$

(and it doesn't matter what d is, as long as it isn't 0)

Proof: $\frac{d}{d} \sim \frac{d'}{d'}$ since $dd' = d'd$. So it doesn't matter what d is, and:

$$\left[\frac{a}{b}\right]\left[\frac{d}{d}\right] = \left[\frac{ad}{bd}\right] = \left[\frac{a}{b}\right]$$

by Proposition 1.2.4. This is what we needed to prove. Again, it is easy to see that this is the **only** multiplicative identity.

Notation: Once again, we follow mathematical custom and write:

$$1$$

for the multiplicative identity.

Every rational (except 0) has a multiplicative inverse. Proof: Every rational number other than 0 is of the form $\left[\frac{a}{b}\right]$ where $a \neq 0$. Then:

$$\left[\frac{a}{b}\right]\left[\frac{b}{a}\right] = \left[\frac{ab}{ba}\right] = 1$$

so $[\frac{b}{a}]$ is the one and only multiplicative inverse (or **reciprocal**) of $[\frac{a}{b}]$.

Definition of Division (by anything other than 0):

$$\left[\frac{a}{b}\right] \div \left[\frac{c}{d}\right] = \left[\frac{a}{b}\right]\left[\frac{d}{c}\right]$$

(i.e. division is multiplication by the reciprocal)

Finally, I want to talk about one last definition:

Definition of Lowest Terms: An integer fraction (not rational number!)

$$\frac{a}{b}$$

is in lowest terms if $b > 0$ and a and b have no common factors other than 1 and -1 (which are common factors of all integers, hence "uninteresting"!).

Proposition 1.2.5. *Every rational number* $\left[\frac{a}{b}\right]$ *contains exactly one fraction in lowest terms (in the equivalence class).*

Idea of Proof: Start with the fraction $\frac{a}{b}$, which may not be in lowest terms. By cancelling out all the (interesting) common factors of a and b and also -1 if necessary (to make $b > 0$) we arrive at a fraction in lowest terms. This shows that there is **at least** one fraction in lowest terms in the equivalence class $\left[\frac{a}{b}\right]$. To see that there cannot be more than one, we will need to know a bit more about prime numbers, which we will work out later in the course (§2.2).

This allows us to redefine one more time:

$$\mathbb{Q} = \{\text{integer fractions } \tfrac{a}{b} \text{ that are in lowest terms}\}$$

In particular, we get an inclusion of sets:

$$\mathbb{Z} \subset \mathbb{Q}$$

by identifying each integer a with the fraction $\frac{a}{1}$, which is clearly in lowest terms. When we do this, we see something very important. Namely:

$$\left[\frac{a}{1}\right] + \left[\frac{b}{1}\right] = \left[\frac{a+b}{1}\right] \text{ and } \left[\frac{a}{1}\right]\left[\frac{b}{1}\right] = \left[\frac{ab}{1}\right]$$

so addition and multiplication are the same regardless of whether we view a and b as integers, or as rational numbers!

Finally, as in §1.1, we finish with another gem from ancient Greece:

Theorem (Pythagoras): There is no square root of 2 in $\mathbb{Q}$.

Proof: If there were a rational number square root of 2, then:

$$\left[\frac{a}{b}\right]^2 = \left[\frac{a^2}{b^2}\right] = \left[\frac{2}{1}\right]$$

would tell us that:

$$2b^2 = a^2$$

so that 2 divides a^2. But then it would follow that **2 divides a** since the square of an odd number is odd. Thus $a = 2c$ for some c, and then:

$$2b^2 = (2c)^2 = 4c^2 \text{ so } b^2 = 2c^2$$

But then 2 divides b^2 so it would follow as above that **2 divides b**. In other words, 2 would be an interesting common factor of a and b. All this would be true no matter what fraction a/b we chose to represent the rational square root of 2. In other words, **there would be no way to put such a rational number in lowest terms!** This contradicts Proposition 1.2.5, so there cannot be such a rational number.

Recap: Rational numbers are equivalence classes of integer fractions, and they have a very satisfactory arithmetic, with additive inverses and multiplicative inverses (of everything except 0) allowing us to define subtraction and exact division (by anything except 0). On the other hand, from the point of view of geometry, they are less satisfactory, since a perfectly reasonable length ($\sqrt{2}$) cannot be represented by a rational number.

1.2.1 Integer and Rational Number Exercises

In the first three exercises, we consider arithmetic in an abstract setting. The idea is that many of the results of this section are not special properties of the integers or rational numbers, but rather follow from the laws of arithmetic themselves.

2-1 Suppose S is a set and $+$ is an addition rule for elements of S that satisfies:

- the associative law: $(s+t)+u = s+(t+u)$ for all $s, t, u \in S$, and
- the commutative law: $s+t = t+s$ for all $s, t \in S$.

(a) Prove that there is at most one additive identity element in S. That is, prove there is at most one element $z \in S$ such that:

$$s + z = s \text{ for all } s \in S$$

(Mathematicians tell us to rename this element 0)

Hint: If y is another additive identity, think about $y + z$ in two ways.

(b) Assuming that there is an additive identity element (renamed 0), prove that each $s \in S$ has at most one additive inverse in S. That is, prove that there is at most one element $t \in S$ so that:

$$s + t = 0$$

(Mathematicians tell us to rename this element $-s$)

(c) Prove that if s has an additive inverse $-s$, then the additive inverse of $-s$ is s. That is, prove: $-(-s) = s$

Definition: A set S with an addition rule $+$ with a 0 and additive inverses of everything is an **Abelian group**.

2-2 If S is an Abelian group with a multiplication rule $\times$ satisfying:

- the associative law: $(s \times t) \times u = s \times (t \times u)$ and
- the two-sided distributive law with addition:

$$(s+t) \times u = s \times u + t \times u \text{ and } u \times (s+t) = u \times s + u \times t$$

(we will not, for now, assume that multiplication is commutative!)

(a) Prove that $s \times 0 = 0$ and $0 \times s = 0$ for all $s \in S$.

Hint: Consider $s \times (0 + 0)$ and $(0 + 0) \times s$.

(b) Prove that for all $s, t \in S$

$$s \times (-t) = -(s \times t) \text{ and } -(s \times t) = (-s) \times t$$

Hint: Consider $s \times (t + (-t))$ and $(s + (-s)) \times t$.

(c) Prove that $(-s) \times (-t) = s \times t$ for all $s, t \in S$.

Definition: An Abelian group S with an associative and two-sided distributive multiplication rule is called a **ring**.

Examples: $\mathbb{Z}$ and $\mathbb{Q}$ are rings with a commutative multiplication rule. The $n \times n$ matrices (for $n > 1$) with entries in $\mathbb{Z}$ or $\mathbb{Q}$ (or any ring) are themselves a ring with a non-commutative matrix multiplication!

2-3 Suppose S is a ring with a commutative multiplication rule.

(a) Prove that there is at most one multiplicative identity in S.

(Mathematicians tell us to rename this 1)

(b) Prove that each element $s \in S$ has at most one multiplicative inverse (reciprocal) element $t \in S$.

(Mathematicians tell us to rename this $1/s$.)

(c) Prove that 0 does not have a multiplicative inverse (unless $0 = 1$). Discuss what the ring would look like if $0 = 1$.

(d) If s has a multiplicative inverse, prove that:

$$\frac{1}{1/s} = s$$

Hint: Exercise 2.3 is very similar to Exercise 2.1.

Definition: A ring S with a commutative multiplication and a multiplicative identity $1 \in S$, such that every element of S (except 0) has a multiplicative inverse is called a **field**.

Our Only Example of a Field (so far): $\mathbb{Q}$ is a field.

2-4 Prove the following:

(a) If $a, b \in \mathbb{Z}$ and $ab = 0$, then either $a = 0$ or $b = 0$ (or both).

Hint: If a and b are both natural numbers, then $ab \neq 0$ because it is a natural number! What are the other possibilities for a and b?

(b) If $a, b, c \in \mathbb{Z}$ and $ab = cb$ and $b \neq 0$, then $a = c$.

Hint: Find a way to use (a).

2-5 Recall that:

$$\frac{a}{b} \sim \frac{a'}{b'} \text{ exactly when } ba' = ba'$$

We'll check "algebraically" that this really is an equivalence relation.

(i) Reflexive. This is the commutative law for multiplication!

$$\frac{a}{b} \sim \frac{a}{b} \text{ because } ba = ab$$

(ii) Symmetric. This is also the commutative law for multiplication.

$$\text{If } \frac{a}{b} \sim \frac{a'}{b'} \text{ then } ba' = ab' \text{ but then } b'a = a'b \text{ so } \frac{a'}{b'} \sim \frac{a}{b}$$

(iii) Transitive. This is your exercise! You need to explain why

$$ba' = ab' \text{ and } b'a'' = a'b'' \text{ together imply that } ba'' = ab''$$

2-6 Consider the two rational numbers:

$$\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right] \text{ and } \left[\frac{1}{3}\right]$$

Explain carefully why the fact that $\left[\frac{2}{5}\right] \neq \left[\frac{3}{7}\right]$ shows that "dumb" addition:

$$\left[\frac{a}{b}\right] \oplus \left[\frac{c}{d}\right] = \left[\frac{a+c}{b+d}\right]$$

is not well-defined on rational numbers.

2-7 Prove **Pascal's identity**. For natural numbers $m < n$,

$$\frac{n!}{m!(n-m)!} + \frac{n!}{(m-1)!(n-m+1)!} = \frac{(n+1)!}{m!(n-m+1)!}$$

Remark (for your enjoyment): This proves that the "binomial coefficient:"

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

is the $m+1$st number in the $n+1$st row of **Pascal's Triangle:**

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ \vdots \end{array}$$

2-8 The ancient Egyptians had some ideas about fractions, though they apparently didn't like to subtract, didn't like numerators, and didn't like repetitions. The "Egyptian fraction" expansion of a rational number between 0 and 1 is a sum of distinct fractions, all of the form:

$$\frac{1}{n}$$

Here are some examples (I'm going to drop the cumbersome brackets around rational numbers in this problem and from now on!):

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}, \qquad \frac{2}{3} = \frac{1}{2} + \frac{1}{6}$$

($\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$ is not an Egyptian fraction expansion because the $\frac{1}{3}$ repeats)

$$\frac{5}{12} = \frac{1}{3} + \frac{1}{12} = \frac{1}{4} + \frac{1}{6}$$

(so sometimes there is more than one possible expansion).

(a) Find Egyptian fraction expansions for the numbers:

$$\frac{5}{7}, \frac{11}{27}, \frac{19}{49}, \frac{5}{61}$$

(b) Devise a strategy for finding an Egyptian fraction for any m/n (assuming that $m < n$). Hint: You might find induction useful. Apply your strategy to the four numbers above (your calculator will not give you enough accuracy for the last two...you will need a computer!).