A Mathematical Derivation of the General Relativistic Schwarzschild Metric

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ABSTRACT

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We briefly discuss some underlying principles of special and general relativity with the focus on a more geometric interpretation. We outline Einstein's Equations which describes the geometry of spacetime due to the influence of mass, and from there derive the Schwarzschild metric. The metric relies on the curvature of spacetime to provide a means of measuring invariant spacetime intervals around an isolated, static, and spherically symmetric mass M, which could represent a star or a black hole. In the derivation, we suggest a concise mathematical line of reasoning to evaluate the large number of cumbersome equations involved which was not found elsewhere in our survey of the literature.

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1 Introduction to Relativity

A quantitative comprehensive view of the universe was arguably first initiated with Isaac Newton's theory of gravity, a little more than three hundred years ago. It was this theory that first allowed scientists to describe the motion of the heavenly bodies and that of objects on earth with the same principles. In Newtonian mechanics, the universe was thought to be an unbounded, infinite 3-dimensional space modeled by Euclidean geometry, which describes flat space. Thus, any event in the universe could be described by three spatial coordinates and time, generally written as (x, y, z) with the implied concept of an absolute time t.

In 1905, Albert Einstein introduced the Special Theory of Relativity in his paper 'On the Electrodynamics of Moving Bodies.' Special relativity, as it is usually called, postulated two things. First, any physical law which is valid in one reference frame is also valid for any frame moving uniformly relative to the first. A frame for which this holds is referred to as an inertial reference frame. Second, the speed of light in vacuum is the same in all inertial reference frames, regardless of how the light source may be moving.

The first postulate implies there is no preferred set space and time coordinates. For instance, suppose you are sitting at rest in a car moving at constant speed. While looking straight out a side window, everything appears to be moving so quickly! Trees, buildings, and even people are flashing by faster than you can focus on them. However, an observer outside of your vehicle would say that you are the one who appears to be moving. In this case, how should we define the coordinates of you in your car and the observer outside of your car? We could say that the outside observer was simply mistaken, and that you were definitely not moving. Thus, his spatial coordinates were changing while you remained stationary. However, the observer could adamantly argue that you definitely *were* moving, and so it is your spatial coordinates that are changing. Hence, there is no absolute coordinate system that could describe every event in the universe for which all observers would agree and we see that each observer has their own way to measure distances relative to the frame of reference they are in.

It is important to note that special relativity only holds for frames of reference moving uniformly relative to the other, that is, constant velocities and no acceleration. We can illustrate this with a simple example. Imagine a glass of water sitting on a table. According to special relativity, there is no difference in that glass sitting on a table in your kitchen and any other frame with uniform velocity, such as a car traveling at constant speed. The glass of water in the car, assuming a smooth, straight ride with no shaking, turning or bumps, will follow the same laws of physics as it does in your kitchen. In this case, the water in each glass is undisturbed within the glass as time goes on. However, if either reference frame underwent an acceleration, special relativity would no longer hold. For instance, if in your car, you were to suddenly stop, then the water in your glass would likely spill out and you would be forced forward against your seat belt.

1.1 Minkowski Space

Einstein's physical intuition motivated his formulation of special relativity, but his generalization to general relativity would not have occurred without the mathematical formulation given by Hermann Minkowski. In 1907, Minkowski realized the physical notions of Einstein's special relativity could be expressed in terms of events occuring in a universe described with a non-Euclidean geometry. Minkowski took the three spatial dimensions with an absolute time and transformed them into a 4-dimensional manifold that represented spacetime. A manifold is a topological space that is described locally by Euclidean geometry; that is, around every point there is a neighborhood of surrounding points which is approximately flat. Thus, one can think of a manifold as a surface with many flat spacetimes covering it where all of the overlaps are smooth and continuous. A simple example of this is the Earth. Even though the world is known to be spherical, on small scales, such as those we see everyday, it appears to be flat.

In special relativity, the spacetime manifold is actually flat, not just locally but everywhere. However, when we begin the discussion on general relativity, this will not be the case. We define *events* in the spacetime as points on the manifold; in the 4-dimensional spacetime we are dealing with, these points will require four coordinates to uniquely describe an event. These coordinates are usually taken to be a time coordinate and three spatial coordinates. While we could denote these as (t, x, y, z) it is customary to use (x^0, x^1, x^2, x^3) where x^0 refers to the time coordinate. However, we will use them interchangeably as needed. Notice that these are not powers of x, but components of x where x is a position four-vector. Additionally, so that we have the same units for all of the coordinates, we will measure time in units of space. Let us use meters for our units, and notice that if we let $c = 3 \times 10^8 \text{m/s} = 1$ then we can multiply through by seconds and find that 1 second $= 3 \times 10^8$ meters.

Definition 1.1 Minkowski Space The spacetime that Minkowski formulated is called *Minkowski space*. It consists of a description of events characterized by a location (x^0, x^1, x^2, x^3). We can then define an *invariant interval* between two events, a and b, in the spacetime as

$$\begin{aligned} s^2 &= -(x_a^0 - x_b^0)^2 + (x_a^1 - x_b^1)^2 + (x_a^2 - x_b^2)^2 + (x_a^3 - x_b^3)^2 \\ ds^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \end{aligned} \tag{1}$$

It is invariant because another observer using coordinate system (x'^0, x'^1, x'^2, x'^3) would measure the same interval, that is

$$ds^2 = ds'^2 = -(dx'^0)^2 + (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2$$

This invariant interval is analogous to a distance in the flat 3-dimensional space that we are accustomed to. However, one peculiar thing about this "distance" is that it can be negative. We separate the intervals into three types:

$$\begin{aligned} ds^2 &> 0 \quad \text{the interval is spacelike} \\ ds^2 &< 0 \quad \text{the interval is timelike} \\ ds^2 &= 0 \quad \text{the interval is lightlike} \end{aligned}$$

A spacelike interval is one for which an inertial frame can be found such that two events are simultaneous. No material object can be present at two events which are separated by a spacelike interval. However, a timelike interval can describe two events of the same material object. For instance, as an initial event, say you are holding a ball. Suppose you then toss it to a friend, who catches it as a final event. Then the difference between the initial and final events of the ball is a timelike interval. If a ray of light could travel between two events then we say that the interval is lightlike. This is also sometimes referred to as a null geodesic. For a single object, we define the set of all past and future events of that object as the **worldline** of that object. Thus, if two events are on the worldline of a material object, then they are separated by a timelike interval. If two events are on the worldline of a photon, then they are separated by a lightlike interval.

Another way to write equation (1) is in the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

for μ and ν values of $\{0, 1, 2, 3\}$ where we implement Einstein's summation notation. This notation is a simple way in which to condense many terms of a summation. For instance, the above equation could be written as 16 terms

$$ds^2 = \eta_{00} dx^0 dx^0 + \eta_{01} dx^0 dx^1 + \eta_{02} dx^0 dx^2 + \eta_{03} dx^0 dx^3 + \eta_{10} dx^1 dx^0 + \eta_{11} dx^1 dx^1 + ...$$

or more simply as

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu$$

In Einstein's summation notation we simply note that when a variable is repeated in the upper and lower index of a term, then it represents a summation over all possible values. In the above case, μ and ν are in the lower indices of η and the upper indices of x and so we know to sum over all possible values of μ and ν, which in this case would give us 16 terms. We mentioned that this is another expression for equation (1), but note that equation (1) only has four nonzero terms. Then we must constrain the values of $\eta_{\mu\nu}$ such that

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix $\eta_{\mu\nu}$ is referred to as the metric tensor for Minkowski space. As we shall see, the metric tensor plays the major role in characterizing the geometry of the curved spacetime required to describe general relativity. This general form of the metric tensor is often denoted $g_{\mu\nu}$.

Special relativity was not extended to include acceleration until Einstein published 'The Foundation of the General Theory of Relativity' in 1916. In special relativity, observers in different inertial frames cannot agree on distances, and they certainly cannot agree on forces depending on the distance between two objects. Such is the case with Newtonian gravitation, as it describes gravity as an instantaneous force between two particles dependent on their distance from one another. With this in mind, Einstein desired to formulate gravity so that observers in any frame would agree on the definition, regardless of how they were moving in relation to each other. Einstein accomplished this by defining gravity as a curvature of spacetime rather than a force. We can then think of falling objects, planets in orbits, and rays of light as objects following paths in a curved spacetime known as geodesics, which are fully described in Section 1.3. As we will see, this then implies that gravity and acceleration are essentially equivalent.

Imagine a two dimensional flat world represented by a rubber sheet stretched out to infinity and imagine that a one dimensional line object of finite length inhabits this world. In this imaginary setting, there is no gravity, no notion of up or down, and there is no height dimension. The line object living in this world is flat, regardless of where or when it is. Now let some three dimensional being hit this flat world with a hammer, producing ripples in the rubber plane. As the ripples propagate through the region where the line object resides, they produce a geometric force pushing the line object with the curvature of the waves. That is, the line object would feel a force, and from our vantage point outside of the rubber sheet, we would see the line object bend and stretch. Similarly, the curvature in a four dimensional universe acts as a force that pushes three dimensional objects.

General relativity is often summarized with a quote by physicist John Wheeler:

> *"Spacetime tells matter how to move,*
> *and matter tells spacetime how to curve."*

The curvature of spacetime defines a gravitational field and that field acts on nearby matter, causing it to move. However, the matter, specifically it's mass, determines the geometric properties of spacetime, and thus it's curvature. So in general relativity, an object's position in spacetime is unaffected by the object's mass (assuming that it is not large enough to significantly alter the curvature of spacetime) and relies only on the geometry of the spacetime.

1.2 **What is a black hole?**

A black hole is an object so dense that it sufficiently bends the spacetime around it so that nothing can escape. The maximum distance from the center of the black hole for which nothing can escape is called the *event horizon*. From a physical point of view, we can picture black holes in terms of escape velocity. For example, on the Earth, if we were to toss a ball into the air, the overwhelming force of gravity due to the mass of the Earth would cause it to fall back to the ground. However, suppose we had a launcher that could shoot the ball at much larger velocities. As we increase the velocity, the ball will go higher before it falls back down. With a launcher powerful enough, we could even shoot the ball with such a velocity that it would leave the atmosphere of the Earth and continue on into space. The minimum velocity required for the ball to leave and not fall back to Earth is called the *escape velocity*.

Let us continue with a geometrical interpretation of this. Notice that for an object to be a black hole, it must have a sufficiently large *density* not mass. Now, we could think of a spacetime without mass as a large flat frictionless rubber sheet, similar to the surface of a trampoline. If we were to place a bowling ball on this surface, then the sheet would flex in the region around the ball, but would be flat everywhere else. This is analogous to adding an isolated static spherically symmetric mass into the spacetime, such as a star. Suppose we place a marble adjacent to the bowling ball, and then tap it so that it rolls up the flexed region. If we tap it softly, it will roll up the curvature, but then roll back down to the bowling ball. As we tap it harder, it will roll further up the curvature before falling back until we tap it hard enough so that it reaches the flat region and continues on in a straight direction. Like the example above, the minimum velocity of the marble for which it escapes the flexed region is called the *escape velocity*.

Now, let us increase the density of the bowling ball by increasing its mass while leaving the size the same. In doing this, the region around the bowling ball will be deeper and the slope of the flexed region will increase. Then we will have to tap the marble harder in order for it to make it to the flat region, that is, the escape velocity will increase. Now let us increase the mass of the bowling ball (while keeping its size fixed) so that the flexed region is so deep and the slope is so steep, that the ball can never escape. We know that there exists a density of the bowling ball such that the marble can never escape because nothing can travel faster than the speed of light. For heuristic purposes, we can even pretend photons of light are particles with mass $\frac{E}{c^2}$ as given by the relation from Einstein's special theory of relativity. Thus, there is a finite limit to how fast an object can go, but there is no limit to how large an escape velocity can be. So, if an object is so dense that its escape velocity is greater than the speed of light, then nothing can escape the gravity of that object and it is called a *black hole*.

We can now take that geometrical interpretation and make a few predictions using classical Newtonian physics. The kinetic energy of an object of mass m with a velocity v is given by

$$\frac{1}{2}mv^2$$

The gravitational potential energy for an object of mass M at a radius r is given by

$$-\frac{GMm}{r}$$

Then, for an object of mass m to escape from a mass M, it's kinetic energy must be greater than the magnitude of the gravitational potential energy. So if we had an object with a maximum velocity, that is $v = c$ and then set the kinetic energy for that equal to the magnitude of the gravitational potential energy, we would have a potential from which nothing could escape.

$$\frac{1}{2}mc^2 = \frac{GMm}{r}$$

We could then solve to find a radius r in terms of mass M for which nothing could escape.

$$r = \frac{2GM}{c^2}$$

Recall that previously we set $c = 1$. Similarly, we can set $G = 6.67 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}} = 1$. Now we can also define mass in terms of meters. If we use 1 second $= 3 \times 10^8$ meters from earlier, then we see $G = 7.41 \times 10^{-28}\,\mathrm{m\,kg^{-1}} = 1$. We then find that one meter $= 1.34 \times 10^{27}$ kilograms. Setting $c = G = 1$ is also referred to as geometrized units. If we use geometrized units and suppose that m is not large enough to significantly affect spacetime, then we can make the approximation

$$r = 2M$$

Thus, if a massive static object M is condensed into a spherical region with a radius r, as measured in mass, less than $2M$, then that object is a black hole. As we will see later, this value of r, which we derived using classical arguments, happens to be the actual value for the *Schwarzschild radius*, which coincides with the event horizon of a static spherically symmetric black hole.

1.3 Geodesics and Christoffel Symbols

In general relativity, gravity is formulated as a geometric interpretation, and as such, we must discard the classical Newtonian view of gravity. Instead, we can think of an object in a gravitational field as traveling along a geodesic in the semi-Riemannian manifold that represents 4-dimensional spacetime. Due to this geometric interpretation, geodesics are very important in describing motion due to gravity. A geodesic is commonly defined as the shortest distance between two points. We are familiar with a geodesic in flat Euclidean geometry; it is simply a line between the two points. However, as we move to 4-dimensional spacetime, it is not always so simple. In calculating the geodesic on a curved manifold, the curvature must be taken into account. For instance, let us return to our example of the bowling ball on the frictionless rubber sheet. Suppose we roll a marble towards the flexed region. Assume that the marble starts on a flat part of the surface, and that it does not run into the bowling ball. Then the marble would initially roll straight toward the flexed region, but upon entering the curvature, it would appear to bend with the surface and exit the region heading straight out in a different direction. This is analogous to the deflection of a comet's trajectory by the gravitational influence of the Sun. In this instance, the marble and the comet are both following "straight" paths on the curved surface which are both geodesics.

One way to describe geodesics is by a concept called parallel transport. In this description, a path is considered a geodesic if it parallel transports its own tangent vectors at all points on the path. That is, if $\vec{A}$ is a tangent vector at some point P then there is a function that transports $\vec{A}$ to some other point Q as $D\vec{A}$. The transported vector $D\vec{A}$ at point Q is then parallel to the tangent vector $\vec{B}$ at point Q. The act of parallel transporting a tangent vector relies specifically on how the curvature changes from point to point. As such, there is a natural way in which we can define a geodesic based on the intrinsic properties of that curvature. Mathematically, we denote this as

$$\frac{d^2x^\lambda}{d\rho^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\rho}\frac{dx^\nu}{d\rho} = 0 \tag{2}$$

which is called the *geodesic equation*. In the above equation, $\Gamma^\lambda_{\mu\nu}$ is a Christoffel symbol and is defined by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right) \tag{3}$$

where $g_{\mu\nu}$ is a component of the metric tensor previously alluded to. In 4-dimensional spacetime, λ, μ, and ν can be 0, 1, 2, or 3 and so, the above equation represents 64 values. However, in the specific case we will look at, most of those values will be zero. We'll return to these equations frequently in the next few sections, but let us consider this parallel transport idea without equations first. Let us begin with a geodesic in a flat geometry, which is a straight line. Then a tangent vector at any point on the line is identically parallel to the line, and thus trivially parallel to every other tangent vector. For curved geometry, let us make a simple analogy to riding in a car. In this case, suppose you are riding in the car and the direction of your gaze is a tangent vector. Now consider looking straight ahead as the car travels in a straight path. Your gaze does not change and is trivially parallel throughout. Now consider looking straight ahead while the car travels on a road with twists and turns. In this case, your gaze changes directions, but it is due to the change in direction of the car rather than the movement of your body. We could then find a relation between the change in your gaze and a change the car made. For example, if the car was initially going North and then turned East, we could simply "transport" your original gaze Eastward and it would be parallel to your gaze while the car headed East. Then by definition, we could consider your gaze to be the tangent vectors of a path that is a geodesic in a sense. You did not make any additional movements; your gaze was only altered due to the change in the car's direction. However, if we consider the case that the car travels in a straight path, but your gaze varies from left to right or any other way, then no matter how straight of a path the car took, your gaze would no longer represent the tangent vectors of a geodesic, because your gaze deviated without respect to the car's changing direction. Thus, a geodesic in a curved space is simply a path for which the tangent vectors only change due to the changing geometry of that space.

2 Einstein's Field Equations and Requirements for a Solution

In the introduction, we stated that Einstein formulated gravity as a geometry of spacetime. We now know that spacetime tells matter how to move, and matter tells spacetime how to curve. We even alluded to the metric tensor $g_{\mu\nu}$ and its role in characterizing the geometry of curved spacetime. However, we have not yet described in any detail in what way matter, and specifically mass, influences the curvature of spacetime. This relation will be described by Einstein's Field Equations.

We will now look back to Newton's Law of Gravitation to give a brief motivation for the solution to Einstein's Field Equations as outlined by Faber (1983). We will continue to use geometrized units, that is, $c = G = 1$. Suppose a mass M is located at the origin of a 3-dimensional system (x, y, z) with position vector $\vec{X} = \langle x(t), y(t), z(t)\rangle$. Let $r = \sqrt{x^2 + y^2 + z^2}$ and define $\vec{u}_r = -\vec{X}/r$ to be the unit radial vector, that is, a vector which points from $\vec{X}$ to the mass M at the origin. Then the force $\vec{F}$ on a particle of mass m located at $\vec{X}$ is

$$\vec{F} = -\frac{Mm}{r^2}\vec{u}_r = m\frac{d^2\vec{X}}{dt^2}$$

where the second equality comes from Newton's second law. We then see that

$$\frac{d^2\vec{X}}{dt^2} = -\frac{M}{r^2}\vec{u}_r$$

Now let us define $\Phi = \Phi(r)$ as the potential function

$$\Phi(r) = -\frac{M}{r}$$

Now let us notice that using the chain rule gives

$$\frac{\partial r}{\partial x^i} = \frac{\partial}{\partial x^i}\left[\left(\vec{X}\cdot\vec{X}\right)^{1/2}\right] = \frac{2x^i}{2\left(\vec{X}\cdot\vec{X}\right)^{1/2}} = \frac{x^i}{r}$$

and

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x^i}$$

We may then write

$$\begin{aligned} -\nabla\Phi &= -\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right) \\ &= -\frac{M}{r^2}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = -\frac{M}{r^2}\vec{u}_r = \frac{d^2\vec{X}}{dt^2} \end{aligned}$$

We can then compare the individual components of the vectors above which gives us

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i} \tag{4}$$

Notice that the left hand side of the above equation looks remarkably like a term in the geodesic equation given by equation (2). Let us now write the geodesic equation as

$$\frac{d^2 x^\lambda}{d\tau^2} = -\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{5}$$

We can then make some general insights on the similarities between equations (4) and (5). For instance, equation (4) relies on the first partial derivatives of the potential function and equation (5) relies on the first partial derivatives of the components of $g_{\mu\nu}$. Then, we could imagine that the coefficients given by the metric tensor in general relativity is analogous to the gravitational potentials of Newton's theory. In a similar line of reasoning, we would want a corresponding result for Laplace's equation which describes gravitational potentials in empty space. The potential function satisfies Laplace's equation which is given by

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{6}$$

In general relativity, for the analogy to hold, we would need an equation involving the second partial derivatives of the metric tensor components $g_{\mu\nu}$. Additionally, we want the equation to be invariant, so that it is independent of the coordinate system used. From the treatment given by Faber (1983), the above requirements force our equation to be a function of $R^{\lambda}_{\mu\nu\sigma}$, which are components of the Riemann curvature tensor, and $g_{\mu\nu}$. The Riemann curvature tensor is itself a function of the metric coefficients $g_{\mu\nu}$ and their first and second derivatives, and so it relies solely on the intrinsic properties of a surface.

$$R^{\lambda}_{\mu\nu\sigma} = \frac{\partial\Gamma^{\lambda}_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\beta}_{\mu\sigma}\,\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\,\Gamma^{\lambda}_{\beta\sigma} \tag{7}$$

We also want this equation to have the flat spacetime of special relativity as one solution. One might suggest that the Riemann curvature tensor would be a good candidate for our equation, however, to allow for the solution of flat spacetime the curvature tensor must be zero.

$$R^{\lambda}_{\mu\nu\sigma} = 0$$

for λ, μ, ν, $\sigma \in \{0, 1, 2, 3\}$. If the curvature tensor, and thus the curvature of our surface, was zero though, we would *only* have flat spacetime and there would be no gravitational fields. This is too restrictive and so we need another equation that allows our spacetime to have curvature. With this in mind, we will now look at Einstein's field equations which do satisfy the above requirements.

2.1 Einstein's Field Equations

We still want the field equations to rely on the metric tensor components $g_{\mu\nu}$ and their first and second partial derivatives. It should relate these components, which describe the curvature of spacetime, to the distribution of matter throughout spacetime. In the previous section, we saw that the Riemann curvature tensor was too restrictive. However, if we set $\sigma = \lambda$ in equation (7) and then sum over λ we obtain the components of the less restrictive *Ricci Tensor*.

Definition 2.1 The *Ricci tensor* is obtained from the curvature tensor by summing over one index:

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\beta}_{\mu\lambda}\, \Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\, \Gamma^{\lambda}_{\beta\lambda}$$

Definition 2.2 *Einstein's vacuum field equations* for general relativity are the system of second order partial differential equations

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\beta}_{\mu\lambda}\, \Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\, \Gamma^{\lambda}_{\beta\lambda} = 0 \tag{8}$$

where Γ was defined in equation (3) as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\beta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right)$$

Hence, the vacuum field equations describe spacetime in the absence of mass, and so are analogous to Laplace's Equation. The field equations are a system of second order partial differential equations in the unknown function $g_{\mu\nu}$. Notice that this relates 16 equations and 16 unknown functions. The $g_{\mu\nu}$ determine the metric form of spacetime and therefore all intrinsic properties of the 4-dimensional semi-Riemannian manifold that is spacetime, including curvature.

3 Derivation of the Schwarzschild Metric

In this section, we will find a solution to Einstein's Field equations that describes a gravitational field exterior to an isolated sphere of mass M assumed to be at rest. Let us place this sphere at the origin of our coordinate system. For simplicity, we will use spherical coordinates ρ, ϕ, θ:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

Starting from the flat Minkowski spacetime of special relativity and changing to spherical coordinates, we have the invariant interval from equation (1)

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + d\rho^2 + \rho^2\, d\phi^2 + \rho^2 \sin^2\phi\, d\theta^2 \end{aligned} \tag{9}$$

Notice that if ρ and t are constant, then we are left with the geometry of a 2-sphere

$$dl^2 = \rho^2 \left(d\phi^2 + \sin^2\phi\, d\theta^2\right)$$

Definition 3.1 A spacetime is *spherically symmetric* if every point in the spacetime lies on a 2-D surface which is a 2-sphere. Using spacetime coordinates (ρ, t, θ, ϕ), then the interval is

$$dl^2 = f(\rho, t) \left[d\phi^2 + \sin^2\phi\, d\theta^2\right]$$

where $\sqrt{f(\rho, t)}$ is the *radius of curvature* of the 2-sphere.

Unlike flat spacetime, where the radius of curvature is found to always be the radial spherical coordinate ρ, in curved spacetime there is not always such a simple relation between the angular coordinates of the 2-D sphere and the two remaining coordinates for each point in spacetime. However, we can always define a new radial coordinate, r, which satisfies $r^2 = f(\rho, t)$, which we will do below.

Recall, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, so if we label $x^0 = t, x^1 = r, x^2 = \phi, x^3 = \theta$ then

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{tt} & g_{tr} & g_{t\phi} & g_{t\theta} \\ g_{rt} & g_{rr} & g_{r\phi} & g_{r\theta} \\ g_{\phi t} & g_{\phi r} & g_{\phi\phi} & g_{\phi\theta} \\ g_{\theta t} & g_{\theta r} & g_{\theta\phi} & g_{\theta\theta} \end{pmatrix} \tag{10}$$

In addition to having a spherically symmetric solution to Einstein's field equations we also want it to be static. That is, the gravitational field is unchanging with time and independent of ϕ and θ. If it were not independent of ϕ and θ then we would be able to define a preferred direction in the space, but since this is not so

$$g_{r\theta} = g_{r\phi} = g_{\theta r} = g_{\phi r} = g_{\theta\phi} = g_{\phi\theta} = 0$$

Similarly, to prevent a preferred direction in spacetime, we can further restrict the coefficients

$$g_{t\theta} = g_{t\phi} = g_{\theta t} = g_{\phi t} = 0$$

Also, with a static and unchanging gravitational field, all of the metric coefficients must be independent of t and the metric should remain unchanged if we were to reverse time, that is, apply the transformation $t \to -t$. With that constraint, only dt^2 leaves ds^2 unchanged and thus implies $g_{tr} = g_{rt} = 0$. At this point we have reduced $g_{\mu\nu}$ to the following

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\phi\phi} & 0 \\ 0 & 0 & 0 & g_{\theta\theta} \end{pmatrix} \tag{11}$$

We may then write the general form of the static spherically symmetric spacetime as

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\phi\phi}\, d\phi^2 + g_{\theta\theta} d\theta^2$$

Now let us solve for the $g_{\mu\nu}$ coefficients. We start with a generalization of the invariant interval in flat spacetime from equation (9).

$$ds^2 = -U(\rho)\, dt^2 + V(\rho)\, d\rho^2 + W(\rho)\, (\rho^2\, d\phi^2 + \rho^2 \sin^2\phi\, d\theta^2) \tag{12}$$

where U, V, W are functions of ρ only. Recall from earlier that we can redefine our radial coordinate, so let $r = \rho\sqrt{W(\rho)}$, and then we can define some $A(r)$ and $B(r)$ so that the above becomes

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + r^2\, d\phi^2 + r^2 \sin^2\phi\, d\theta^2 \tag{13}$$

We next define functions $m = m(r)$ and $n = n(r)$ so that

$$A(r) = e^{2m(r)} = e^{2m} \text{ and } B(r) = e^{2n(r)} = e^{2n}$$

We set $A(r)$ and $B(r)$ equal to exponentials because we know that they must be strictly positive, and with a little foresight, it will make calculations easier later on. Then substituting into equation (13) we have

$$ds^2 = -e^{2m}\, dt^2 + e^{2n}\, dr^2 + r^2\, d\phi^2 + r^2 \sin^2\phi\, d\theta^2 \tag{14}$$

From earlier, we have $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, again we label $x^0 = t, x^1 = r, x^2 = \phi, x^3 = \theta$ and so we have

$$g_{\mu\nu} = \begin{pmatrix} -e^{2m} & 0 & 0 & 0 \\ 0 & e^{2n} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\phi \end{pmatrix} \tag{15}$$

Since $g_{\mu\nu}$ is a diagonal matrix, $g = \det(g_{ij}) = -e^{2m+2n}\, r^4 \sin^2\!\phi$. In order to solve for a static, spherically symmetric solution we must find $m(r)$ and $n(r)$. However, to solve for them we must use the Ricci Tensor from equation (8):

$$R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\,\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\,\Gamma^\lambda_{\beta\lambda} = 0$$

which relies on the Christoffel symbols of equation (3):

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right)$$

From equation (15), we see that $g_{\mu\nu} = 0$ for $\mu \neq \nu$, and so $g^{\mu\mu} = 1/g_{\mu\mu}$ and $g^{\mu\nu} = 0$ if $\mu \neq \nu$. Thus, the coefficient $g^{\lambda\beta}$ is 0 unless $\beta = \lambda$ and substituting this into the above we have

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right) \tag{16}$$

Before going further, let us stop and prove the following Lemma for our specific case, as we will need it to continue the Schwarzschild derivation.

Lemma 3.2 For each μ

$$\frac{\partial}{\partial x^\mu}\left[\ln|g|\right] = \frac{1}{g}\frac{\partial g}{\partial x^\mu} = g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\mu}$$

Proof. As shown above, g is the determinant of a diagonal matrix and so it is simply the product of the diagonal elements. Without loss of generality, suppose these elements are a, b, c, and d so that $g = abcd$. Then we can rewrite $\ln|g|$ as $\ln|a| + \ln|b| + \ln|c| + \ln|d|$. Let us note that for a function $u(x) = u$

$$\frac{\partial \ln|u|}{\partial x} = \frac{u'}{u} \tag{17}$$

Using this, we can then see that

$$\frac{\partial}{\partial x^\mu}\left[\ln|g|\right] = \frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{d'}{d} \tag{18}$$

We then divide out $g = abcd$ to get

$$\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{d'}{d} = \frac{1}{abcd}\left(a'bcd + ab'cd + abc'd + abcd'\right)$$

However, the right half of the above equation is a product rule on four elements divided by all four elements giving us

$$\frac{\partial}{\partial x^\mu}\left[\ln|g|\right] = \frac{1}{abcd}\left(a'bcd + ab'cd + abc'd + abcd'\right) = \frac{1}{g}\frac{\partial g}{\partial x^\mu}$$

Now, using equation (18) and recalling that $g^{\lambda\beta} = 1/g_{\lambda\beta}$ we see

$$\frac{\partial}{\partial x^\mu}\left[\ln|g|\right] = \frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{d'}{d} = \frac{1}{a}\frac{\partial a}{\partial x^\mu} + \frac{1}{b}\frac{\partial b}{\partial x^\mu} + \frac{1}{c}\frac{\partial c}{\partial x^\mu} + \frac{1}{d}\frac{\partial d}{\partial x^\mu} = g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\mu}$$

Thus,

$$\frac{\partial}{\partial x^\mu}\left[\ln|g|\right] = \frac{1}{g}\frac{\partial g}{\partial x^\mu} = g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\mu}$$

■

3.1 Evaluation of the Christoffel Symbols

Now let us return to the Schwarzschild derivation. We had just shown equation (16)

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)$$

Notice that $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ so we have three cases: $\lambda = \nu$, $\mu = \nu \neq \lambda$, and μ, ν, λ distinct. Recall that $g_{\mu\nu} = 0$ for $\mu \neq \nu$ as we will use that to simplify several terms.

Case 1. For $\lambda = \nu$ [using Lemma 3.2]:

$$\begin{aligned}\Gamma^\nu_{\mu\nu} &= \frac{1}{2g_{\nu\nu}}\left(\frac{\partial g_{\mu\nu}}{\partial x^\nu} + \frac{\partial g_{\nu\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\nu}\right)\\ &= \frac{1}{2g_{\nu\nu}}\left(\frac{\partial g_{\nu\nu}}{\partial x^\mu}\right) = \frac{1}{2}\frac{\partial}{\partial x^\mu}\left[\ln|g_{\nu\nu}|\right] \end{aligned} \tag{19}$$

Case 2. For $\mu = \nu \neq \lambda$:

$$\begin{aligned}\Gamma^\lambda_{\mu\mu} &= \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\mu} + \frac{\partial g_{\mu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\mu}}{\partial x^\lambda}\right)\\ &= -\frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\mu}}{\partial x^\lambda}\right)\end{aligned} \tag{20}$$

Case 3. For μ, ν, λ distinct:

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)\\ &= 0\end{aligned} \tag{21}$$

Using the values for $g_{\mu\nu}$ from equation (15) we can calculate the nonzero Christoffel symbols (in terms of $m, n, r,$ and ϕ) where $' \equiv d/dr$

$$\begin{array}{ll}\Gamma^0_{10} = \Gamma^0_{01} = m' & \Gamma^1_{00} = m'e^{2m-2n}\\ \Gamma^1_{11} = n' & \Gamma^1_{22} = -re^{-2n}\\ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} & \Gamma^1_{33} = -re^{-2n}\sin^2\phi\\ \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} & \Gamma^3_{23} = \Gamma^3_{32} = \cot\phi\\ & \Gamma^2_{33} = -\sin\phi\cos\phi\end{array}$$

Now let us see what Lemma 3.2 implies for the Ricci tensor. We begin by noting

$$\ln|g|^{1/2} = \frac{1}{2}\ln|e^{2m+2n}r^4\sin^2\phi| = m + n + 2\ln|r| + \ln|\sin\phi| \tag{22}$$

Now, recall Lemma 3.2

$$\frac{\partial}{\partial x^\mu}\left[\ln|g|\right] = \frac{1}{g}\frac{\partial g}{\partial x^\mu} = g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\mu}$$

We can then see

$$\frac{\partial}{\partial x^\beta}\left[\ln|g^{1/2}|\right] = \frac{1}{2}\frac{\partial}{\partial x^\beta}\left[\ln|g|\right] = \frac{1}{2}g^{\lambda\mu}\frac{\partial g_{\lambda\mu}}{\partial x^\beta} = \frac{1}{2}g^{\lambda\lambda}\frac{\partial g_{\lambda\lambda}}{\partial x^\beta}$$

The last equality follows from the fact that $g_{\mu\nu} = 0$ for $\mu \neq \nu$. Now let us recall equation (3)

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right)$$

Now let $\mu = \beta$, $\nu = \lambda$ and let δ be the dummy variable (which was previously β) and we see

$$\begin{aligned}
\Gamma^\lambda_{\beta\lambda} &= \frac{1}{2}g^{\lambda\delta}\left(\frac{\partial g_{\beta\delta}}{\partial x^\lambda} + \frac{\partial g_{\lambda\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\lambda}}{\partial x^\delta}\right)\\
&= \frac{1}{2}g^{\lambda\lambda}\left(\frac{\partial g_{\beta\lambda}}{\partial x^\lambda} + \frac{\partial g_{\lambda\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\lambda}}{\partial x^\lambda}\right)\\
&= \frac{1}{2}g^{\lambda\lambda}\left(\frac{\partial g_{\lambda\lambda}}{\partial x^\beta}\right)
\end{aligned}$$

The second equality comes from the fact that $g^{\lambda\beta} = 0$ for $\lambda \neq \beta$ and the terms are zero if δ is something other than λ. From the above we then have

$$\frac{\partial}{\partial x^\beta}\left[\ln|g^{1/2}|\right] = \frac{1}{2}g^{\lambda\lambda}\left(\frac{\partial g_{\lambda\lambda}}{\partial x^\beta}\right) = \Gamma^\lambda_{\beta\lambda}$$

Similarly $\frac{\partial}{\partial x^\mu}\left[\ln|g^{1/2}|\right] = \Gamma^\lambda_{\mu\lambda}$. Now let us recall the Field equations (8)

$$R_{\mu\nu} = \frac{\partial\Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\lambda_{\beta\lambda} = 0$$

and therefore we have that the Field equations imply

$$R_{\mu\nu} = \frac{\partial^2}{\partial x^\mu\partial x^\nu}\left[\ln|g^{1/2}|\right] - \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\frac{\partial}{\partial x^\beta}\left[\ln|g^{1/2}|\right] = 0 \qquad (23)$$

3.2 Ricci Tensor Components

However, $R_{\mu\nu}$ for some values of μ and ν will be in terms of m, n, r, and ϕ which we must set equal to zero. Let us calculate those terms now and show that the other terms reduce to zero afterwards. Recall $x^0 = t, x^1 = r, x^2 = \phi, x^3 = \theta$, equation (22), and the nonzero values of the Christoffel symbols presented previously.

$$
\begin{aligned}
R_{00} &= \frac{\partial^2}{\partial t^2}\left[\ln|g^{1/2}|\right] - \frac{\partial \Gamma^{\lambda}_{00}}{\partial x^{\lambda}} + \Gamma^{\beta}_{0\lambda}\Gamma^{\lambda}_{0\beta} - \Gamma^{\beta}_{00}\frac{\partial}{\partial x^{\beta}}\left[\ln|g^{1/2}|\right] \\
&= 0 - \frac{\partial \Gamma^{1}_{00}}{\partial r} + \Gamma^{1}_{00}\Gamma^{0}_{01} + \Gamma^{0}_{01}\Gamma^{1}_{00} - \Gamma^{1}_{00}\frac{\partial}{\partial r}\left[\ln|g^{1/2}|\right] \\
&= -\frac{\partial}{\partial r} m' e^{2m-2n} + 2m'^2 e^{2m-2n} - m' e^{2m-2n}\frac{\partial}{\partial r}\left(m + n + 2\ln|r| + \ln|\sin\phi|\right) \\
&= -m'' e^{2m-2n} - m'(2m' - 2n')e^{2m-2n} + 2m'^2 e^{2m-2n} - m' e^{2m-2n}\left(m' + n' + \frac{2}{r}\right) \\
&= e^{2m-2n}\left(-m'' - 2m'^2 + 2m'n' + 2m'^2 - m'^2 - m'n' - \frac{2m'}{r}\right) \\
&= e^{2m-2n}\left(-m'' + m'n' - m'^2 - \frac{2m'}{r}\right)
\end{aligned}
$$

In the second line, only those terms which involve nonzero Christoffel symbols are shown. Also, recall that $m = m(r)$ and $n = n(r)$ and $' \equiv d/dr$. On the fifth to the sixth line, terms that sum to zero are removed and we find a value for R_{00} in terms of m, n, and r.

$$
\begin{aligned}
R_{11} &= \frac{\partial^2}{\partial r^2}\left[\ln|g^{1/2}|\right] - \frac{\partial \Gamma^{1}_{11}}{\partial r} + \Gamma^{0}_{10}\Gamma^{0}_{10} + \Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{2}_{12}\Gamma^{2}_{12} + \Gamma^{3}_{13}\Gamma^{3}_{13} - \Gamma^{1}_{11}\frac{\partial}{\partial r}\left[\ln|g^{1/2}|\right] \\
&= m'' + n'' - \frac{2}{r^2} - n'' + m'^2 + n'^2 + \left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^2 - n'\left(m' + n' + \frac{2}{r}\right) \\
&= m'' + m'^2 - m'n' - \frac{2n'}{r}
\end{aligned}
$$

From the second to the third line, we again remove the terms that sum to zero and find a value for R_{11} in terms of m, n, and r.

$$
\begin{aligned}
R_{22} &= \frac{\partial^2}{\partial \phi^2}\left[\ln|g^{1/2}|\right] - \frac{\partial \Gamma^1_{22}}{\partial r} + \Gamma^2_{21}\,\Gamma^1_{22} + \Gamma^1_{22}\,\Gamma^2_{21} + \Gamma^3_{23}\,\Gamma^3_{23} - \Gamma^1_{22}\,\frac{\partial}{\partial r}\left[\ln|g^{1/2}|\right] \\
&= \frac{\partial^2}{\partial \phi^2}\left[\ln|\sin\phi|\right] + \left(e^{-2n} - 2n' r e^{-2n}\right) - \frac{re^{-2n}}{r} - \frac{re^{-2n}}{r} + \cot^2\phi + re^{-2n}\left(m' + n' + \frac{2}{r}\right) \\
&= -1 - \frac{\cos^2\phi}{\sin^2\phi} + e^{-2n} - 2n' r e^{-2n} - 2e^{-2n} + \frac{\cos^2\phi}{\sin^2\phi} + m' r e^{-2n} + n' r e^{-2n} + 2e^{-2n} \\
&= -1 + e^{-2n} - 2n' r e^{-2n} + m' r e^{-2n} + n' r e^{-2n} \\
&= e^{-2n}\left(1 + m' r - n' r\right) - 1
\end{aligned}
$$

Using the same techniques as before, we find a value for R_{22}.

$$
\begin{aligned}
R_{33} &= \frac{\partial^2}{\partial \theta^2}\left[\ln|g^{1/2}|\right] - \frac{\partial \Gamma^1_{33}}{\partial r} - \frac{\partial \Gamma^2_{33}}{\partial \phi} + \Gamma^1_{33}\,\Gamma^3_{31} + \Gamma^3_{31}\,\Gamma^1_{33} + \Gamma^2_{33}\,\Gamma^3_{32} + \Gamma^3_{32}\,\Gamma^2_{33} \\
&\qquad - \Gamma^1_{33}\,\frac{\partial}{\partial r}\left[\ln|g^{1/2}|\right] - \Gamma^2_{33}\,\frac{\partial}{\partial \phi}\left[\ln|g^{1/2}|\right] \\
&= 0 + \frac{\partial}{\partial r}\left(re^{-2n}\sin^2\phi\right) + \frac{\partial}{\partial \phi}\left(\sin\phi\cos\phi\right) - \frac{re^{-2n}\sin^2\phi}{r} - \frac{re^{-2n}\sin^2\phi}{r} - \left(\sin\phi\cos\phi\right)\cot\phi \\
&\qquad - \left(\sin\phi\cos\phi\right)\cot\phi + re^{-2n}\sin^2\phi\left(m' + n' + \frac{2}{r}\right) + \sin\phi\cos\phi\left(\frac{\cos\phi}{\sin\phi}\right) \\
&= \left(e^{-2n}\sin^2\phi - 2n' r e^{-2n}\sin^2\phi\right) + \left(\cos^2\phi - \sin^2\phi\right) - 2e^{-2n}\sin^2\phi - 2\cos^2\phi \\
&\qquad + \sin^2\phi\left(m' r e^{-2n} + n' r e^{-2n} + 2e^{-2n}\right) + \cos^2\phi \\
&= \sin^2\phi\left(e^{-2n} - n' r e^{-2n} - 1 + m' r e^{-2n}\right) \\
&= \left[e^{-2n}\left(1 - n' r + m' r\right) - 1\right]\sin^2\phi = R_{22}\sin^2\phi
\end{aligned}
$$

From line three to four we remove several terms that sum to zero and then pull out a $\sin^2\phi$ term. This allows us to find the value of R_{33} in terms of ϕ and R_{22}. Now that we have found all of the nonzero values of $R_{\mu\nu}$ we will show that the remaining are identically zero. Recall equation (23)

$$R_{\mu\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left[\ln|g^{1/2}|\right] - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\frac{\partial}{\partial x^\beta}\left[\ln|g^{1/2}|\right] = 0$$

We will now look at the case that $\mu \neq \nu$ for all μ and ν term by term. We notice that the first term is going to be zero. Recall equation (22) for $\ln|g^{1/2}|$.

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu}\left[\ln|g^{1/2}|\right] = \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(m + n + 2\ln|r| + \ln|\sin\phi|\right) = 0$$

This is because each term is a function of only one variable, but we are taking the derivative with respect to two distinct variables since $\mu \neq \nu$. Thus, the first term is zero for all $R_{\mu\nu}$ with $\mu \neq \nu$. Now, let us look at the second term in its full expansion as it is currently written in Einstein's summation notation. While it can be confusing, the only summation occurs on the left side below. The right side terms may look identical but they are terms involving four *specific* variables while the left hand side sums over a single dummy variable.

$$-\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} = -\frac{\partial}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial}{\partial x^\beta}\Gamma^\beta_{\mu\nu} - \frac{\partial}{\partial x^\mu}\Gamma^\mu_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\nu_{\mu\nu}$$

While we could directly calculate the Christoffel symbols using equation (16), most of the terms are zero and there is no real understanding gained from it. We will simply defer to the calculations of the three cases listed previously in equations (19), (20), and (21). The first two Christoffel symbols fall under Case 3, since λ, μ, and ν are all distinct cases. Hence, they are both zero. The last two Christoffel symbols are both

Case 1 and so they reduce and we have

$$-\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} = -\frac{\partial}{\partial x^{\mu}}\left[\frac{1}{2g_{\mu\mu}}\left(\frac{\partial g_{\mu\mu}}{\partial x^{\nu}}\right)\right] - \frac{\partial}{\partial x^{\nu}}\left[\frac{1}{2g_{\nu\nu}}\left(\frac{\partial g_{\nu\nu}}{\partial x^{\mu}}\right)\right]$$

Using the chain rule we have

$$\begin{aligned}-\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} &= -\frac{\partial}{\partial x^{\mu}}\left(\frac{1}{2g_{\mu\mu}}\right)\left(\frac{\partial g_{\mu\mu}}{\partial x^{\nu}}\right) - \frac{1}{2g_{\mu\mu}}\left[\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}(g_{\mu\mu})\right] \\ &\quad -\frac{\partial}{\partial x^{\nu}}\left(\frac{1}{2g_{\nu\nu}}\right)\left(\frac{\partial g_{\nu\nu}}{\partial x^{\mu}}\right) - \frac{1}{2g_{\nu\nu}}\left[\frac{\partial^2}{\partial x^{\nu}\partial x^{\mu}}(g_{\nu\nu})\right]\end{aligned}$$

Recall equation (15) and notice that $g_{\beta\beta}$ for β of 0, 1, or 2 can only be functions of one variable. Therefore, the second and fourth terms above are zero if μ or ν is 0, 1, or 2. Similar to earlier, this is because we have functions of only one variable, and we are taking derivatives with respect to two distinct variables. Now let us look at the case that either μ or ν is 3. Without loss of generality, suppose $\mu = 3$. In this case, $g_{\mu\mu} = g_{33} = g_{\theta\theta} = r^2 \sin\phi$ which is a function of two variables, r and ϕ. However, we take the partial derivative with respect to μ and since $\mu = 3$ we have $x^{\mu} = x^3 = \theta$, and so the second and fourth terms would still be zero. Thus, we are left with the first and third terms

$$-\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} = -\frac{\partial}{\partial x^{\mu}}\left(\frac{1}{2g_{\mu\mu}}\right)\left(\frac{\partial g_{\mu\mu}}{\partial x^{\nu}}\right) - \frac{\partial}{\partial x^{\nu}}\left(\frac{1}{2g_{\nu\nu}}\right)\left(\frac{\partial g_{\nu\nu}}{\partial x^{\mu}}\right)$$

Notice that in the remaining two terms μ and ν are not specific variables, and so if we were to swap μ for ν and ν for μ, then the two terms would be equivalent. Thus, if we show that one terms goes to zero, then the other will equivalently go to zero. Without loss of generality, let us consider the first term. Notice that it consists of a derivative of $g_{\mu\mu}$ with respect to x^{μ} and a derivative of $g_{\mu\mu}$ with respect to x^{ν}. If μ is equal to 0, 1, or 2 then $g_{\mu\mu}$ is a function of only one variable, r. Since we have derivatives with respect to two distinct variables, then the one that is not r gives us a value of zero and the whole term becomes zero. If μ is 3, then $g_{\mu\mu}$ is a function of r and ϕ. However, one derivative is with respect to $x^\mu = x^3 = \theta$ and so the term would be zero. Thus, the second term of $R_{\mu\nu}$ is always zero for $\mu \neq \nu$.

$$-\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} = 0$$

We now have for equation (23)

$$\begin{aligned} R_{\mu\nu} &= \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left[\ln |g^{1/2}|\right] - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\,\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\,\frac{\partial}{\partial x^\beta}\left[\ln |g^{1/2}|\right] = 0 \\ &= 0 - 0 + \Gamma^\beta_{\mu\lambda}\,\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\,\frac{\partial}{\partial x^\beta}\left[\ln |g^{1/2}|\right] = 0 \end{aligned}$$

Now let us look at the third term in its full expansion. Using Einstein's summation notation it appears to be just a product of two Christoffel symbols; however, it really is summing over λ and β each of which could be four *specific* variables giving us 16 terms total. As with the second $R_{\mu\nu}$ term, recall that the while the left hand side is a summation with dummy variables β and λ, the right hand side is expanded and contains no further summations, regardless of how similar they may appear.

$$\begin{aligned} \Gamma^\beta_{\mu\lambda}\,\Gamma^\lambda_{\nu\beta} &= \Gamma^\lambda_{\mu\lambda}\,\Gamma^\lambda_{\nu\lambda} + \Gamma^\beta_{\mu\lambda}\,\Gamma^\lambda_{\nu\beta} + \Gamma^\lambda_{\mu\beta}\,\Gamma^\beta_{\nu\lambda} + \Gamma^\mu_{\mu\lambda}\,\Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\mu\mu}\,\Gamma^\mu_{\nu\lambda} + \Gamma^\nu_{\mu\lambda}\,\Gamma^\lambda_{\nu\nu} + \Gamma^\lambda_{\mu\nu}\,\Gamma^\nu_{\nu\lambda} \\ &\quad + \Gamma^\beta_{\mu\beta}\,\Gamma^\beta_{\nu\beta} + \Gamma^\mu_{\mu\beta}\,\Gamma^\beta_{\nu\mu} + \Gamma^\beta_{\mu\mu}\,\Gamma^\mu_{\nu\beta} + \Gamma^\nu_{\mu\beta}\,\Gamma^\beta_{\nu\nu} + \Gamma^\beta_{\mu\nu}\,\Gamma^\nu_{\nu\beta} + \Gamma^\mu_{\mu\mu}\,\Gamma^\mu_{\nu\mu} + \Gamma^\nu_{\mu\mu}\,\Gamma^\mu_{\nu\nu} \\ &\quad + \Gamma^\mu_{\mu\nu}\,\Gamma^\nu_{\nu\mu} + \Gamma^\nu_{\mu\nu}\,\Gamma^\nu_{\nu\nu} \\ &= \Gamma^\lambda_{\mu\lambda}\,\Gamma^\lambda_{\nu\lambda} + 2\Gamma^\beta_{\mu\lambda}\,\Gamma^\lambda_{\nu\beta} + 2\Gamma^\mu_{\mu\lambda}\,\Gamma^\lambda_{\nu\mu} + 2\Gamma^\nu_{\mu\lambda}\,\Gamma^\lambda_{\nu\nu} + \Gamma^\beta_{\mu\beta}\,\Gamma^\beta_{\nu\beta} + 2\Gamma^\mu_{\mu\beta}\,\Gamma^\beta_{\nu\mu} + 2\Gamma^\nu_{\mu\beta}\,\Gamma^\beta_{\nu\nu} \\ &\quad + \Gamma^\mu_{\mu\mu}\,\Gamma^\mu_{\nu\mu} + 2\Gamma^\nu_{\mu\mu}\,\Gamma^\mu_{\nu\nu} + \Gamma^\nu_{\mu\nu}\,\Gamma^\nu_{\nu\nu} \end{aligned}$$

On the second line we gather the terms that are symmetric; we are able to do this since Christoffel symbols commute. Notice that there are indeed 16 total. As before, rather than writing out all of the work for each Christoffel symbol, let us use the cases that were derived before in equations (19), (20), and (21). Using those, we'll first notice that the Christoffel symbols that have three distinct variables (those of Case 3) reduce to zero. Below we write only those terms that remain.

$$\Gamma^{\beta}_{\mu\lambda}\,\Gamma^{\lambda}_{\nu\beta} = \Gamma^{\lambda}_{\mu\lambda}\,\Gamma^{\lambda}_{\nu\lambda} + \Gamma^{\beta}_{\mu\beta}\,\Gamma^{\beta}_{\nu\beta} + \Gamma^{\mu}_{\mu\mu}\,\Gamma^{\mu}_{\nu\mu} + \Gamma^{\nu}_{\mu\nu}\,\Gamma^{\nu}_{\nu\nu} + 2\Gamma^{\nu}_{\mu\mu}\,\Gamma^{\mu}_{\nu\nu}$$

Here, the first four terms are each a product of two Case 1 Christoffel symbols and the last term is a product of two Case 2 Christoffel symbols. Using those equations, we can now write out the third $R_{\mu\nu}$ term explicitly.

$$\begin{aligned}
\Gamma^{\beta}_{\mu\lambda}\,\Gamma^{\lambda}_{\nu\beta} \quad = \quad & \frac{1}{2g_{\lambda\lambda}}\left[\frac{\partial}{\partial x^{\mu}}(g_{\lambda\lambda})\right]\frac{1}{2g_{\lambda\lambda}}\left[\frac{\partial}{\partial x^{\nu}}(g_{\lambda\lambda})\right] + \frac{1}{2g_{\beta\beta}}\left[\frac{\partial}{\partial x^{\mu}}(g_{\beta\beta})\right]\frac{1}{2g_{\beta\beta}}\left[\frac{\partial}{\partial x^{\nu}}(g_{\beta\beta})\right] \\
& + \frac{1}{2g_{\mu\mu}}\left[\frac{\partial}{\partial x^{\mu}}(g_{\mu\mu})\right]\frac{1}{2g_{\mu\mu}}\left[\frac{\partial}{\partial x^{\nu}}(g_{\mu\mu})\right] + \frac{1}{2g_{\nu\nu}}\left[\frac{\partial}{\partial x^{\mu}}(g_{\nu\nu})\right]\frac{1}{2g_{\nu\nu}}\left[\frac{\partial}{\partial x^{\nu}}(g_{\nu\nu})\right] \\
& + 2\left(\frac{-1}{2g_{\nu\nu}}\right)\left[\frac{\partial}{\partial x^{\nu}}(g_{\mu\mu})\right]\left(\frac{-1}{2g_{\mu\mu}}\right)\left[\frac{\partial}{\partial x^{\mu}}(g_{\nu\nu})\right]
\end{aligned}$$

By looking carefully at the terms above, we see they all contain a product of a derivative of a $g_{\alpha\alpha}$ term, where $\alpha \in \{\lambda, \beta, \mu, \nu\}$, with respect to x^{μ} and x^{ν}. Earlier we saw that $g_{\alpha\alpha}$ terms are functions of either just r or r and ϕ, and so to prevent both derivatives from being zero, μ and ν cannot be 0 or 3. If they were, then the derivatives would be taken with respect to $x^0 = t$ or $x^3 = \theta$, both of which would be zero for *any* $g_{\alpha\alpha}$ terms, trivially forcing the entire right hand side to be zero. Thus, μ must be 1 or 2 and ν must be 1 or 2. Notice that for each term, we could swap μ and ν and each term would remain unchanged. That is, each term for $\mu = 1$ and

$\nu = 2$ is equivalent to that of $\mu = 2$ and $\nu = 1$. Therefore, we can swap μ and ν and the entire right hand side will remain unchanged. Additionally, the first two terms are the only ones that contain λ and β which we could also swap. Since we are able to swap these variables without affecting the value of the equation, we can then find a value for just one case and know that it is equivalent to the others. Thus, without loss of generality, let $\mu = 1 \Rightarrow x^\mu = r$, $\nu = 2 \Rightarrow x^\nu = \phi$, $\lambda = 3 \Rightarrow x^\lambda = \theta$, and $\beta = 0 \Rightarrow x^\beta = t$. We then have

$$
\begin{aligned}
\Gamma^{\beta}_{1\lambda}\,\Gamma^{\lambda}_{2\beta} \quad = \quad & \frac{1}{2g_{\theta\theta}}\left[\frac{\partial}{\partial r}\left(g_{\theta\theta}\right)\right]\frac{1}{2g_{\theta\theta}}\left[\frac{\partial}{\partial \phi}\left(g_{\theta\theta}\right)\right] + \frac{1}{2g_{tt}}\left[\frac{\partial}{\partial r}\left(g_{tt}\right)\right]\frac{1}{2g_{tt}}\left[\frac{\partial}{\partial \phi}\left(g_{tt}\right)\right] \\
& + \frac{1}{2g_{rr}}\left[\frac{\partial}{\partial r}\left(g_{rr}\right)\right]\frac{1}{2g_{rr}}\left[\frac{\partial}{\partial \phi}\left(g_{rr}\right)\right] + \frac{1}{2g_{\phi\phi}}\left[\frac{\partial}{\partial r}\left(g_{\phi\phi}\right)\right]\frac{1}{2g_{\phi\phi}}\left[\frac{\partial}{\partial \phi}\left(g_{\phi\phi}\right)\right] \\
& + 2\left(\frac{-1}{2g_{\phi\phi}}\right)\left[\frac{\partial}{\partial \phi}\left(g_{rr}\right)\right]\left(\frac{-1}{2g_{rr}}\right)\left[\frac{\partial}{\partial r}\left(g_{\phi\phi}\right)\right]
\end{aligned}
$$

Substituting the values from equation (15) and taking derivatives, we see the above reduces to

$$
\begin{aligned}
\Gamma^{\beta}_{1\lambda}\,\Gamma^{\lambda}_{2\beta} \quad &= \quad \frac{1}{2\,r^2\sin^2\phi}\frac{\partial}{\partial r}\left(r^2\sin^2\phi\right)\frac{1}{2\,r^2\sin^2\phi}\frac{\partial}{\partial \phi}\left(r^2\sin^2\phi\right) + 0 + 0 + 0 + 0 \\
&= \quad \frac{1}{4\,r^4\sin^4\phi}\left(2\,r\sin^2\phi\right)\left(r^2\,2\sin\phi\cos\phi\right) \\
&= \quad \frac{\cos\phi}{r\sin\phi} = \frac{\cot\phi}{r}
\end{aligned}
$$

In the second to the third line, most of the terms wind up canceling and we are left with just $\dfrac{\cot\phi}{r}$ for the third term of $R_{12} = R_{21}$. In deriving this, we also showed that the third term of $R_{\mu\nu}$ is trivially zero for all other $\mu \neq \nu$. We now have only the fourth term of $R_{\mu\nu}$ to calculate. Let us again expand Einstein's summation notation for the fourth term. This sums over β and so we expect four terms. Recall, as before, the left hand side is a summation whereas the right hand side uses *specific* variables.

$$\begin{aligned}
-\Gamma^{\beta}_{\mu\nu}\frac{\partial}{\partial x^{\beta}}\left[\ln|g^{1/2}|\right] &= -\Gamma^{\beta}_{\mu\nu}\frac{\partial}{\partial x^{\beta}}\left[\ln|g^{1/2}|\right]-\Gamma^{\lambda}_{\mu\nu}\frac{\partial}{\partial x^{\lambda}}\left[\ln|g^{1/2}|\right]-\Gamma^{\mu}_{\mu\nu}\frac{\partial}{\partial x^{\mu}}\left[\ln|g^{1/2}|\right]\\
&\quad -\Gamma^{\nu}_{\mu\nu}\frac{\partial}{\partial x^{\nu}}\left[\ln|g^{1/2}|\right]
\end{aligned}$$

We can quickly simplify the above by using equation (21) (Case 3) for the Christoffel symbols. Then the first two terms, which have Christoffel symbols of three distinct variables, reduce to zero leaving

$$\begin{aligned}
-\Gamma^{\beta}_{\mu\nu}\frac{\partial}{\partial x^{\beta}}\left[\ln|g^{1/2}|\right] &= -\Gamma^{\mu}_{\mu\nu}\frac{\partial}{\partial x^{\mu}}\left[\ln|g^{1/2}|\right]-\Gamma^{\nu}_{\mu\nu}\frac{\partial}{\partial x^{\nu}}\left[\ln|g^{1/2}|\right]\\
&= \frac{-1}{2g_{\mu\mu}}\left[\frac{\partial}{\partial x^{\nu}}(g_{\mu\mu})\right]\frac{\partial}{\partial x^{\mu}}\left[\ln|g^{1/2}|\right]+\frac{-1}{2g_{\nu\nu}}\left[\frac{\partial}{\partial x^{\mu}}(g_{\nu\nu})\right]\frac{\partial}{\partial x^{\nu}}\left[\ln|g^{1/2}|\right]
\end{aligned}$$

Using equation (19) (Case 1) for the Christoffel symbols, we have the above. As before, we are taking derivatives with respect to x^{μ} and x^{ν} of $g_{\mu\mu}$ and $g_{\nu\nu}$ but this time there is an additional twist. We are also taking derivatives of $\ln|g^{1/2}|$. However, this too is a function of only r and ϕ, and so our techniques from the previous terms will work. Thus, in order for the derivatives to be trivially nonzero, μ and ν must be either 1 or 2. Once again, we can swap μ and ν for the two remaining terms and so, without loss of generality, let $\mu = 1 \Rightarrow x^{\mu} = r$ and $\nu = 2 \Rightarrow x^{\nu} = \phi$.

$$\begin{aligned}
-\Gamma^{\beta}_{12}\frac{\partial}{\partial x^{\beta}}\left[\ln|g^{1/2}|\right] &= \frac{-1}{2g_{rr}}\left[\frac{\partial}{\partial \phi}(g_{rr})\right]\frac{\partial}{\partial r}\left[\ln|g^{1/2}|\right]+\frac{-1}{2g_{\phi\phi}}\left[\frac{\partial}{\partial r}(g_{\phi\phi})\right]\frac{\partial}{\partial \phi}\left[\ln|g^{1/2}|\right]\\
&= 0+\frac{-1}{2\,r^2}\left[\frac{\partial}{\partial r}\left(r^2\right)\right]\frac{\partial}{\partial \phi}\left[\ln|\sin\phi|\right]\\
&= \frac{-1}{2\,r^2}(2r)\left(\frac{\cos\phi}{\sin\phi}\right)=\frac{-\cos\phi}{r\sin\phi}=\frac{-\cot\phi}{r}
\end{aligned}$$

We substitute the values from equation (15) and take derivatives to see that the fourth term for $R_{12} = R_{21} = \dfrac{-\cot\phi}{r}$. In addition, we know that $R_{\mu\nu}$ for all other $\mu \neq \nu$ is zero because the derivatives are trivially zero. Now we have calculated all four terms of $R_{\mu\nu}$ for $\mu \neq \nu$.

$$R_{\mu\nu} = \begin{cases} 0 + 0 + \dfrac{\cot\phi}{r} - \dfrac{\cot\phi}{r} = 0 & \text{for } \mu, \nu = \{1, 2\} \\ 0 + 0 + 0 + 0 & \text{for all other } \mu \neq \nu \end{cases}$$

We also calculated the terms for $\mu = \nu$

$$R_{00} = e^{2m-2n}\left(-m'' + m'n' - m'^2 - \frac{2m'}{r}\right) \tag{24}$$

$$R_{11} = m'' + m'^2 - m'n' - \frac{2n'}{r} \tag{25}$$

$$R_{22} = e^{-2n}(1 + m'r - n'r) - 1 \tag{26}$$

$$R_{33} = \left[e^{-2n}(1 - n'r + m'r) - 1\right]\sin^2\phi = R_{22}\sin^2\phi \tag{27}$$

3.3 Solving for the Coefficients

Since we are finding the solution for an isolated sphere of mass, M, the Field Equations from equation (23) imply that outside this mass, all components of the Ricci tensor are zero. We can then set the above equations equal to zero and solve the system for m and n so we can find an exact metric that describes the gravitational field of a static spherically symmetric spacetime. Hence, we need

$$-m'' + m'n' - m'^2 - \frac{2m'}{r} = 0 \tag{28}$$

$$m'' + m'^2 - m'n' - \frac{2n'}{r} = 0 \tag{29}$$

$$e^{-2n}(1 + m'r - n'r) - 1 = 0 \tag{30}$$

$$R_{22}\sin^2\phi = 0 \tag{31}$$

Adding equations (28) and (29) we have

$$\frac{-2\,(m' + n')}{r} = 0$$

which implies

$$m + n = \text{constant}$$

However, at large distances from the mass, the metric must reduce to the flat Minkowski spacetime of special relativity. Therefore we have the following boundary conditions

$$\text{as } r \to \infty,\ \ e^{2m} \to 1 \text{ and } e^{2n} \to 1$$

and so

$$\text{as } r \to \infty,\ \ m \to 0 \text{ and } n \to 0$$

Thus, $m + n = 0$ and $n = -m$ so we can eliminate n and rewrite equation (30) as

$$1 = (1 + 2\,rm')\,e^{2m} = e^{2m} + 2\,rm'e^{2m}$$

Now notice that

$$\frac{d}{dr}\left(re^{2m}\right) = e^{2m} + 2\,rm'e^{2m} = 1$$

Integrating this with respect to r gives us

$$\begin{aligned} re^{2m} &= r + \alpha \\ e^{2m} &= 1 + \frac{\alpha}{r} \end{aligned} \tag{32}$$

for some constant α. We will need to solve for α to find the exact metric. With this in mind, suppose we release a 'test' particle with so little mass that it does not disturb the spacetime metric. Also, suppose we release it from rest so that initially

$$dx^{\mu} = 0 \text{ for } \mu = 1, 2, 3$$

Then, we can rewrite equation (14) as

$$\begin{aligned} ds^2 &= -e^{2m}\,dt^2 + e^{2n}\,dr^2 + r^2\,d\phi^2 + r^2\sin^2\phi\,d\theta^2 \\ &= -e^{2m}dt^2 + 0 + 0 + 0 \end{aligned}$$

However, for timelike intervals, we can relate proper time τ between two events as

$$d\tau^2 = -\frac{ds^2}{c^2} = -ds^2 \tag{33}$$

The second equality follows from our use of geometrized units; we set $c = 1$ in the formulation of Minkowski spacetime. We then have

$$d\tau^2 = -ds^2 = e^{2m}dt^2$$

from which follows

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{e^{2m}}$$

which implies

$$\frac{dt}{d\tau} = e^{-2m/2} = e^{-m} \tag{34}$$

Recall the geodesic equation from equation (2)

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Notice that it relies on $\dfrac{dx^\mu}{d\tau}$ which motivates our previous change from ds to $d\tau$. Recall that we are attempting to solve for α and that we want our metric to reduce to Newtonian gravitation at the weak field limit. Towards that end, and having the benefit of a little foresight, let us find the geodesic equation for $x^\lambda = x^1 = r$ at the instant that we release our 'test' particle. Remember that we are releasing this particle from rest and so $\dfrac{dx^\lambda}{d\tau} = 0$ for $\lambda \neq 0$ and so we are left with only the following in our summation

$$\frac{d^2r}{d\tau^2} + \Gamma^1_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

Now we see more of the motivation for switching to proper time τ as we can easily substitute in from equation (34) and move the right hand term to the other side to have

$$\begin{aligned}
\frac{d^2r}{d\tau^2} &= -\left(m' e^{2m-2n}\right) e^{-2m} \\
&= -m' e^{-2n} = -\frac{dm}{dr} e^{-2n} \\
&= -\frac{dm}{dr} e^{2m} = -\frac{d}{dr}\left[\frac{1}{2} e^{2m}\right] \\
&= -\frac{d}{dr}\frac{1}{2}\left[1 + \frac{\alpha}{r}\right] \\
&= -\frac{1}{2}\left(0 - \frac{\alpha}{r^2}\right) = \frac{\alpha}{2r^2} \qquad (35)
\end{aligned}$$

In the first step we simply substitute using our value for the Christoffel symbol and equation (34). For the second line, we cancel out an exponential term and then rewrite m' recalling that $' \equiv d/dr$. The third line follows from $m = -n$. We then notice that the left hand side of the third line is equal to a derivative. We then use equation (32) to arrive at the fourth line, from which the final answer follows. We now have an equation relating α to the acceleration along r. Conveniently, we know that this must reduce to the predictions of Newtonian gravity for the limit of a weak gravitational field and so

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} \qquad (36)$$

where M is the mass of the object around which we are attempting to determine the metric. Setting equation (35) equal to equation (36) we trivially find the solution for α. Recall that we are using geometrized units, and so $G = 1$ as we showed previously. We then have

$$\alpha = -2M$$

Using this to substitute back into equation (32) and recalling that $m = -n$ we can find an exact solution for equation (14)

$$\begin{aligned} ds^2 &= -e^{2m}\,dt^2 + e^{2n}\,dr^2 + r^2\,d\phi^2 + r^2\sin^2\phi\,d\theta^2 \\ ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2\,d\phi^2 + r^2\sin^2\phi\,d\theta^2 \end{aligned} \tag{37}$$

Thus, equation (37) is the *Schwarzschild Metric* which defines the way we measure invariant intervals around a mass M in a static spherically symmetric spacetime.

3.4 Circular Orbits of the Schwarzschild Metric

Now that we have the Schwarzschild Metric, we can find exact equations of motion for objects in a spacetime described by the metric. For instance, we can find orbits around a star or black hole. Since we are interested in the equation of motion, perhaps it would be best to use the geodesic equation as a starting point. Recall that the geodesic equation from equation (2) is

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

for $\lambda \in \{0, 1, 2, 3\}$. Since we will be using the geodesic equation which relates dx^α to $d\tau$ for some value α, let us use equation (33) and rewrite the Schwarzschild Metric as

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2 = -d\tau^2$$

so that we can divide by $d\tau^2$ and arrive at

$$-1 = g_{tt}\frac{dt^2}{d\tau^2} + g_{rr}\frac{dr^2}{d\tau^2} + g_{\phi\phi}\frac{d\phi^2}{d\tau^2} + g_{\theta\theta}\frac{d\theta^2}{d\tau^2} \tag{38}$$

The equation above gives us a glimpse of how to proceed. We need to find an expression for $\frac{dr}{d\tau}$ and set it equal to zero. In the previous section, we found the analytic solutions for $g_{\mu\mu}$ so now we only need the $\frac{dx^\mu}{d\tau^2}$. Let us use the geodesic equation to arrive at relations for these values. Recall the nonzero Christoffel symbols of Section 3.1 and let $\lambda = 2 \Rightarrow x^2 = \phi$. Then the geodesic equation becomes

$$\begin{aligned} \frac{d^2\phi}{d\tau^2} + \Gamma^2_{12}\frac{dr}{d\tau}\frac{d\phi}{d\tau} + \Gamma^2_{21}\frac{d\phi}{d\tau}\frac{dr}{d\tau} - \Gamma^2_{33}\frac{d\theta^2}{d\tau^2} &= 0 \\ \frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} - \sin\phi\cos\phi\frac{d\theta^2}{d\tau^2} &= 0 \end{aligned}$$

Notice that one solution to the above equation follows from ϕ being constant and a multiple of $\pi/2$. If that were the case, then $\frac{d\phi}{d\tau} = \frac{d^2\phi}{d\tau^2} = 0$ and $\sin\phi\cos\phi = 0$ so all of the terms would be zero. Without loss of generality, let us place our orbit in the plane of $\phi = \pi/2$, which is the equatorial plane of our coordinate system. Once the orbit is in this plane, ϕ will not change because we are dealing with a static spherically symmetric spacetime. Thus, we have a solution for one of the terms we are interested in solving. Now let $\lambda = 0 \Rightarrow x^0 = t$ and recall that $m' = \frac{dm}{dr}$, then we can write the geodesic equation as

$$\begin{aligned} \frac{d^2t}{d\tau^2} + \Gamma^0_{10}\frac{dr}{d\tau}\frac{dt}{d\tau} + \Gamma^0_{01}\frac{dt}{d\tau}\frac{dr}{d\tau} &= 0 \\ \frac{d^2t}{d\tau^2} + 2m'\frac{dr}{d\tau}\frac{dt}{d\tau} = \frac{d^2t}{d\tau^2} + 2\frac{dm}{dr}\frac{dr}{d\tau}\frac{dt}{d\tau} &= 0 \\ \frac{d^2t}{d\tau^2} + 2\frac{dm}{d\tau}\frac{dt}{d\tau} &= 0 \end{aligned}$$

We can then divide out a $\dfrac{dt}{d\tau}$ so that we have

$$\frac{d^2t/d\tau^2}{dt/d\tau} + 2\frac{dm}{d\tau} = 0$$

However, if we recall equation (17), and let $u = \dfrac{dt}{d\tau}$ then we see

$$\frac{d}{d\tau}\left[\ln\left|\frac{dt}{d\tau}\right|\right] = \frac{d^2t/d\tau^2}{dt/d\tau}$$

Using the above to substitute back into the previous equation we can rearrange the terms to find

$$\frac{d}{d\tau}\left[\ln\left|\frac{dt}{d\tau}\right|\right] = -2\frac{dm}{d\tau}$$

If we integrate both sides we have

$$\ln\left|\frac{dt}{d\tau}\right| = -2m + C$$

where C is a constant of integration. We can then exponentiate to get

$$\frac{dt}{d\tau} = e^{-2m+C} = e^{C}\,e^{-2m}$$

From here, notice that $e^{-2m} = -1/g_{tt}$ and let $e^{C} = k$ where k is a strictly positive constant, then we are left with

$$\frac{dt}{d\tau} = \frac{k}{1 - \frac{2M}{r}} \tag{39}$$

which gives us a value for another of the terms we are interested in solving. Now let $\lambda = 3 \Rightarrow x^3 = \theta$ so that the geodesic equation becomes

$$\begin{aligned}
\frac{d^2\theta}{d\tau^2} + \Gamma^3_{13}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \Gamma^3_{31}\frac{d\theta}{d\tau}\frac{dr}{d\tau} + \Gamma^3_{23}\frac{d\phi}{d\tau}\frac{d\theta}{d\tau} + \Gamma^3_{32}\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} &= 0 \\
\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + 2\cot\phi\frac{d\phi}{d\tau}\frac{d\theta}{d\tau} &= 0
\end{aligned}$$

However, previously we set $\phi = \pi/2$ and so $\cot\phi = \cot(\pi/2) = 0$. Then the third term is zero and we are left with

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} = 0$$

One solution to this equation would be $\dfrac{d\theta}{d\tau} = 0$ however, since ϕ is fixed, θ cannot be constant or we would not have an orbit. In fact, it would be the radial geodesic of an infalling object. Therefore, assume that $\dfrac{d\theta}{d\tau} \neq 0$. Since it is a nonzero value, then we can divide through by $\dfrac{d\theta}{d\tau}$ giving us

$$\frac{d^2\theta/d\tau^2}{d\theta/d\tau} + \frac{2}{r}\frac{dr}{d\tau} = 0$$

We again use equation (17) and see that

$$\begin{aligned} \frac{d}{d\tau}\left[\ln\left|\frac{d\theta}{d\tau}\right|\right] &= \frac{d^2\theta/d\tau^2}{d\theta/d\tau} \\ \frac{d}{d\tau}\left[\ln\left|r^2\right|\right] &= 2\,r\frac{dr}{d\tau}\left(\frac{1}{r^2}\right) = \frac{2}{r}\frac{dr}{d\tau} \end{aligned}$$

If we use the above two equations and substitute we have that

$$\frac{d}{d\tau}\left[\ln\left|\frac{d\theta}{d\tau}\right|\right] + \frac{d}{d\tau}\left[\ln\left|r^2\right|\right] = 0$$

integrating yields

$$\ln\left|\frac{d\theta}{d\tau}\right| + \ln\left|r^2\right| = \ln\left|r^2\frac{d\theta}{d\tau}\right| = C$$

for some constant C. Exponentiating, we then have

$$r^2\frac{d\theta}{d\tau} = e^C = h$$

where h is a strictly positive constant. We can then rewrite this to give us a solution to the last term we need, that is

$$\frac{d\theta}{d\tau} = \frac{h}{r^2} \qquad (40)$$

Now we have all the values necessary to solve for $\dfrac{dr^2}{d\tau^2}$ in equation (38). We will use the Schwarzschild Metric for the $g_{\mu\mu}$ values, equations (39) and (40) for the differentials, and recall that we set $\phi = \pi/2$ so that $\dfrac{d\phi}{d\tau} = 0$ and $\sin^2\phi = 1$. This gives us

$$\begin{aligned}
-1 &= g_{tt}\frac{dt^2}{d\tau^2} + g_{rr}\frac{dr^2}{d\tau^2} + g_{\phi\phi}\frac{d\phi^2}{d\tau^2} + g_{\theta\theta}\frac{d\theta^2}{d\tau^2} \\
&= -\left(1 - \frac{2M}{r}\right)\left(\frac{k}{1 - \frac{2M}{r}}\right)^2 + \left(\frac{1}{1 - \frac{2M}{r}}\right)\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{h}{r^2}\right)^2 \\
&= -\frac{k^2}{1 - \frac{2M}{r}} + \left(\frac{1}{1 - \frac{2M}{r}}\right)\left(\frac{dr}{d\tau}\right)^2 + \frac{h^2}{r^2}
\end{aligned}$$

Where we have just simplified our substitutions for the last line. Now let's solve this for $\left(\dfrac{dr}{d\tau}\right)^2$.

$$\left(\frac{1}{1 - \frac{2M}{r}}\right)\left(\frac{dr}{d\tau}\right)^2 = -1 + \frac{k^2}{1 - \frac{2M}{r}} - \frac{h^2}{r^2}$$

We can then multiply everything by $\left(1 - \frac{2M}{r}\right)$ giving us

$$\left(\frac{dr}{d\tau}\right)^2 = \left(-1 + \frac{2M}{r}\right) + k^2 - \left(\frac{h^2}{r^2} - \frac{h^2}{r^2}\frac{2M}{r}\right)$$

which we can rearrange to find

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r} + \frac{2Mh^2}{r^3} \tag{41}$$

This is the equation of motion for the coordinate radius, r, of a test mass orbiting a mass, M, as given by the Schwarzschild Metric. We now need to find solutions to the strictly positive constants h and k. Let us differentiate equation (41) with respect to τ. Recall that $r = r(\tau)$ and $\dfrac{d}{d\tau} = \dfrac{d}{dr}\dfrac{dr}{d\tau}$. Then using the chain rule we have

$$\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right)^2 = \frac{d^2r}{d\tau^2}\frac{dr}{d\tau} + \frac{dr}{d\tau}\frac{d^2r}{d\tau^2} = \frac{2d^2r}{d\tau^2}\frac{dr}{d\tau}$$

This gives us the derivative from the left hand side of equation (41), now let us set it equal to the derivative of the right hand side. Recall that h and k are constants.

$$\frac{2d^2r}{d\tau^2}\frac{dr}{d\tau} = \frac{d}{d\tau}\left[k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r} + \frac{2Mh^2}{r^3}\right] = \left(\frac{2h^2}{r^3} - \frac{2M}{r^2} - \frac{6Mh^2}{r^4}\right)\frac{dr}{d\tau}$$

We then divide both sides by $\dfrac{2dr}{d\tau}$ and find

$$\frac{d^2r}{d\tau^2} = \frac{h^2}{r^3} - \frac{M}{r^2} - \frac{3Mh^2}{r^4} \tag{42}$$

Now we can use equations (41) and (42) to solve for the constants h and k. Since we are interested in circular orbits we let $r = a$, where a is some constant, and set $\dfrac{dr}{d\tau} = \dfrac{d^2r}{d\tau^2} = 0$. We then have for equation (41)

$$\begin{aligned} 0 &= \frac{h^2}{a^3} - \frac{M}{a^2} - \frac{3Mh^2}{a^4} \\ 0 &= h^2 - Ma - \frac{3Mh^2}{a} \\ Ma &= h^2\left(1 - \frac{3M}{a}\right) \end{aligned}$$

We multiply both sides by a^3 to get the second line and then rearrange the terms. The solution for h^2 follows directly as

$$h^2 = \frac{Ma}{\left(1 - \frac{3M}{a}\right)} \tag{43}$$

Now we use equation(41) and substitute in our value for h^2 found above

$$\begin{aligned} 0 &= k^2 - 1 - \frac{h^2}{a^2} + \frac{2M}{a} + \frac{2Mh^2}{a^3} \\ &= k^2 - 1 - \frac{Ma}{a^2\left(1 - \frac{3M}{a}\right)} + \frac{2M}{a} + \frac{2M^2a}{a^3\left(1 - \frac{3M}{a}\right)} \end{aligned}$$

Which leads to

$$
\begin{aligned}
k^2 &= 1 + \frac{M}{a\left(1-\frac{3M}{a}\right)} - \frac{2M}{a} - \frac{2M^2}{a^2\left(1-\frac{3M}{a}\right)} \\
&= \frac{\left(1-\frac{3M}{a}\right) + \frac{M}{a} - \left(\frac{2M}{a} - \frac{6M^2}{a^2}\right) - \frac{2M^2}{a^2}}{\left(1-\frac{3M}{a}\right)} \\
&= \frac{1-\frac{4M}{a}+\frac{4M^2}{a^2}}{\left(1-\frac{3M}{a}\right)}
\end{aligned}
$$

We find a common denominator for all the terms and then sum them to find a condensed value for k^2. However, notice that the numerator looks like a square. In fact, we see that

$$\left(1-\frac{2M}{a}\right)^2 = 1-\frac{4M}{a}+\frac{4M^2}{a^2}$$

and so we can write

$$k^2 = \frac{\left(1-\frac{2M}{a}\right)^2}{\left(1-\frac{3M}{a}\right)} \tag{44}$$

The interesting thing about these constants and their values is that they were added as constants of integration, and so they are real values. However, if $a < 3M$ then we would have h^2 and k^2 as negative values, and so h and k would be imaginary, which does not hold for the Schwarzschild Metric. Thus, there are no stable circular orbits closer than $r = 3M$ for a static spherically symmetric spacetimes around a mass, M. In most cases, this limit for circular orbits does not make a big impact on everyday star life. For instance, the Sun's mass in kilometers is a mere 1.477 km, and so $3M_\odot$ is less than 4.5 km. This is miniscule compared to the actual radius of the Sun which is almost 696,000 km!

4 Eddington-Finkelstein Coordinates

As shown previously, the solution to Einstein's Field equations for an isolated spherically symmetric mass M with radius r_B at the origin of our coordinate system is the *Schwarzschild Metric*:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\, d\phi^2 - r^2 \sin^2 \phi\, d\theta$$

However, this solution is not perfect. Notice for $r = 2M$, the metric coefficient at g_{11} is undefined, and so the equation is not valid at that point. Additionally, the metric does not hold for $r < 2M$. Recall that we used a "boundary condition" to find a solution for the metric, specifically that the metric would reduce to flat Minkowski space as r went off to infinity. That boundary condition does not hold for $r = 2M$ because it is a singularity, i.e. undefined. Also, we do not have another boundary condition for $r < 2M$ so our solution does not work for that case. Therefore, the metric in the form above only holds for $r > 2M$. In fact, the solution was derived on the assumption that we were only considering points *outside* of an isolated mass, and so the Schwarzschild solution is only valid for $r > \max\{2M, r_B\}$ where r_B is the radius of the spherical mass.

Definition 4.1 For a spherically symmetric mass M as above, we define $r_S = 2M$ as the *Schwarzschild radius* of the mass. If the radius of the mass is less than the Schwarzschild radius, then the object is called a *black hole*. If the object is a black hole, then the Schwarzschild radius is also referred to as the *event horizon*. It is the distance from the center of the mass for which nothing can escape the gravitational pull, not even light (see Section 1.2).

Recall that in the previous section, we discussed the smallest stable orbits possible around a mass for which the Schwarzschild metric holds. We noticed that the mass of the Sun in kilometers is 1.477 kilometers, and so its Schwarzschild radius is $2M_{\odot} = 2 \times 1.477 = 2.954$ kilometers. The Schwarzschild radius for the Earth is about 9 millimeters but it is about 3 billion kilometers for a 10^9 solar mass black hole, which is a size of black hole thought to inhabit the centers of most galaxies. If a black hole of that size were at the center of our solar system, then its event horizon would extend out to Uranus.

Now let us compare these two black holes. If an astronaut were to happen upon a black hole with the mass of the Earth, then it would be quite small, about the width of a dime. Now recall that the force of gravity between two objects is proportional to $\frac{1}{r^2}$. So, if the astronaut were to come close to the black hole with their feet forward, then the difference between the astronaut's head and feet would be quite large compared to the distance from the astronaut to the black hole. For instance, supposing the astronaut is 2 meters tall, then the astronaut's feet would feel a force sixteen times greater than the astronaut's head when the feet are .5 meters away from the black hole (we get this by relating $1/.25 = 4$ to $1/4 = .25$). So before even reaching the event horizon, an astronaut would be greatly stretched lengthwise into a long noodle shape by the tidal forces resulting in what has come to be termed *spaghettification* [5]. However, since the 2 meter height of an astronaut is small compared to almost 3 billion kilometers, an astronaut could fall through the event horizon of a supermassive black hole with no initial damaging effects, though his future after that is rather bleak. However, we cannot chart the course of the astronaut upon entering the event horizon with the Schwarzschild metric because it does not hold for $r \leq 2M$.

Despite the existence of a singularity at $r = 2M$ for the Schwarzschild metric, there is nothing that physically distinguishes the event horizon in space. Unless previously calculated, an astronaut would unknowingly fall through the event horizon of a supermassive black hole. It is simply a coordinate singularity. Simply put, the spherical coordinates (t, r, ϕ, θ) are inadequate for $r \leq r_S$. Therefore, we will introduce a new coordinate which will provide metric coefficients that are valid for all $r > r_B$. In essence, this will give us a metric that allows us to explore what happens inside the event horizon of a black hole.

Now, let us make a change of coordinates and derive a new metric. We will keep r, ϕ, and θ but replace t with

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

Now let us find dt so we can substitute back into the Schwarzschild Metric in equation (37). We begin by taking the derivative of t above.

$$\begin{aligned} dt &= dv - dr - \frac{dr}{\frac{r}{2M} - 1} = dv - \left(\frac{\frac{r}{2M} - 1 + 1}{\frac{r}{2M} - 1}\right) dr \\ &= dv - \frac{rdr}{r - 2M} = dv - \left(\frac{1}{1 - \frac{2M}{r}}\right) dr \end{aligned}$$

After simplifying terms, it follows that

$$dt = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \qquad (45)$$

and so we find

$$dt^2 = dv^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 \tag{46}$$

Now let us substitute this directly into the Schwarzschild Metric so that we have

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\,d\phi^2 + r^2 \sin^2\phi\,d\theta^2 \\
&= -\left(1 - \frac{2M}{r}\right)\left[dv^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr + \left(1 - \frac{2M}{r}\right)^{-2} dr^2\right] \\
&\quad + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\,d\phi^2 + r^2 \sin^2\phi\,d\theta^2 \\
&= -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\
&\quad + r^2\,d\phi^2 + r^2 \sin^2\phi\,d\theta^2
\end{aligned}$$

and then we have

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr + r^2\,d\phi^2 + r^2 \sin^2\phi\,d\theta^2 \tag{47}$$

The above equation represents the *Eddington-Finkelstein coordinates*. Notice that in this new coordinate system, we no longer have a singularity at $r = 2M$ and so this holds for all $r > r_B$ where r_B is the radius of the black hole. Thus, we are able to explore what happens inside the event horizon of a static spherically symmetric black hole. Notice that we still have a singularity at $r = 0$; however, this singularity is a real physical singularity, and not due to the coordinates used.

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