

Interest Rate Formulas

Mathematics 210G

1 Simple Interest

If you put a sum of money in the bank and let the interest accumulate, the amount of money you will have some time in the future is given by the formula

$$A = P(1 + r)^t$$

where

P is the initial investment,

r is the interest rate per period (converted to a decimal),

t is the number of periods,

A is the amount of money in the bank after t periods.

If you know any three of the four numbers A, P, r, t you can solve for the fourth by using the following formulas

$$\begin{aligned} P &= A(1 + r)^{-t} \\ r &= (A/P)^{1/t} - 1 \\ t &= \frac{\ln(A/P)}{\ln(1 + r)} = \frac{\log(A/P)}{\log(1 + r)}. \end{aligned}$$

Note that you can use \log instead of \ln in the calculation for t ; you will get the same number even though the numerator and denominator of the fraction will be different.

2 Present and Future Value

If you want to know how inflation affects the value of your money, the formula

$$F = P(1 + r)^t$$

will determine this. Here

P is the present value of your money,
 F is the future value of your money,
 r is the inflation rate per period (converted to a decimal),
 t is the number of periods.

In other words, P dollars in today's dollars is equivalent to F dollars after t periods in the future. If you want to know how much money was worth in the past, use

$$P = F(1 + r)^{-t},$$

where F is the current value of your money and P is the equivalent in past dollars.

3 Loans

If you are taking out a loan, you can use the following formula

$$P = \frac{Lr}{(1 - (1 + r)^{-t})}$$

where

L is the amount you borrow,
 P is the payment per period,
 r is the interest rate per period (converted to a decimal),
 t is the number of periods.

If you know what you can make in payments, at what rate you will get, then you can determine how much you can borrow with the formula

$$L = \frac{P(1 - (1 + r)^{-t})}{r},$$

which we get from the previous formula by solving for L . If you change your payments to a new payment P' , you can determine the new number of periods by the formula

$$t = \frac{\ln\left(\frac{P' - Lr}{P'}\right)}{\ln(1 + r)} = \frac{\log\left(\frac{P' - Lr}{P'}\right)}{\log(1 + r)}.$$

4 Credit Cards

If you buy items with a credit card, you have the option of either paying off the entire balance or paying some part of the balance. If you do the latter, then, in effect, you are borrowing money from the credit card company. Credit card interest rates are usually much higher than savings account rates and car and house loan rates; often they are 18% per year or higher.

Typical credit cards require you to pay a minimum payment, which will be the larger of some fixed amount or a fixed percentage of your balance. If you have a monthly interest rate of r (converted to a decimal) and you have to pay p percent of your balance for your minimum payment, then starting with a balance of B and paying the minimum payment, your balance one month later will be

$$\begin{aligned} & (B - pB) + (B - pB)r \\ = & (B - pB)(1 + r) \\ = & (1 + r)(1 - p)B \end{aligned}$$

And, more generally, after t months of paying the minimum payment, your balance will be $[(1 + r)(1 - p)]^t B$.

5 Annuities

5.1 Paying into an annuity

For annuities, the formulas are similar as those for loans. If you need a certain amount of money in the future and want to put away some money every month (or in some other frequency), the formula is

$$P = \frac{Ar}{((1 + r)^t - 1)},$$

where

P is how much you put in the bank each period,
 r is the interest rate per period,
 t is the number of periods,
 A is how much money you need in the future.

If you know how much you wish to invest each period and want to find out how much money you have in the future, we solve this formula for A to get

$$A = \frac{P((1 + r)^t - 1)}{r}.$$

Keep in mind that inflation will eat away at your money, so if you need \$10,000 now, you will need more 20 years from now.

5.2 Collecting money from an annuity

If you have an amount of money invested and you receive regular payments from this investment, the appropriate formula is the same as the loan formula,

$$P = \frac{Ar}{(1 - (1 + r)^{-t})},$$

where P is your income per period, A is the amount of money you started with in the bank, r is the interest rate per period, and t is the number of payments you receive. If you know the amount to invest and you want to determine the amount of money you started with, use

$$A = \frac{P(1 - (1 + r)^{-t})}{r}.$$

If you have a perpetual annuity, one that pays you without a fixed ending date, then your income P is given by the equation

$$P = Ar.$$

This is the same as if you have an amount A of money in the bank collecting interest at a rate of r , and you withdraw the interest each period. If you have a desired income P , you can find out how much money A you need in the bank by using $A = P/r$.

6 Examples

To help you use these formulas, here are a few examples.

Example 1. If you invest \$1,000 in the bank at 6%, compounded yearly, after 7 years you have

$$1000(1.06)^7 = \$1503.63.$$

If the interest rate is compounded monthly instead of yearly, then in 7 years you have

$$1000(1 + .06/12)^{84} = \$1520.37.$$

If you hoped to have \$1,750 in the bank 7 years later, you need a yearly interest rate of

$$r = (1750/1000)^{1/7} - 1 = .083,$$

or 8.3%. On the other hand, if your interest rate was 7.5% compounded monthly, the amount of time it will take for your \$1,000 to increase to \$1,750 is

$$t = \frac{\ln(1750/1000)}{\ln(1 + .075/12)} = 89.8,$$

and this number is in months since our interest rate, which is $.075/12$, is the yearly rate of 7.5% converted into a monthly rate. Therefore, it takes $89.8/12 = 7.48$ years, or almost 7 and a half years, for this to happen.

Example 2. You need \$75,000 now to send your kid through college. However, he or she won't go to college for 12 more years, and the college estimates its tuition and other expenses will rise at the rate of 9% per year. Thus, in 12 years you will need

$$75000(1 + .09)^{12} = \$210,949.86$$

for college expenses.

For another example, suppose inflation in the last fifty years has averaged 3%. If you can buy a house today for \$100,000, what would be the equivalent amount in 1935 dollars? This value is

$$100000(1 + .03)^{-50} = \$22,810.70,$$

so, at this rate of inflation, a house costing \$22,811 fifty years ago would cost \$100,000 today.

Example 3. Suppose your credit card has an annual interest rate of 21% and the minimum payment is the larger of \$20 or 2% of your balance. If you have a balance of \$1500, then 2% of that is \$30, so that is your minimum payment. If you make the minimum payment of \$30, your new balance is \$1470, and the credit card company will then charge you 1.75% interest on that amount, or $\$1470 \times .0175 = \25.73 . Your new balance is then $\$1470 + \$25.73 = \$1495.73$.

To do this calculation in one step, if B is our balance, and we pay 2%, we then pay $.02B$, leaving us with a new balance of $B - .02B = .98B$. We then get charged 1.75% of this, or $.0175 \times (.98B) = .01715B$. So, our new balance is the old balance of $.98B$ plus this interest, or

$$.98B + .01715B = .99715B$$

So, to go from one month to the next, we take the current balance and multiply it by $.99715$ to get the new balance. To find the balance after two months, we would repeat this calculation, getting

$$\begin{aligned} .99715(.99715B) &= .99715^2B \\ &= .99431B \end{aligned}$$

In general, after n months, our balance would be $.99715^n B$. For instance, by paying just the minimum payment each month, after 12 months of payments, our balance would be $.99715^{12}B = .9026B$, or around 90% of our original balance.

Example 4. You need \$50,000 to buy a used Jaguar, but you only have \$5,000 for a down payment. If the bank will loan you the rest at 11% for a five year loan, then your monthly payment will be

$$\frac{45000(.11/12)}{(1 - (1 + .11/12)^{-60})} = \$978.41$$

Note that we had to convert the interest rate to a monthly rate. Also, five years has 60 months, so you will make 60 payments of \$978.41. If you are able to pay \$1,050 each month, you will pay off the loan in

$$\frac{\ln\left(\frac{1050}{1050 - 45000(.11/12)}\right)}{\ln(1 + .11/12)} = 54.7,$$

or in just under 55 monthly payments. At \$978.41 for 60 months you pay a total of $\$978.41 \cdot 60 = \$58,704.60$, while at \$1,050 for 55 months you pay a total of $\$1,050 \cdot 55 = \$57,750.00$. So, you save about \$1,000 over the life of the loan.

Example 5. To get back to the college expenses problem, you need to have approximately \$211,000 in 12 years for your kid's college expenses. If you can invest money each month at an 8% yield, each month you need to deposit

$$\frac{211000(.08/12)}{((1 + .08/12)^{144} - 1)} = \$877.31.$$

Again, we had to convert the interest rate to a monthly rate and use the number of months in 12 years.

Example 6. You have saved for your retirement, and today you have \$400,000 in your retirement plan. If your plan is earning 4% interest per year, and you want to receive monthly payments for the next 20 years, you will receive

$$\frac{400000(.04/12)}{(1 - (1 + .04/12)^{-240})} = \$2,423.92$$

each month. If you only want to receive money for 15 years you will receive

$$\frac{400000(.04/12)}{(1 - (1 + .04/12)^{-180})} = \$2,958.75$$

per month. However, if you wish to receive a perpetual annuity, you will receive

$$400000(.04/12) = \$1,333.33$$

each month, assuming you continue to earn 4% per year.