# Section 1.7: Interval Notation and Linear Inequalities

- Linear Inequalities

## Linear Inequalities

An inequality in the variable $x$ is **linear** if each term is a constant or a multiple of $x$.
The inequality will contain an inequality symbol:

| | |
|---|---|
| $<$ | is less than |
| $\leq$ | is less than or equal to |
| $>$ | is greater than |
| $\geq$ | is greater than or equal to |

To solve an inequality containing a variable, find all values of the variable that make the inequality true.

## Rules for Solving Inequalities:

In solving inequalities, isolate the variable on one side of the inequality symbol by using the following rules.

**1)** $A \leq B$ is equivalent to $A + C \leq B + C$

Adding the same quantity to both sides of an inequality produces an equivalent inequality.

**2)** $A \leq B$ is equivalent to $A - C \leq B - C$

Subtracting the same quantity from both sides of an inequality produces an equivalent inequality.

**3)** If $C > 0$, then $A \leq B$ is equivalent to $CA \leq CB$

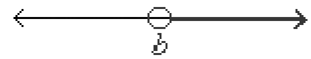
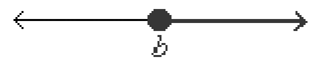
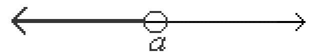
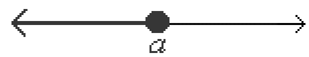
Multiplying both sides of an inequality by a positive number produces an equivalent inequality.

**4)** If $C < 0$, then $A \leq B$ is equivalent to $CA \geq CB$

Multiplying both sides of an inequality by a negative number reverses the direction of the inequality.

# Interval Notation:

After solving a linear inequality, we graph the solution set on the real number line and write the solution in interval notation.

| Interval Notation | Description of Interval | Graph |
|---|---|---|
| $(b, \infty)$ | includes all real numbers $x$ such that $x$ is greater than $b$ $(x > b)$ |  |
| $[b, \infty)$ | includes all real numbers $x$ such that $x$ is greater than or equal to $b$ $(x \geq b)$ |  |
| $(-\infty, a)$ | includes all real numbers $x$ such that $x$ is less than $a$ $(x < a)$ |  |
| $(-\infty, a]$ | includes all real numbers $x$ such that $x$ is less than or equal to $a$ $(x \leq a)$ |  |
| $(-\infty, \infty)$ | includes all real numbers $x$ |  |
| $(a, b)$ | includes all real numbers $x$ such that $x$ is between $a$ and $b$ $(a < x < b)$ |  |
| $[a, b)$ | includes all real numbers $x$ such that $x$ is greater than or equal to $a$ and $x$ is less than $b$ $(a \leq x < b)$ |  |
| $(a, b]$ | includes all real numbers $x$ such that $x$ is greater than $a$ and $x$ is less than or equal to $b$ $(a < x \leq b)$ |  |
| $[a, b]$ | includes all real numbers $x$ such that $x$ is between and including $a$ and $b$ $(a \leq x \leq b)$ |  |

**Example:**

Solve the inequality $2x - 5 > 3$. Give the answer in interval notation and graph the solution set.

**Solution:**

$$2x - 5 > 3$$

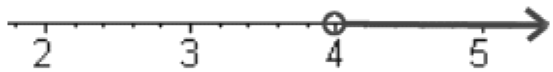
$$2x - 5 + 5 > 3 + 5$$ Add 5 to both sides.

$2x > 8$ Simplify.

$\frac{\cancel{2}x}{\cancel{2}} > \frac{8}{2}$ Divide both sides by 2.

$x > 4$ Simplify.

The inequality is true for all values of $x$ that are greater than 4. In interval notation the solution is $(4,\infty)$. The graph of the solution is sketched below.




**Example:**

Solve the inequality $3x+11 \ge 6x+8$. Give the answer in interval notation and graph the solution set.

**Solution:**

$3x+11 \ge 6x+8$

$3x+11-6x \ge 6x+8-6x$ Subtract $6x$ from both sides.

$-3x+11 \ge 8$ Simplify.

$-3x+11-11 \ge 8-11$ Subtract 11 from both sides.

$-3x \ge -3$ Simplify.

$\frac{\cancel{-3}x}{\cancel{-3}} \le \frac{-3}{-3}$ Divide both sides by $-3$. Reverse the direction of the inequality.

$x \le 1$ Simplify.

The inequality is true for all values of $x$ that are less than or equal to 1. In interval notation the solution is $(-\infty,1]$. The bracket at 1 indicates that 1 is in the set. The graph of the solution is sketched below.


**Example:**

Solve the double inequality $-1 < 2x-5 \le 3$. Give the answer in interval notation and graph the solution set.

**Solution:**

The solution set consist of all values of $x$ that satisfy both inequalities:

$$-1 < 2x - 5 \text{ and } 2x - 5 \le 3$$

To solve $-1 < 2x - 5 \le 3$, use the rules for inequalities to isolate $x$ in the middle.

$$-1 < 2x - 5 \le 3$$

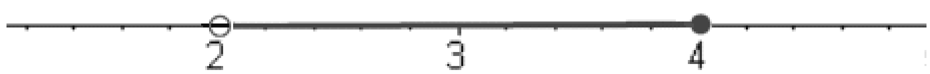
$$-1 + 5 < 2x - 5 + 5 \le 3 + 5 \quad \text{Add 5.}$$

$$4 < 2x \le 8 \quad \text{Simplify.}$$

$$\frac{4}{2} < \frac{\cancel{2}x}{\cancel{2}} \le \frac{8}{2} \quad \text{Divide by 2.}$$

$$2 < x \le 4 \quad \text{Simplify.}$$

The solution in interval notation is $(2,4]$. The graph of the solution is shown below.




**Additional Example 1:**

Let $S = \{-4, 0, 3\}$. What elements of $S$, if any, satisfy the inequality $3x + 5 > x$?

**Solution:**

Substitute $x = -4$ into the inequality.

$$3x + 5 > x$$

$$3(-4) + 5 \overset{?}{>} -4$$

$$-12 + 5 \overset{?}{>} -4$$

$$-7 \not> -4$$

$-4$ does not satisfy the inequality.

Substitute $x = 0$ into the inequality.

$$3x + 5 > x$$

$$3(0) + 5 \overset{?}{>} 0$$

$$0+5 \overset{?}{>} 0$$
$$5 \overset{\checkmark}{>} 0$$

0 satisfies the inequality.

Substitute $x = 3$ into the inequality.

$$3x+5 > x$$
$$3(3)+5 \overset{?}{>} 3$$
$$9+5 \overset{?}{>} 3$$
$$14 \overset{\checkmark}{>} 3$$

3 satisfies the inequality.

**Additional Example 2:**

Solve the inequality $4x-9<11$. Write the solution in interval notation and graph the solution on the real number line.

**Solution:**

$$4x-9<11$$
$$4x-9+9<11+9 \qquad \text{Add 9 to both sides.}$$
$$4x<20 \qquad \text{Simplify.}$$
$$\frac{\cancel{4}x}{\cancel{4}}<\frac{20}{4} \qquad \text{Divide both sides by 4.}$$
$$x<5 \qquad \text{Simplify.}$$

The inequality is true for all values of $x$ that are less than 5. In interval notation the solution is $(-\infty, 5)$. The graph of the solution is sketched below.

**Additional Example 3:**

Solve the inequality $7-3x \geq 9$. Write the solution in interval notation and graph the solution on the real number line.

**Solution:**

$$7-3x \geq 9$$

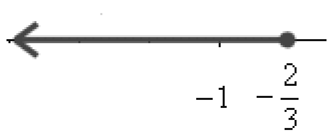
$7-3x-7 \geq 9-7$ Subtract 7 from both sides.

$-3x \geq 2$ Simplify.

$\frac{\cancel{-3}x}{\cancel{-3}} \leq \frac{2}{-3}$ Divide both sides by $-3$. Reverse the direction of the inequality.

$x \leq -\frac{2}{3}$ Simplify.

The inequality is true for all values of $x$ that are less than or equal to $-\frac{2}{3}$. In interval notation the solution is $\left(-\infty,-\frac{2}{3}\right]$. The bracket at $-\frac{2}{3}$ indicates that $-\frac{2}{3}$ is in the set. The graph of the solution is sketched below.




**Additional Example 4:**

Solve the inequality $3(4x-1) \leq 15x+12$. Write the solution in interval notation and graph the solution on the real number line.

**Solution:**

$$3(4x-1) \leq 15x+12$$

$12x-3 \leq 15x+12$ Use the distributive property on the LHS.

$12x-3-15x \leq 15x+12-15x$ Subtract $15x$ from both sides.

$-3x-3 \leq 12$ Simplify.

$-3x-3+3 \leq 12+3$ Add 3 to both sides.

$-3x \leq 15$ Simplify.

$\frac{\cancel{-3}x}{\cancel{-3}} \geq \frac{15}{-3}$ Divide both sides by $-3$. Reverse the direction of the inequality.

$x \geq -5$ Simplify.

The inequality is true for all values of $x$ that are greater than or equal to $-5$. In interval notation the solution is $[-5,\infty)$. The bracket at $-5$ indicates that $-5$ is in the set. The graph of the solution is sketched below.


**Additional Example 5:**

Solve the double inequality $1 \le 7+2x \le 9$. Write the solution in interval notation and graph the solution on the real number line.

**Solution:**

| | |
|---|---|
| $1 \le 7+2x \le 9$ | Isolate $x$ in the middle. |
| $1-7 \le 7+2x-7 \le 9-7$ | Subtract 7. |
| $-6 \le 2x \le 2$ | Simplify. |
| $\frac{-6}{2} \le \frac{\cancel{2}x}{\cancel{2}} \le \frac{2}{2}$ | Divide by 2. |
| $-3 \le x \le 1$ | Simplify. |

In interval notation the solution is $[-3,1]$. The brackets at $-3$ and 1 indicate that $-3$ and 1 are in the set. The graph of the solution is sketched below.


**Additional Example 6:**

Solve the double inequality $-2 < \frac{4-x}{5} \le \frac{3}{5}$. Write the solution in interval notation and graph the solution on the real number line.

**Solution:**

| | |
|---|---|
| $-2 < \frac{4-x}{5} \le \frac{3}{5}$ | Isolate $x$ in the middle. |
| $(5)(-2) < \cancel{5} \cdot \frac{4-x}{\cancel{5}} \le \cancel{5} \cdot \frac{3}{\cancel{5}}$ | Multiply by 5. |

| | |
|---|---|
| $-10 < 4 - x \le 3$ | Simplify. |
| $-10 - 4 < 4 - x - 4 \le 3 - 4$ | Subtract 4. |
| $-14 < -x \le -1$ | Simplify. |
| $(-1)(-14) > (-1)(-x) \ge (-1)(-1)$ | Multiply by $-1$. Reverse the direction of the inequalities. |
| $14 > x \ge 1$ | Simplify. |

$14 > x \ge 1$ can be written as $1 \le x < 14$.

In interval notation the solution is $[1, 14)$. The bracket at 1 indicates that 1 is in the set. The graph of the solution is sketched below.


**Additional Example 7:**
A rental car company offers two options. Option 1 is \$100 per week plus 10 cents for each mile. Option 2 is \$125 per week plus 5 cents for each mile. How many miles per week would a person need to drive to make Option 2 more economical than Option 1?

**Solution:**
Let $x$ = the number of miles per week.
$100 + .10x$ = the weekly cost of Option 1.
$125 + .05x$ = the weekly cost of Option 2.

Write an inequality. We want Option 2 more economical than Option 1.

(The cost of Option 2) is less than (the cost of Option 1).

| | |
|---|---|
| $125 + .05x < 100 + .10x$ | |
| $125 + .05x - .10x < 100 + .10x - .10x$ | Subtract $.10x$ from both sides. |
| $125 - .05x < 100$ | Simplify. |
| $125 - .05x - 125 < 100 - 125$ | Subtract 125 from both sides. |
| $-.05x < -25$ | Simplify. |
| $\dfrac{\cancel{-.05}x}{\cancel{-.05}} > \dfrac{-25}{-.05}$ | Divide both sides by $-.05$. Reverse the direction of the inequality. |
| $x > 500$ | Simplify. |

A person must drive more than 500 miles per week for Option 2 to be more economical than Option 1.

# Exercise Set 1.7: Interval Notation and Linear Inequalities

**For each of the following inequalities:**
**(a) Write the inequality algebraically.**
**(b) Graph the inequality on the real number line.**
**(c) Write the inequality in interval notation.**

**1.** $x$ is greater than 5.

**2.** $x$ is less than 4.

**3.** $x$ is less than or equal to 3.

**4.** $x$ is greater than or equal to 7.

**5.** $x$ is not equal to 2.

**6.** $x$ is not equal to $-5$.

**7.** $x$ is less than $-1$.

**8.** $x$ is greater than $-6$.

**9.** $x$ is greater than or equal to $-4$.

**10.** $x$ is less than or equal to $-2$.

**11.** $x$ is not equal to $-8$.

**12.** $x$ is not equal to 3.

**13.** $x$ is not equal to 2 and $x$ is not equal to 7.

**14.** $x$ is not equal to $-4$ and $x$ is not equal to 0.

**Write each of the following inequalities in interval notation.**

**15.** $x > 3$

**16.** $x \geq -5$

**17.** $x \leq -2$

**18.** $x < 7$

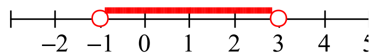
**19.** $3 < x \leq 5$

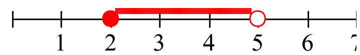
**20.** $-7 \leq x \leq 2$

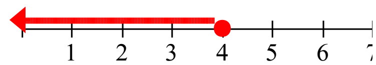
**21.** $x \neq -7$

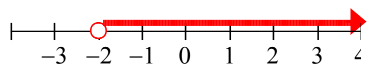
**22.** $x \neq 9$

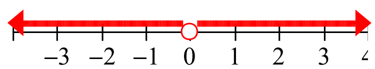
**Write each of the following inequalities in interval notation.**

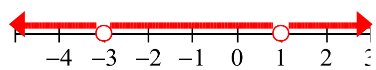
**23.** 


**24.** 


**25.** 


**26.** 


**27.** 


**28.** 


**Given the set $S = \left\{2, 4, -3, \frac{1}{3}\right\}$, use substitution to determine which of the elements of $S$ satisfy each of the following inequalities.**

**29.** $2x + 5 \leq 10$

**30.** $4x - 2 > -14$

**31.** $-2x + 1 > -7$

**32.** $-3x + 1 \geq 0$

**33.** $x^2 + 1 < 10$

**34.** $\frac{1}{x} \leq \frac{2}{5}$

**For each of the following inequalities:**
**(a) Solve the inequality.**
**(b) Graph the solution on the real number line.**
**(c) Write the solution in interval notation.**

**35.** $2x < 10$

**36.** $3x \geq 24$

**37.** $-5x \geq 30$

**38.** $-4x < 40$

**39.** $2x - 5 \geq -11$

**40.** $3x + 4 \leq -17$

**41.** $8 - 3x > 20$

**42.** $10 - x > 0$

**43.** $4x - 11 < 7x + 4$

**44.** $5 - 9x \leq 3x - 7$

**45.** $10x - 7 \geq 2x + 6$

**46.** $8 - 4x < 6 - 5x$

**47.** $5 - 8x \geq 4x + 1$

**48.** $x + 10 \geq 8x - 9$

**49.** $-3(4 + 5x) < -2(7 - x)$

**50.** $-4(3 - 2x) \leq -(x + 20)$

**51.** $\frac{5}{6} - \frac{1}{3}x \leq \frac{1}{2}(x + 5)$

**52.** $\frac{2}{5}\left(x + \frac{1}{2}\right) > -\frac{1}{3}(10 - x)$

**53.** $-10 \leq 3x + 2 < 8$

**54.** $-9 < 2x - 3 < 13$

**55.** $-4 \leq 3 - 7x \leq 17$

**56.** $-19 < 5 - 4x \leq -3$

**57.** $\frac{2}{3} < \frac{3x-10}{15} < \frac{4}{5}$

**58.** $\frac{3}{4} > \frac{5-2x}{6} > -\frac{5}{3}$

**Which of the following inequalities can never be true?**

**59.** **(a)** $5 \leq x \leq 9$
**(b)** $9 \leq x \leq 5$
**(c)** $-3 < x \leq 7$
**(d)** $-5 \geq x > -3$

**60.** **(a)** $3 > x > 5$
**(b)** $-8 \leq x < 1$
**(c)** $-2 < x \leq -8$
**(d)** $-7 \geq x > -10$

**Answer the following.**

**61.** You go on a business trip and rent a car for \$75 per week plus 23 cents per mile. Your employer will pay a maximum of \$100 per week for the rental. (Assume that the car rental company rounds to the nearest mile when computing the mileage cost.)

**(a)** Write an inequality that models this situation.
**(b)** What is the maximum number of miles that you can drive and still be reimbursed in full?

**62.** Joseph rents a catering hall to put on a dinner theatre. He pays \$225 to rent the space, and pays an additional \$7 per plate for each dinner served. He then sells tickets for \$15 each.

**(a)** Joseph wants to make a profit. Write an inequality that models this situation.
**(b)** How many tickets must he sell to make a profit?

**63.** A phone company has two long distance plans as follows:

Plan 1: \$4.95/month plus 5 cents/minute
Plan 2: \$2.75/month plus 7 cents/minute

How many minutes would you need to talk each month in order for Plan 1 to be more cost-effective than Plan 2?

**64.** Craig's goal in math class is to obtain a "B" for the semester. His semester average is based on four equally weighted tests. So far, he has obtained scores of 84, 89, and 90. What range of scores could he receive on the fourth exam and still obtain a "B" for the semester? (Note: The minimum cutoff for a "B" is 80 percent, and an average of 90 or above will be considered an "A".)