BURGERS EQUATION

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Abstract

In this paper we present the Burgers equation in its viscous and non-viscous version. After submitting, as a motivation, some applications of this paradigmatic equations, we continue with the mathematical analysis of them. This will lead us to confront one of the main problems linked to non-linear PDE: The appearance of shocks. Finally, the paper concludes presenting several numerical schemes for solving these equations and their corresponding implementation in MATLAB.

Contents

1 Introduction

In this paper we will consider the viscid Burgers equation to be the nonlinear parabolic PDE

$$u_t + uu_x = \epsilon u_{xx} \tag{1}$$

where $\epsilon > 0$ is the constant of viscosity. This is the simplest PDE combining both nonlinear propagation effects and diffusive effects. When the right term is removed from (1) we obtain the hiperbolic PDE

$$u_t + uu_x = 0. \tag{2}$$

We will refer to (2) as the inviscid Burgers equation. Note that equation (2) can be rewritten in the form

$$u_t + [f(u)]_x = 0 \quad \text{with} \quad f(u) = \frac{u^2}{2}, \tag{3}$$

where is easily recognizable the structure of a scalar hiperbolic conservation law. Many of the ideas presented in this paper (relating to mathematical treatment, numerical methods,...) can be formulated in the framework of the theory of scalar hiperbolic conservation laws so for generality we will often refer to (2) in the form (3) and the developments will be valid for a general f in (3).

There is an important connection between the above equations: Equation (2) is the limit as $\epsilon \to 0$ of (1). This is true from the formal point of view but also in the most rigorous sense. This fact will be studied in more detail in the paragraph called *Vanishing viscosity approach*.

2 Burgers Model

Burgers equations appear often as a simplification of a more complex and sophisticated model. Hence it is usually thought as a *toy model*, namely, a tool that is used to understand some of the inside behavior of the general problem. Here we will present two examples.

2.1 Navier Stokes equations simplification

Consider the Navier Stokes equations

$$\begin{cases} \nabla \cdot \mathbf{v} = 0, & (4.1) \\ (\rho\mathbf{v})_t + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) + \nabla p - \mu\nabla^2\mathbf{v} = 0. & (4.2) \end{cases} \tag{4}$$

It is well known that when ρ is consider to be the density, p the pression, $\mathbf{v}$ the velocity and μ the viscosity of a fluid, equations (4) describe the dynamics of a divergence free (4.1) incompressible ($\rho_t = 0$) flow where gravitational effects are negligible.

Simplification in (4.2) of the x componet of the velocity vector, which we will call v^x, gives

$$\rho\frac{\partial v^x}{\partial t} + \rho v^x\frac{\partial v^x}{\partial x} + \rho v^y\frac{\partial v^x}{\partial y} + \rho v^z\frac{\partial v^x}{\partial z} + \frac{\partial p}{\partial x} - \mu\left(\frac{\partial^2 v^x}{\partial x^2} + \frac{\partial^2 v^x}{\partial y^2} + \frac{\partial^2 v^x}{\partial z^2}\right) = 0.$$

If we consider a 1D problem with no pressure gradient, the above equation reduces to

$$\rho\frac{\partial v^x}{\partial t} + \rho v^x\frac{\partial v^x}{\partial x} - \mu\frac{\partial^2 v^x}{\partial x^2} = 0. \tag{5}$$

If we use now the traditional variable u rather than v^x and take ϵ to be the kinematic viscosity, i.e, $\epsilon = \frac{\mu}{\rho}$, then the last equation becomes just the viscid Burgers equation as it has been presented

in (1).

When the viscosity μ of the fluid is almost zero, one could think, as an idealization, to simply remove the second-derivative term in (5). This would lead to

$$\rho\frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} = 0 \tag{6}$$

which, after making $u = v^x$ and dividing by ρ, becomes the inviscid Burgers equation as it is shown in (2). It turns out that, in order to use (6) as a model for the dynamics of an *inviscous* fluid, it has to be supplemented with other physical conditions (section 3.1) which will prevent equation (6) from developing physical meaningless solutions. This extension is worth because working with (6) is much easier than dealing with (5).

2.2   Traffic Flow

Consider the flow of cars on a highway and let $\rho(x,t)$ denote the density of cars and $f(x,t)$ the traffic flow. We will also consider ρ^* to be the restriction of ρ to a certain range, $0 \le \rho^* \le \rho_{max}$, where ρ_{max} is the value at which cars are bumper to bumper.

Since cars are conserved, the density of cars and the flow must be related by the continuity equation

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial f}{\partial x^*} = 0. \tag{7}$$

Obviously, the first expression in which one thinks for the flow is $f = v\rho^*$ where v is the velocity. However, it turns out that in order to reflect the fact that drivers will reduce their speed to account for an increasing density ahead we should suppose that f is a function of the density gradient as well. A simple assumption is to take

$$f(\rho^*) = \rho^* v(\rho^*) - D\frac{\partial \rho^*}{\partial x^*} \quad \text{where } D \text{ is a constant.} \tag{8}$$

We are assuming also that the velocity v is a given function of ρ^*: On a highway we would optimally like to drive at some speed v_{max} (the speed limit perhaps) but with heavy traffic we slow down, with velocity decreasing as density increases. The simplest relation that is aware of this is

$$v(\rho^*) = \frac{v_{max}}{\rho_{max}}(\rho_{max} - \rho^*). \tag{9}$$

Putting (8) and (9) into (7) leads to

$$\frac{\partial \rho^*}{\partial t^*} + \frac{d}{dx^*}\left[\frac{v_{max}}{\rho_{max}}(\rho_{max} - \rho^*)\rho^*\right] = D\frac{\partial \rho^{*2}}{\partial x^{*2}}. \tag{10}$$

Scaling through $v_{max} = \frac{x_0}{t_0}$, $\rho = \rho_{max}\rho^*$, $x = x_0 x^*$ and $t = t_0 t^*$ results in

$$\rho_t + [(1 - \rho)\rho]_x = \epsilon\rho_{xx} \quad \text{with} \quad \epsilon = \frac{D}{v_{max}x_0} \text{ and } 0 \le \rho \le 1. \tag{11}$$

The transformation $u = 2\rho - 1$ leads to the viscid Burgers equation as it is shown in (1) with the conditions $-1 \le u \le 1$.

3 Mathematical analysis

From the mathematical point of view Burgers equations are a very interesting and suggestive topic: It turns out that a study of them leads to most of the ideas that arises in the field of nonlinear hiperbolyc waves.

3.1 Inviscid

We will focus first on equation (2). Specifically, we will deal with the initial value problem

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{12}$$

As it as has been suggested previously, although (12) seems to be a very *innocent* problem *a priori* it hides many unexpected phenomena.

Characteristic method. The similarity with the advection equation suggests considering, as a first approach to solve (12), the characteristic method. In this case the characteristic equation would be

$$\begin{cases} x'(t) = u(x(t), t), & t > 0, \\ x(0) = x_0. \end{cases} \tag{13}$$

If $x(t)$ and $u(x,t)$ ($\in C^1$) are solutions of (13) and (12) respectively, then

$$\frac{d}{dt}\left[u(x(t),t)\right] = u_t(x(t),t) + x'(t)u_x(x(t),t) = u_t(x(t),t) + u(x(t),t)u_x(x(t),t) = 0,$$

i.e, u is constant along the characteristic curve $x(t)$ and therefore

$$u(x(t),t) = u(x(0),0) = u_0(x_0), \tag{14}$$

which considering the sistem (13), leads to conclude that the characteristic curves are straight lines determined by initial data :

$$x = x_0 + u_0(x_0)t, \ t > 0. \tag{15}$$

In principle, one could invert (15) to obtain $x_0 = x_0(x,t)$. Then, using (14), one would obtain the solution $u(x,t) = u_0(x_0(x,t))$. However, as the required inversion usually can not be accomplished analytically, one could use a symbolic calculation program to construct a discretized solution of (12) by dragging the initial data $u_0(x_0)$ along the characteristic line (15). This strategy is followed by the following program in MATHEMATICA:

```
Clear[f0, fval, x, f]
f0[x_] = Exp[-(2 (x - 1))^2];   (*Condición inicial*)
x[t_, x0_] = x0 + f0[x0] t;
f[t_, x0_] = f0[x0];
fval[t_] := Table[{x[t, x0], f[t, x0]}, {x0, -.5, 3, .1}]

(*Dibujo de las características*)
Plot[Table[x[t, x0], {x0, -.5, 3, .1}], {t, 0, 2},AxesLabel -> {Text[Style["t", Italic, 23]],
   Text[Style["x", Italic, 23]]}]

(*Dibujo de la solución u(x,t)*)
ListPlot3D[{Table[Table[{x[t, x0], t, f[t, x0]}, {x0, -.5, 3, .1}], {t, 0, 2, 0.1}]},
```

```
ColorFunction -> "DarkRainbow", PlotStyle -> Directive[Opacity[0.9]], MeshFunctions -> {#2 &},
Mesh -> 5, PlotRange -> All, Axes -> {True, True, True} ,
 Boxed -> False, ImageSize -> 600,AxesLabel -> {Text[Style["x", Italic, 23]],
   Text[Style["t", Italic, 23]], Text[Style["u", Italic, 23]]}]
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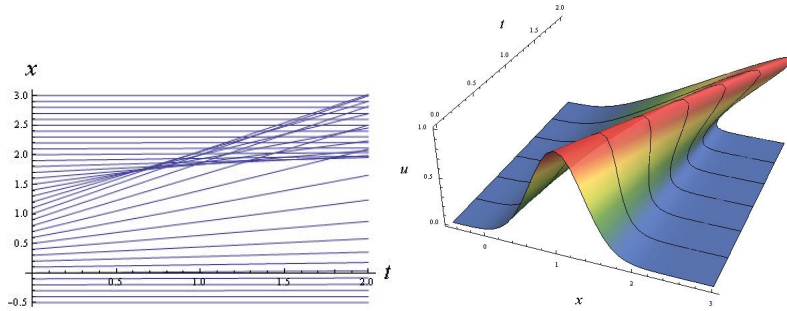

Figure 1: Output of the programme for $u_0(x) = e^{-(2(x-1))^2}$. Some characteristic lines are presented in the first picture.

Looking to this example one quickly finds that problem (12) exibits (under certain initial conditions) what is called the wavebreaking phenomenon: The peak of the pulse moves the fastest because wave speed increases with increasing amplitude. This can also be understood by looking to the corresponding slope of the characteristic line in the (x, t) plane. Eventually the peak overtakes the rest of the pulse, or the characteristic cross with another one, and the solution becomes multiple valued. The time value t_B at which this happens for the first time is called the breaking time.

Breaking Time. In order to determine the breaking time, let us consider two characteristics that arise from initial conditions x_1 and $x_2 = x_1 + \Delta x$. According to (15), these characteristics will cross when
$$x_1 + u_0(x_1)t = x_2 + u_0(x_2)t.$$

Solving for t leads to

$$t = -\frac{x_1 - x_2}{u_0(x_1) - u_0(x_2)} = \frac{\Delta x}{u_0(x_1) - u_0(x_1 + \Delta x)}. \tag{16}$$

When $\Delta x \to 0$ then the time in (16) converges to $t = -\frac{1}{u_0'(x_1)}$. To find the breaking time t_B we find the minimum (positive) value for t,

$$t_B = \min_{x \in \mathbb{R}} \left[-\frac{1}{u_0'(x)} \right].$$

Although the solution obtained via the characteristic method seems to be valid for $t < t_B$, it is clear that mathematically multiple valued solutions in a region is unphysical in most of the situations (think of u as density) so it can not be accepted for $t > t_B$. What is then the solution

of (12) after the breaking time? This question is in the basis of what is called the shock-fitting problem:

The only way to establish a solution after the breaking time is to allow discontinuities of u. Such discontinuity is called a shock. This requires some mathematical extension of what we mean by a solution to (12), since strictly speaking the derivatives of u will not exist at a discontinuity. It can be done through the concept of a *weak solution*. However, by expanding our class of solutions to include discontinuous solutions, we can no longer guarantee the uniqueness of the solution to (12). The nonuniqueness can be resolved only by appeal to physically inspired criteria. More specifically, we need some shock conditions relating the jumps of u across the discontinuity.

The Rankine-Hugoniot jump condition. The Rankine-Hugoniot jump condition determines the position of a shock at a given time. This condition emerges when one consider the equation (3) in integral form by integrating respect to x over $x_2 < x < x_1$,

$$\frac{d}{dt} \int_{x_2}^{x_1} u(x,t)\, dx + f(u)_{x_2}^{x_1} = 0. \tag{17}$$

The physical background is clear, the equation (17) is the traditional form of a conservation law: The rate of change of the total amount of u in any section $x_2 < x < x_1$ must be balanced by the net inflow across x_1 and x_2. Suppose now that there is a discontinuity at $x = s(t)$ and that x_1 and x_2 are chosen so that $x_2 < s(t) < x_1$. Suppose u and its first derivative are continuous in $x_1 \geq x > s(t)$ and in $s(t) > x \geq x_2$, and have finite limits as $x \to s(t)$ from above and below. Then, if we assume (17) we have:

$$f(u)_{x_2} - f(u)_{x_1} = \frac{d}{dt} \int_{x_2}^{s(t)} u(x,t)\, dx + \frac{d}{dt} \int_{s(t)}^{x_1} u(x,t)\, dx =$$

$$u_2 \dot{s} - u_1 \dot{s} + \int_{x_2}^{s(t)} u_t(x,t)\, dx + \int_{s(t)}^{x_1} u_t(x,t)\, dx$$

where u_2, u_1 are the value of u as $x \to s(t)$ from below and above respectively and $\dot{s} = s'(t)$. Since u_t is bounded in each of the intervals separately, the integral tend to zero in the limit as $x_1 \to s^+, x_2 \to s^-$ and, therefore,

$$f(u)_2 - f(u)_1 = (u_2 - u_1)\dot{s}.$$

Note that the argument above is valid for a general hiperbolic conservation law of the form $u_t + [f(u)]_x = 0$. If we focus on equation (3) we obtain

$$\dot{s} = \frac{f(u)_2 - f(u)_1}{u_2 - u_1} = \frac{\frac{1}{2}u_2^2 - \frac{1}{2}u_1^2}{u_2 - u_1} = \frac{1}{2}(u_1 + u_2) \tag{18}$$

This is the first condition that the shock of our discontinuous single valued solution will have to satisfy.

Note on weak solutions. In this section we will work within hiperbolic conservation laws, i.e., as it has been told in the introduction, equations in the form

$$u_t + [f(u)]_x = 0 \tag{19}$$

with some f (see (3)). Mathematically, a discontinuous solution with continuously differentiable parts satisfying (19) and such that $\dot{s} = \frac{f(u)_2 - f(u)_1}{u_2 - u_1}$ can be considered a weak solution of (19). As a motivation for the idea, let $\phi(x, t)$ be an infinitely smooth function in $\mathbb{R} \times [0, +\infty)$ that vanishes on the boundary. Let $u(x, t)$ be a smooth solution of the hyperbolic conservation law given. Then

$$0 = \int_0^\infty \int_{-\infty}^\infty (\phi u_t + \phi\,[f(u)]_x)\ dxdt \overset{\text{Int. by parts}}{=}$$

$$\int_{-\infty}^\infty (\phi u|_0^\infty\ dx + \int_0^\infty (\phi f(u)|_{-\infty}^\infty\ dt - \int_0^\infty \int_{-\infty}^\infty (\phi_t u + \phi_x f(u))\ dxdt \overset{\phi = 0 \text{ on bound.}}{=}$$

$$-\int_0^\infty \int_{-\infty}^\infty (\phi_t u + \phi_x f(u))\ dxdt$$

In view of this development it is natural to say that $u(x, t)$ is a weak solution of the conservation law if for any $\phi(x, t)$ the equation

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t u + \phi_x f(u))\ dxdt = 0 \tag{20}$$

holds. If we consider now a weak solution of (19) with a simple jump discontinuity across $\Omega = \{(s(t), t)\}$ it is easy to show, dividing the integral domain in (20) by Ω, that $\dot{s} = \frac{f(u)_2 - f(u)_1}{u_2 - u_1}$.

At first sight the weak solution concept appears to be the appropriate mathematical tool to work with. However, it turns out that there are still some situations in which this concept is not enough to guarantee the uniqueness. Becouse of that, there is an additional condition, the so called entropy condition, introduced to eliminate nonphysical weak solutions. There is a number of variations in which this condition can be presented, we will mention only the simplest one.

Entropy condition.   For the equation (19), a discontinuity propagating with speed $\dot{s} = \frac{f(u)_2 - f(u)_1}{u_2 - u_1}$ satisfies the entropy condition if $f'(u_2) > s > f'(u_1)$. For the Burgers equation this entropy condition reduces to the requirement that if a discontinuity is propagating with speed $\dot{s}$ then $u_2 > u_1$.

As its name suggests, this condition is the mathematical translation of the condition that says that in every physical process the entropy of the system is nondecreasing. As is well known, this is a fundamental assumption in thermodynamics.

Examples.

  1. **Riemann problem**. The initial value problem (12) with discontinuous initial condition of the form
$$u_0(x) = \begin{cases} u_L, & x \leq 0, \\ u_R, & x > 0 \end{cases}$$
  is called a Riemann problem. As a first example we will take the simple assumption $u_L = 1$ and $u_R = 0$. If we use the characteristic method presented above we obtain
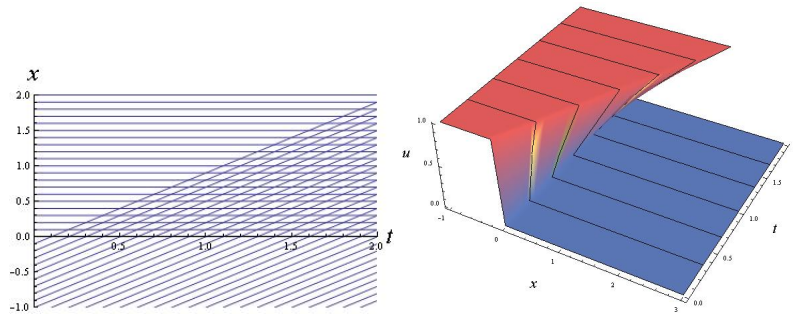
Figure 2:

It is obvious that this solution needs a shock fitting just from the beginnig. Obeying the Rankine-Hugoniot condition we obtain that the discontinuity must travel with the speed $\dot{s} = \frac{1}{2}$ and therefore the solution becomes

$$u(x,t) = \left\{ \begin{array}{ll} 1, & x \leq \frac{t}{2}, \\ 0, & x > \frac{t}{2}. \end{array} \right.$$

Note that this solution also satisfies the entropy condition. In this case it can be shown that, in fact, this is the unique weak solution for the problem.

If we consider now just the opposite problem, i.e, $u_L = 0$ and $u_R = 1$, the hole situation changes. There ere at least two solutions satisfying the Rankine-Hugoniot condition:

$$u_1(x,t) = \left\{ \begin{array}{ll} 0, & x \leq 0, \\ \frac{x}{t}, & 0 < x \leq t, \quad \text{and} \quad u_2(x,t) = \left\{ \begin{array}{ll} 0, & x \leq \frac{t}{2}, \\ 1, & x > \frac{t}{2}. \end{array} \right. \\ 1, & x > t. \end{array} \right.$$

In this situation, the fact that the second one does not satisfy the entropy condition allows us to discard it and keep the continuous solution u_1.

2. In order to get a more accurate idea of what it means to fit a shock we will consider the initial condition

$$u_0(x) = \left\{ \begin{array}{ll} 1, & x < 0 \\ 1 - x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{array} \right. \quad (21)$$

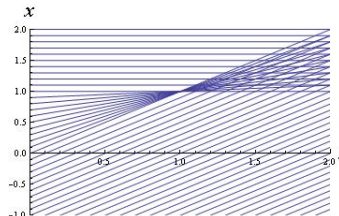for (12). If we draw some characteristics lines we obtain

Figure 3:

As it is seen in the figure we have $t_B = 1$ so it is from there that we have to *fix* the solution. Appealing again to the R-H condition, the shock path for $t > t_B$ must be $s(t) = \frac{t+1}{s}$. In the following figure it is shown the solution to the problem: Until $t_B = 1$ we can keep the solution obtained via the characteristic method but then we have to rid it and fix it.
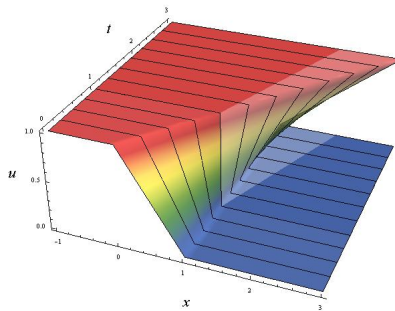

Figure 4: In this figure we can see the part of the *characteristic solution* that has to be replaced in order to obtain the solution we are lookin for (the darker part of the figure).

Note that the entropy condition is satisfied by this solution.

Vanishing viscosity approach. Fitting a shock in the region of overlapped characteristics following the Rankine-Hugoniot and the entropy condition seems mathematical unnatural. However, there is a more plausible and natural approach to construct the discontinuous entropy solution. This is the so called vanishing viscosity approach. The first step is to add the viscosity-dispersion term ϵu_{xx} to obtain equation (1). In the next section we will see that adding this term has an important effect: it suppresses the wavebreaking. The basic reason is easy to understand: dispersion causes waves to spread and this acts against the steepening effect of the nonlinearity. Hence we expect to obtain a smooth solution of (1) even with discontinuous initial data. The vanishing viscosity approach is based on the natural intuition that this smooth solution of (1) will approach a shock wave as $\epsilon \to 0$. On the physical interpretation point of view this is the natural way of setting the correct solution of problem (12): As it has been said in the subsection (2.1), the inviscid Burgers equation usually appears as an idealization of a more precise model where there is always some degree of viscosity. That is why the only relevant solutions are those that can be obtained with this approach. There is a result due to Kruzkov that guarantees the uniqueness of the solution obtained by this way.

The vanishing viscosity approach would be completely useless if one can not find the solution to the correspondig initial value problem (the problem (12) where it has been added the viscosity-dispersion term). Fortunately the Cole-Hopf transformation, introduced by Cole (1951) and Hopf (1950) independently, transforms the equation (1) into the heat equation and therefore it can be solved. The Cole-Hopf transformation is introduced in the next section.

3.2 Viscid

Consider now the initial value problem for the viscid Burgers equation

$$\begin{cases} u_t + uu_x = \epsilon u_{xx}, & x \in \mathbb{R}, \quad t > 0, \quad \epsilon > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{22}$$

The Cole-Hopf transformation.   The Cole-Hopf transformation is defined by

$$u = -2\epsilon \frac{\varphi_x}{\varphi}. \tag{23}$$

Operating in (23) we find that

$$u_t = \frac{2\epsilon \left(\varphi_t \varphi_x - \varphi \varphi_{xt}\right)}{\varphi^2} \quad , \quad uu_x = \frac{4\epsilon^2 \varphi_x \left(\varphi \varphi_{xx} - \varphi_x^2\right)}{\varphi^3} \quad \text{and}$$

$$\epsilon u_{xx} = -\frac{2\epsilon^2 \left(2\varphi_x^3 - 3\varphi \varphi_{xx}\varphi_x + \varphi^2 \varphi_{xxx}\right)}{\varphi^3}.$$

Substituting this expressions into (1),

$$\frac{2\epsilon \left(-\varphi \varphi_{xt} + \varphi_x \left(\varphi_t - \epsilon \varphi_{xx}\right) + \epsilon \varphi \varphi_{xxx}\right)}{\varphi^2} = 0 \iff -\varphi \varphi_{xt} + \varphi_x \left(\varphi_t - \epsilon \varphi_{xx}\right) + \epsilon \varphi \varphi_{xxx} = 0 \iff$$

$$\varphi_x \left(\varphi_t - \epsilon \varphi_{xx}\right) = \varphi \left(\varphi_{xt} - \epsilon \varphi_{xxx}\right) = \varphi \left(\varphi_t - \epsilon \varphi_{xx}\right)_x.$$

Therefore, if φ solves the heat equation $\varphi_t - \epsilon \varphi_{xx} = 0$ $x \in \mathbb{R}$, then $u(x,t)$ given by transformation (23) solves the viscid Burgers equation (1).

To completely transform the problem (22) we still have to work with the initial condition function. To do this, note that (23) can be written as

$$u = -2\epsilon \left(\log \varphi\right)_x.$$

Hence

$$\varphi(x,t) = e^{\left(-\int \frac{u(x,t)}{2\epsilon} \, dx\right)}.$$

It is clear from (23) that multiplying φ by a constant does not affect u, so we can write the last equation as

$$\varphi(x,t) = e^{\left(-\int_0^x \frac{u(y,t)}{2\epsilon} \, dy\right)}. \tag{24}$$

The initial condition on (22) must therefore be transformed by using (24) into

$$\varphi(x,0) = \varphi_0(x) = e^{\left(-\int_0^x \frac{u_0(y)}{2\epsilon} \, dy\right)}.$$

In summary, we have reduced the problem (22) to this one

$$\begin{cases} \varphi_t - \epsilon \varphi_{xx} = 0, & x \in \mathbb{R}, \quad t > 0, \quad \epsilon > 0, \\ \varphi(x,0) = \varphi_0(x) = e^{\left(-\int_0^x \frac{u_0(y)}{2\epsilon} \, dy\right)}, & x \in \mathbb{R}. \end{cases} \tag{25}$$

Heat equation.   The general solution of the initial value problem for the heat equation is well known and can be handled by a variety of methods. Taking the Fourier transform with respect to x for both heat equation and the initial condition $\varphi_0(x)$ in problem (25) we obtain the first order ODE

$$\begin{cases} \widehat{\varphi}_t = \xi^2 \epsilon \widehat{\varphi}, & \xi \in \mathbb{R} \quad t > 0 \quad \epsilon > 0 \\ \widehat{\varphi}(\xi,0) = \widehat{\varphi}_0(\xi) & \xi \in \mathbb{R}, \end{cases} \tag{26}$$

where $\widehat{\varphi}(\xi,t) = \int_{-\infty}^{\infty} \varphi(x,t)e^{i\xi x} \, dx$. The solution for this problem is

$$\widehat{\varphi}(\xi,t) = \widehat{\varphi}_0(\xi)e^{\xi^2 \epsilon t}.$$

To recover $\varphi(x,t)$ we have to use the inverse Fourier transformation F^{-1}, namely,

$$\varphi(x,t) = F^{-1}(\widehat{\varphi}(\xi,t)) = F^{-1}(\widehat{\varphi}_0(\xi)e^{\xi^2 \epsilon t}) = \varphi_0(x) * F^{-1}(e^{\xi^2 \epsilon t}),$$

where $*$ denotes the convolution product.

On the other hand

$$F^{-1}(e^{\xi^2 \epsilon t}) = \frac{1}{2\sqrt{\pi \epsilon t}} e^{-\frac{x^2}{4\epsilon t}},$$

so the initial value problem (25) has the analytic solution

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi \epsilon t}} \int_{-\infty}^{\infty} \varphi_0(\xi) e^{-\frac{(x-\xi)^2}{4\epsilon t}} \, d\xi.$$

Finally, from (23), we obtain the analytic solution for the problem (22)

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} \varphi_0(\xi) e^{-\frac{(x-\xi)^2}{4\epsilon t}} \, d\xi}{\int_{-\infty}^{\infty} \varphi_0(\xi) e^{-\frac{(x-\xi)^2}{4\epsilon t}} \, d\xi}. \tag{27}$$

Examples.   We will use now (27) to draw the exact solution of (22) with different initial conditions. In particular we will take the same initial conditions that we took for the inviscid case and we will vary the viscosity parameter ϵ in order to visualize the vanishing viscosity effect.
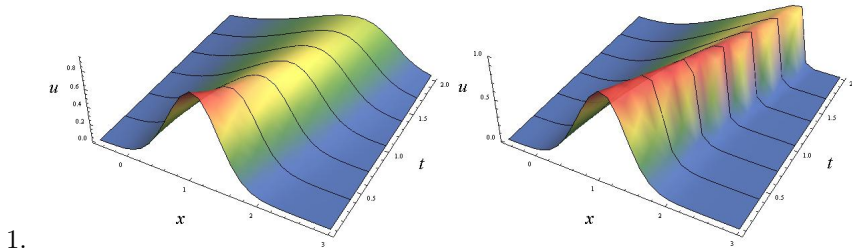

1.

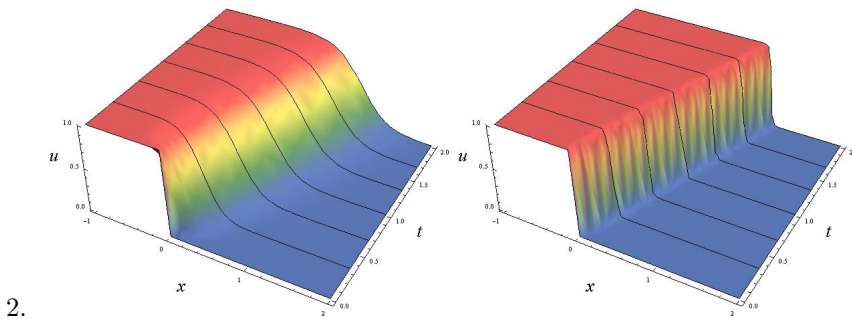Figure 5: $u_0(x) = e^{-(2(x-1))^2}$ with $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively.

2.

Figure 6: Riemann problem with $u_L = 1$, $u_R = 0$ and $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively.
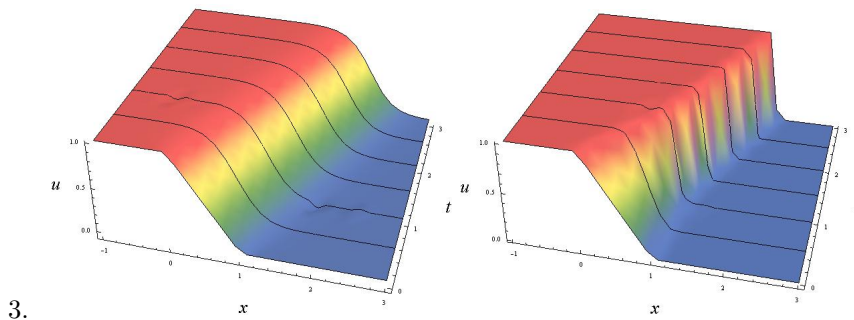
Figure 7: Same initial condition that the second example of last section with $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively. We can see how the solution approaches the fixed one (figure 4) as $\epsilon \to 0$.

4   Numerical methods

At this point we should be convinced of the complexity that nonlinearity hides from the point of view of the mathematical analysis. This complexity also arises when one trie to solve Burgers equation using numerical methods. Major problems arise when one tries to approximate the solutions for which we have admit discontinuities: It could happen that the method converges to another weak solution of our original equation (or that is the wrong weak solution, i.e, does not satisfy the entropy condition). In order to avoid that to happend, numerical schemes must satisfy certain non-obvious conditions.

4.1   Inviscid

Up-wind nonconservative.   If we consider the inviscid burgers equation in the quasilinear form

$$u_t + uu_x = 0, \tag{28}$$

then a natural finite difference method obtained by a forward in time and backward in space discretization of the derivatives is

$$\boxed{U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n \left(U_j^n - U_{j-1}^n \right).} \tag{29}$$

We will refer to (29) as the Up-wind nonconservative scheme. Although this method is consistent with (4.1) and is adequate for smooth solutions, it will not converge in general to a discontinuous weak solution as the grid is refine.

To prevent a method from converging to non-solutions, there is a simple condition that we can require : The method will have to be in **conservation form**, i.e, in the form

$$U_j^{n+1} = U_j^n - \frac{k}{h} \left[F \left(U_{j-p}^n, U_{j-p+1}^n, \dots, U_{j+q}^n \right) - F \left(U_{j-p-1}^n, U_{j-p}^n, \dots, U_{j+q-1}^n \right) \right] \tag{30}$$

where F is some function of $p + q + 1$ arguments called the numerical flux function. Methods that conform to this scheme are called conservative methods. The following methods belong all to this class. We will first present the schemes for a general scalar conservation law $u_t + [f(u)]_x = 0$ and then particularize them to the inviscid Burgers equation showing how thay are going to be implemented in MATLAB.

Up-wind conservative.   If we consider a general scalar conservation law

$$u_t + [f(u)]_x = 0$$

and use the standard finite difference discretizations, we obtain the conservation method called Up-wind conservative, namely,

$$U_j^{n+1} = U_j^n - \frac{k}{h} \left[f\left(U_j^n\right) - f\left(U_{j-1}^n\right) \right]. \tag{31}$$

Note that this is in the form (30) with $p = 0$, $q = 1$ and $F(U,V) = f(U)$.

For the Burgers equation we have

$$U_j^{n+1} = U_j^n - \frac{k}{h} \left[\frac{1}{2} \left(U_j^n\right)^2 - \frac{1}{2} \left(U_{j-1}^n\right)^2 \right]. \tag{32}$$

Lax-Friedrichs.   The Lax-Friedrichs method to nonlinear systems takes the form

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - \frac{k}{2h} \left[f(U_{j+1}^n) - f(U_{j-1}^n) \right]$$

Note that this is in the form (30) with $p = 0$, $q = 1$ and $F(U,V) = \frac{h}{2k}(U - V) + \frac{1}{2}(f(U) + f(V))$.

For the Burgers equation we have

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - \frac{k}{2h} \left[\frac{1}{2} \left(U_{j+1}^n\right)^2 - \frac{1}{2} \left(U_{j-1}^n\right)^2 \right]. \tag{33}$$

Lax-Wendroff.   The methods consider hitherto are all of first order. The Lax-Wendroff method to nonlinear conservation laws is a second order method and takes the form

$$U_j^{n+1} = U_{j+1}^n - \frac{k}{2h} \left(f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right) \right) + \frac{k^2}{2h^2} \left[A_{j+\frac{1}{2}} \left(f(U_{j+1}^n) - f(U_j^n) \right) - A_{j-\frac{1}{2}} \left(f(U_j^n) - f(U_{j-1}^n) \right) \right]$$

where $A_{j\pm\frac{1}{2}}$ is the jacobian matrix $A(u) = f'(u)$ evaluated at $\frac{1}{2}(U_j^n + U_{j\pm1}^n)$.
For Burgers equation we have $f'(u) = u$ so

$$\begin{aligned} U_j^{n+1} &= U_{j+1}^n - \frac{k}{2h} \left(\frac{1}{2} \left(U_{j+1}^n\right)^2 - \frac{1}{2} \left(U_{j-1}^n\right)^2 \right) + \\ & \quad \frac{k^2}{2h^2} \left[\left(\frac{1}{2}(U_j^n + U_{j+1}^n) \right) \left(\frac{1}{2} \left(U_{j+1}^n\right)^2 - \frac{1}{2} \left(U_j^n\right)^2 \right) - \left(\frac{1}{2}(U_j^n + U_{j-1}^n) \right) \left(\frac{1}{2} \left(U_j^n\right)^2 - \frac{1}{2} \left(U_{j-1}^n\right)^2 \right) \right]. \end{aligned} \tag{34}$$

MacCormack.   Another method of the same type is known as MacCormack's method. This method uses first forward differencing and then backward differencing to achieve second order accuracy:

$$\begin{aligned} U_j^* &= U_j^n - \frac{k}{h} \left[f\left(U_{j+1}^n\right) - f\left(U_j^n\right) \right] \\ U_j^{n+1} &= \frac{1}{2}(U_j^n + U_j^*) - \frac{k}{2h} \left[f\left(U_j^*\right) - f\left(U_{j-1}^*\right) \right] \end{aligned} \tag{35}$$

Godunov. The last method for solving Burgers equation that will be presented in this paper belongs to the so called finite volume methods. The idea of Godunov's method is the following. Let U_j^n be a numerical solution on the n-th layer. Then we define a function $\widehat{u}^n(x,t)$ for $t_n < t < t_{n+1}$ as follows. At $t = t_n$,

$$\widehat{u}^n(x,t) = U_j^n, \ x_j - \frac{h}{2} < x < x_j + \frac{h}{2}, \ j = 2, \ldots, n-1.$$

Then we define $\widehat{u}(x,t)$ to be the solution of the collection of Riemann problems on the interval $[t_n, t_{n+1}]$. If k is small enough so that the characteristics starting at the points $x_{j\pm\frac{h}{2}}$ do not intersect within this interval, then $\widehat{u}(x,t_{n+1})$ is determined unambiguously. Then the numerical solution on the next layer, U_j^{n+1} is defined by averaging $\widehat{u}(x,t_{n+1})$ over the intervals $x_j - \frac{h}{2} < x < x_j + \frac{h}{2}$. This idea reduces to a very simple conservative scheme:

$$U_j^{n+1} = U_j^n - \frac{k}{h} \left[F\left(U_j^n, U_{j+1}^n \right) - F\left(U_{j-1}^n, U_j^n \right) \right]$$

where the numerical flux F is defined by $F(U,V) = \frac{(u^*)^2}{2}$ where u^* is defined as follows

If $U \geq V$ then

$$u^* = \begin{cases} U, & \text{if } \frac{U+V}{2} > 0, \\ V, & \text{in other case.} \end{cases}$$

If $U < V$ then

$$u^* = \begin{cases} U, & \text{if } U > 0, \\ V, & \text{if } V < 0, \\ 0, & \text{if } U \leq 0 \leq V. \end{cases}$$

4.2 Viscid

Parabolic Method. Consider equation (1) in the form

$$u_t + [f(u)]_x = \epsilon u_{xx} \quad \text{with} \quad f(u) = \frac{u^2}{2}. \tag{36}$$

If we formally integrate (36) from $x_{j-\frac{1}{2}}$ to $x_{j+\frac{1}{2}}$ and rewrite the equation, we obtain

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u_t \, dx - [\epsilon u_x]_{x_{j-1/2}}^{x_{j+1/2}} = -[f(u)]_{x_{j-1/2}}^{x_{j+1/2}} \tag{37}$$

We will now look for approximations of each of the terms in (37). We have that

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u_t \, dx \approx \frac{du}{dt}(x_j, t) h,$$

$$-[\epsilon u_x]_{x_{j-1/2}}^{x_{j+1/2}} = \epsilon \left[u_x(x_{j-1/2}, t) - u_x(x_{j+1/2}, t) \right] \approx \epsilon \left[\frac{u(x_j, t) - u(x_{j-1}, t)}{h} - \frac{u(x_{j+1}, t) - u(x_j, t)}{h} \right] =$$

$$-\epsilon \frac{u(x_{j+1}, t) - 2u(x_j, t) + u(x_{j-1}, t)}{h}$$

and

$$-[f(u)]_{x_{j-1/2}}^{x_{j+1/2}} = f(u(x_{j-1/2}, t)) - f(u(x_{j+1/2}, t))$$

Substituting these expressions into (37), dividing by h and taking $U_j(t)$ to be the function $u(x_j, t)$, we obtain the system of ordinary differential equations

$$\frac{dU_j}{dt} - \epsilon \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = \frac{f(U_{j-\frac{1}{2}}) - f(U_{j+\frac{1}{2}})}{h}.$$

As a computational method we discretize the time derivative in the last expression by a forward difference to obtain the explicit method

$$U_j^{n+1} = U_j^n + k \left(\epsilon \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + \frac{f(U_{j+\frac{1}{2}}^n) - f(U_{j-\frac{1}{2}}^n)}{h} \right). \tag{38}$$

where we take $f(U_{j-\frac{1}{2}})$ to be the average of $f(U_j)$ and $f(U_{j\pm 1})$.

4.3 Code

```
%                      ------------Burgers program-------------
% This program computes numerical solutions for viscid and inviscid Burgers equation.
% It needs the functions df.m, f.m, nf.m and uinit.m .

clear all;

%Selection of equation and method.
type_burger = menu('Choose the equation:', ...
'Inviscid Burgers equation','Viscid Burgers equation');

if (type_burger == 1)
method = menu('Choose a numerical method:', ...
'Up-wind nonconservative','Up-wind conservative','Lax-Friedrichs','Lax-Wendroff','MacCormack','Godunov');
else
method = menu('Choose a numerical method:', ...
'Parabolic Method');
end

ictype = menu('Choose the initial condition type:', ...
'Piecewise constant (shock)','Piecewise constant (expansion)','Gaussian','Piecewise continuous');

%Selection of numerical parameters (time step, grid spacing, etc...).
xend = 2;    % x-axis size.
tend = 2;    % t-axis size.
N = input('Enter the number of grid points: ');
dx = xend/N; % Grid spacing
dt = input('Enter time step dt: ');

x   = 0:dx:xend;
nt = floor(tend/dt);
dt = tend / nt;

%Set up the initial solution values.
u0  = uinit(x,ictype);  %Call to the function "uinit".
u   = u0;
unew = 0*u;

%Implementation of the numerical methods.
if (type_burger == 1)
 for i = 1 : nt,
  switch method
```

```
  case 1  %Up-wind nonconservative
   unew(2:end) = u(2:end) - dt/dx * u(2:end) .* (u(2:end) - u(1:end-1));
   unew(1)  = u(1);    %u(3:end): subvector de u desde 3 hasta el final.

  case 2  %Up-wind conservstive
   unew(2:end)= u(2:end) - dt/dx * (f(u(2:end)) - f(u(1:end-1)));
   unew(1)   = u(1);

  case 3  %Lax-Friedrichs
    unew(2:end-1) = 0.5*(u(3:end)+u(1:end-2)) - 0.5*dt/dx * ...
(f(u(3:end)) - f(u(1:end-2)));
   unew(1)  = u(1);
   unew(end) = u(end);

  case 4  %Lax-Wendroff
    unew(2:end-1) = u(2:end-1) ...
- 0.5*dt/dx * (f(u(3:end)) - f(u(1:end-2))) ...
+ 0.5*(dt/dx)^2 * ...
( df(0.5*(u(3:end) + u(2:end-1))) .* (f(u(3:end)) - f(u(2:end-1))) - ...
  df(0.5*(u(2:end-1) + u(1:end-2))) .* (f(u(2:end-1)) - f(u(1:end-2))) );
   unew(1)  = u(1);
   unew(end) = u(end);

  case 5  %MacCormack
   us   = u(1:end-1) - dt/dx * (f(u(2:end)) - f(u(1:end-1)));
   unew(2:end-1)= 0.5*(u(2:end-1) + us(2:end)) - ...
0.5*dt/dx* (f(us(2:end)) - f(us(1:end-1)));
   unew(1)  = u(1);
   unew(end) = u(end);

  case 6  %Godunov
   unew(2:end-1) =u(2:end-1)- dt/dx*(nf(u(2:end-1),u(3:end)) - nf(u(1:end-2),u(2:end-1)));
   unew(1)  = u(1);
   unew(end) = u(end);

 end
  u = unew;
  U(i,:) = u(:);
 end

else

 for i = 1 : nt,
  switch method

   case 1  %Parabolic method
    D=0.01;
    fminus = 0.5*( f(u(2:end-1)) + f(u(1:end-2)) );
    fplus  = 0.5*( f(u(2:end-1)) + f(u(3:end)) );
    unew(2:end-1) = u(2:end-1) + dt*(D*(u(3:end)-2*u(2:end-1)+u(1:end-2))/(dx)^2  ...
        - (fplus-fminus)/dx );
    unew(1)  = u(1);
    unew(end) = u(end);

  end
  u = unew;
  U(i,:) = u(:);
 end

end
```

```
U=[u0;U];
T=0:dt:tend;

%Plot of the solutions.
figure(1)
surf(x,T,U)
shading interp
xlabel('x'), ylabel('t'), zlabel ('u(x,t)');
grid on
colormap('Gray');
```

The initial condition is introduced through `uinit.m`:

```
function ui = uinit( x, ictype )
xshift = 0;
if (ictype==1)        %Shock (uL > uR)
  uL = 1;
  uR = 0;
  ui = uR + (uL-uR) * ((x-xshift) <= 0.0);

elseif (ictype==2)   %Expansion (uL < uR)
  uL = 0.5;
  uR = 1.0;
  ui = uR + (uL-uR) * ((x-xshift) <= 0.0);

elseif (ictype==3)   %Gaussian
  ui = exp(-2*(x - 1).^2);

elseif (ictype==4)   %Piecewise continuous
    for i=1:length(x)
     if (x(i) < 0)
         ui(i)=1;
      elseif (0<= x(i) &&  x(i) < 1)
         ui(i)=1-x(i);
      elseif (x(i) >= 1)
         ui(i)=0;
     end
    end
end
```

The functions `f.m`, `df.m` and `nf.m` (used on Godunov):

```
function ret = f( u )
ret = 0.5 * u.^2;

function ret = df( u )
ret = u;

function ret = nf( u, v )
 for i = 1:length(u)
  if (u(i) >= v(i))
     if ((u(i)+v(i))/2 > 0)
         ustar(i)=u(i);
```

```
    else
        ustar(i)=v(i);
    end
 else
    if (u(i)>0)
        ustar(i)=u(i);
    elseif (v(i)<0)
        ustar(i)=v(i);
    else
        ustar(i)=0;
    end
 end
end
ret =f(ustar);
```

4.4 Numerical experiments
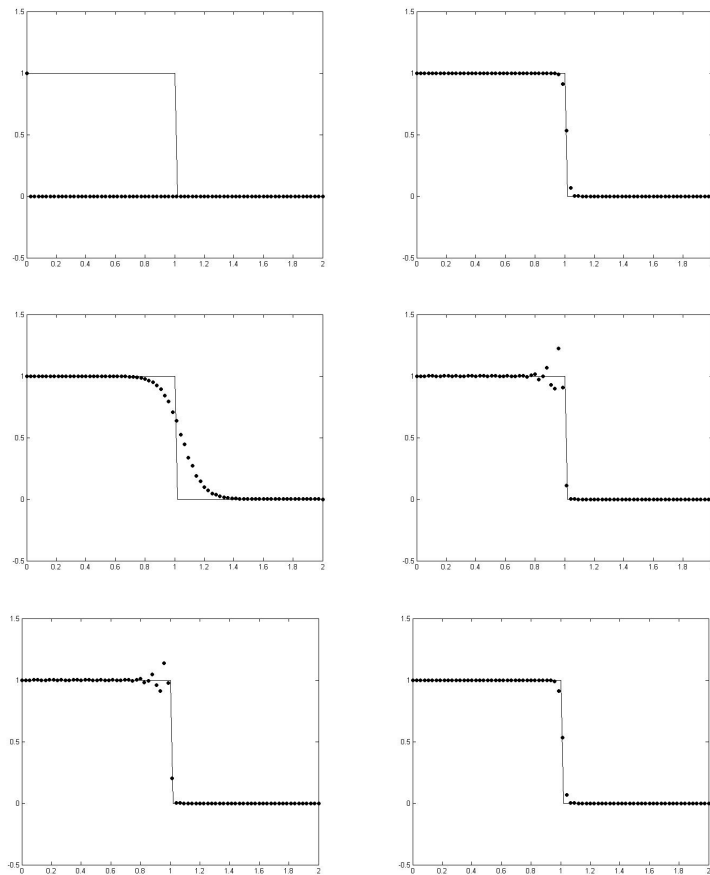
1. Riemann problem with $u_L = 1$ and $u_R = 0$.

Figure 8: Up-wind non conservative, Up-wind conservative, Lax-Friedrichs, Lax-Wendroff, Mac-Cormack and Godunov respectively.
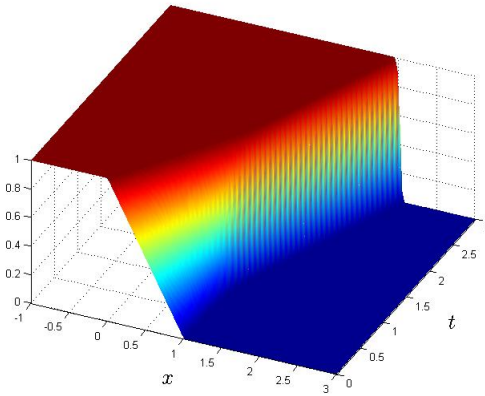
2. The inviscid Burgers equation with the initial condition (21) solved by Godunov method:

Figure 9:

Note that the obtained solution agrees with the one presented in the second example of section 3.

3. The next examples have been obtained using the parabolic method with $N = 100$ and $dt = 0.001$. In order to display the effectiveness of the method we can compare the results with the exact solutions shown in section 3.2.
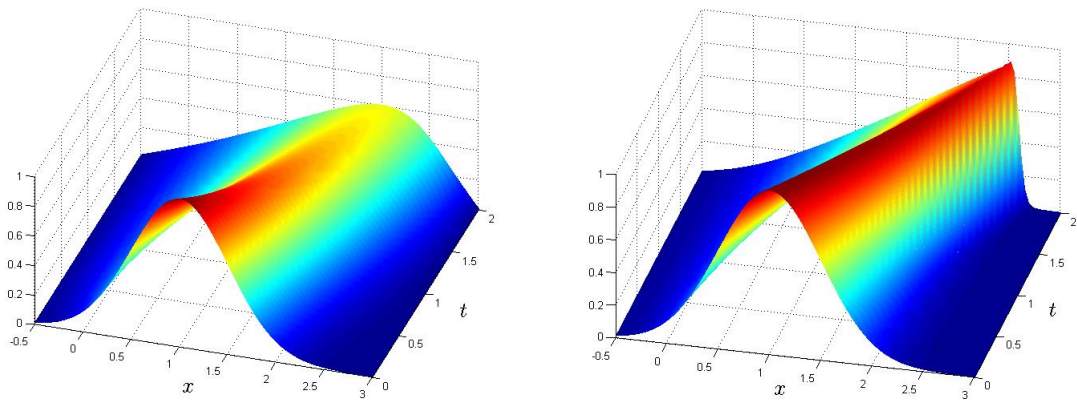

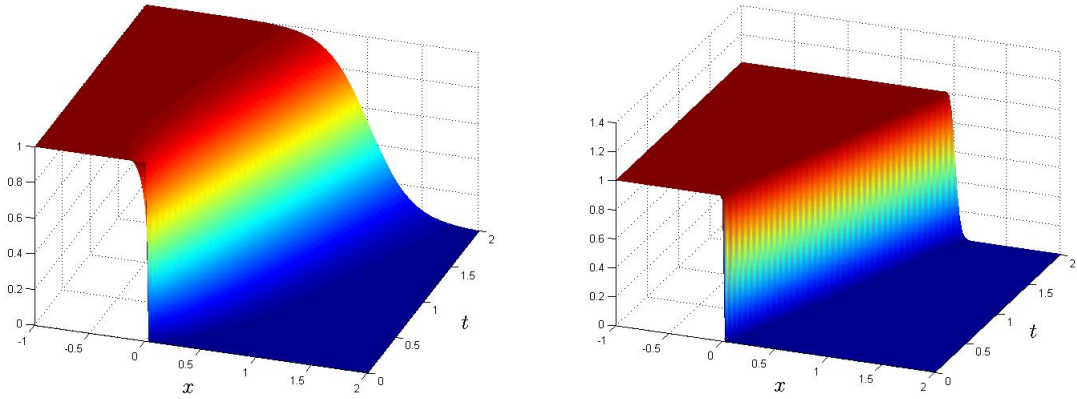Figure 10: $u_0(x) = e^{-(2(x-1))^2}$ with $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively.

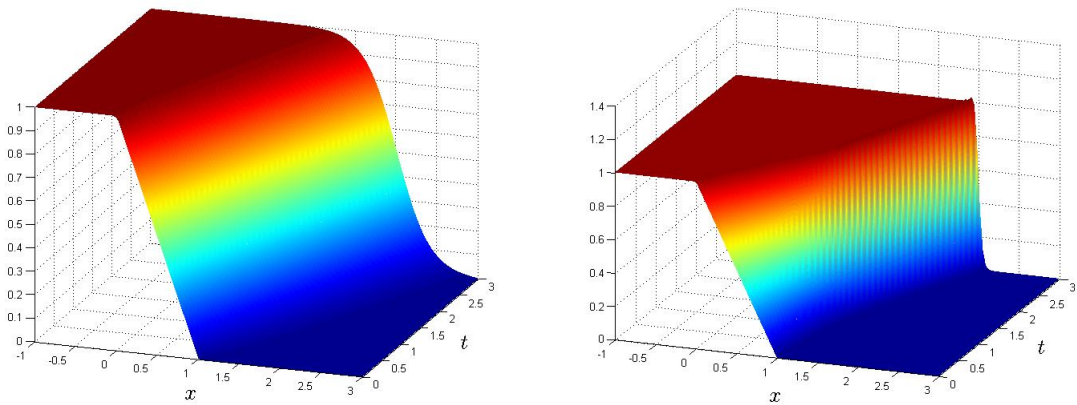Figure 11: Riemann problem with $u_L = 1$, $u_R = 0$ and $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively.

Figure 12: Same initial condition that the second example of last section with $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively.

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