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PERSPECTIVES

Defining Risk

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Financial markets are becoming increasingly sophisticated in pricing, isolating, repackaging, and transferring risks. Tools such as derivatives and securitization contribute to this process, but they pose their own risks. The failure of accounting and regulation to keep abreast of developments introduces yet more risks, with occasionally spectacular consequences.

Practical applications—including risk limits, trader performance-based compensation, portfolio optimization, and capital calculations—all depend on the measurement of risk. In the absence of a definition of risk, it is unclear what, exactly, such measurements reflect. With financial decisions hanging in the balance, debates flare on trading floors and in industry magazines.

A search of the financial literature yields many discussions of risk but few definitions. To understand risk, we must explore two streams flowing through the 20th century. One is subjective probability. The other is operationalism. Where they meet, we can understand risk. Interestingly, both streams have origins in the same source—the empiricism of David Hume.

Frank Knight

The most famous definition of risk is that provided by Frank Knight (1921), who wrote during a period of active research into the foundations of probability. Contemporaneous research includes John Maynard Keynes (1921), Richard von Mises (1928), and Andrey Kolmogorov (1933). One debate from this period relates to subjective versus objective interpretations of probability. According to objective interpretations, probabilities are real. We may discover them by logic or estimate them through statistical analyses. According to subjective interpretations, probabilities are human beliefs. They are not intrinsic to nature. Individuals specify them to characterize their own uncertainty.

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The philosophical roots of subjective interpretations of probability may be traced to Hume (1748):

> Though there be no such thing as *Chance* in the world; our ignorance of the real cause of any event has the same influence on the understanding, and begets a like species of belief or opinion. (p. 55)

Groundbreaking accounts of subjective probability include those of Frank Ramsey (1931), Bruno de Finetti (1937), and Leonard Savage (1954). The subjectivist position is aptly summarized by de Finetti (1970), who admonished:

> My thesis, paradoxically, and a little provocatively, but nonetheless genuinely, is simply this:
>
> PROBABILITY DOES NOT EXIST.
>
> The abandonment of superstitious beliefs about the existence of Phlogiston, the Cosmic Ether, Absolute Space and Time, . . . or Fairies and Witches, was an essential step along the road to scientific thinking. Probability, too, if regarded as something endowed with some kind of objective existence, is no less a misleading misconception, an illusory attempt to exteriorize or materialize our true probabilistic beliefs. (1974 translation, vol. 1, p. x)

Knight wrote from the competing objectivist perspective. He believed that propositions have intrinsic probabilities of being true or false. Probabilities may seem to depend on an observer's particular lack of information, but Knight distinguished between necessary and mere factual ignorance. He illustrated this distinction with the example of an urn filled with red and black balls. One man is ignorant of the number of each. Another man knows that the proportion is three red to one black:

> It may be argued that "to the first man" the probability of drawing a red ball is fifty-fifty, while to the second man it is seventy-five to twenty-five. Or it may be contended that the probability is "really" in the latter ratio, but that the first man simply does not know it. . . . The doctrine of real probability, if it is to be valid, must, it seems, rest upon inherent unknowability in the factors, not merely the fact of ignorance. (pp. 218–219)

Knight distinguished between probabilities obtained in two manners:

- *A priori* probabilities are derived from inherent symmetries, as in the throw of a die.
- Statistical probabilities are obtained through analysis of homogenous data.

He recognized that, in the absence of symmetries or homogenous data, people may still somehow quantify their uncertainty. Returning to his example of balls in an urn, he considered a situation in which we do not know the proportion of red to black balls but we are allowed to look inside the urn and form our own estimate of that proportion. According to Knight, opinions such as these, made in the absence of symmetry or homogeneity, are the basis for most business decisions:

> Business decisions . . . deal with situations which are far too unique, generally speaking, for any sort of statistical tabulation to have any value for guidance. (p. 231)

Knight was loath to attach the label "probabilities" to opinions formed in the absence of symmetry or homogenous data. He suggested that *a priori* and statistical probabilities reflect "measurable uncertainty" and opinions represent "unmeasurable uncertainty." He acknowledged that other authors mark this distinction with the terms "objective probability" and "subjective probability," but he preferred his own terminology:

> To preserve the distinction . . . between the measurable uncertainty and an unmeasurable one we may use the term "risk" to designate the former and the term "uncertainty" for the latter. (p. 233)

This statement is Knight's famous definition of risk. Risk relates to objective probabilities. Uncertainty relates to subjective probabilities.

Knight acknowledged that his use of both the terms "risk" and "uncertainty" did not conform to common usage. In this article, I use these terms more in accordance with common usage, which I will clarify shortly.

Critique of Knight's Definition

A criticism of Knight's definition is the obvious one that it really is not a definition of risk. According to common usage, risk entails both uncertainty and exposure—possible consequences. Knight's distinction addresses only the uncertainty.

His definition is based on a particular objectivist interpretation of probability. To Knight, probability is intrinsic to a proposition and depends only on necessary ignorance. Comparing this conception of probability with Keynes' contemporaneous conception is informative. According to Keynes, probabilities apply not to propositions but to pairs of propositions:

- One proposition is not known to be true or false.
- The other is the evidence for the first.

A probability, then, is a relationship between two propositions. In Knight's original example of balls in an urn, Keynes would say that the probability for the first man really is fifty-fifty. This is the probability that logically relates the proposition "a red ball will be drawn" to the proposition representing the limited evidence available to that man.

Keynes' interpretation of probability is objectivist because he stipulates that probability relationships are "rationally determinate." According to him, if two individuals consider the same evidence for a proposition, they must assign the same probability based on that evidence. Given any two propositions, a probability relationship between them is a product of logic.

Like Knight, Keynes accepted that in some situations of uncertainty, objective probabilities cannot be assigned. Accordingly, Keynes' interpretation of probability is amenable to Knight's distinction between risk and uncertainty. But the distinction for Keynes is more complicated. For Knight, propositions are categorized as either risks or uncertainties. For Keynes, *pairs* of propositions must be so categorized.

If we embrace a subjectivist interpretation of probability, Knight's definition of risk becomes empty. In the absence of objective probabilities (however defined), there can be no risks under his definition.

Intellectually, objectivist interpretations of probability are difficult to defend. Knight's conception of probabilities depends on the notion of necessary ignorance, but how do we distinguish necessary from factual ignorance in any given situation?

Ramsey criticized Keynes' view that probabilities are rationally determinate relationships between pairs of propositions. Commenting on these logical relationships, Ramsey noted:

> I do not perceive them, and if I am to be persuaded that they exist it must be by argument; moreover I shrewdly suspect that others do not perceive them either, because they are able to come to so very little agreement as to which of them relates any two given propositions. All we appear to know about them are certain general propositions, the laws of addition and multiplication; it is as if everyone knew the laws of geometry but no one could tell whether any given object were round or square; and I find it hard to imagine how so large a general body of knowledge can be combined with so slender a stock of particular facts. (pp. 161–162)

Even Knight's *a priori* probabilities—those based on some symmetry of a problem—are suspect. One issue is the fact that problems can exhibit multiple symmetries. Savage offered the example of an urn that contains two balls: Both may be white; both may be black; or one may be white and the other black. Based on one symmetry, we might ascribe each possibility a 1/3 probability. An alternative would be to perceive four equally likely possibilities: (1) the "first" ball is black and the "second" ball is black, (2) the "first" ball is black and the "second" ball is white, and so on. From this perspective, there is a 1/4 probability of both balls being black, a 1/4 probability of both being white, and a 1/2 probability of one being black and the other being white.

A final criticism of Knight's definition is that it appears to have only parochial relevance. For economists, Knight's distinction parallels divisions between types of economic activity. His notion of risk (measurable uncertainty) conforms to many contingencies that are the province of insurance. His notion of uncertainty (unmeasurable uncertainty) conforms to many contingencies that confront entrepreneurs or speculators. Accordingly, economists have found it useful to embrace some form of distinction between measurable and unmeasurable uncertainty.

The validity or usefulness of such a distinction continues to be a topic of debate among economists. In other contexts, Knight's distinction is less relevant. In finance, it has played essentially no role.

Harry Markowitz

Finance emerged as a subject independent of economics during the 20th century. Some authors, including Louis Bachelier (1900) and Alfred Cowles (1933, 1944), published papers we would today consider finance, but they are recognized as such more in retrospect than they were in their day. The *Journal of Finance* was launched with little fanfare in 1946. The event that marks the emergence of finance as an independent subject seems to be Harry Markowitz's defense of his doctoral thesis in the University of Chicago economics department.

Markowitz's thesis comprised his theory of portfolio selection. At its defense, his doctorate was in jeopardy. Sitting on the examination committee was Milton Friedman, who protested that the thesis was not about economics. Neither was it about math or business administration. Friedman was not sure how to categorize it, but he was averse to granting a doctorate in economics based on a thesis that was not about economics. Fortunately, Friedman's protests did not sway other members of the committee, and Markowitz received his degree.[1] His 1952 paper, which he had previously published in the *Journal of Finance*, spawned the field of portfolio theory.

Portfolio theory is generally perceived as a body of models that describe how investors may balance risk and reward in constructing investment portfolios. It is interesting that Markowitz offered no definition of risk in 1952; he simply proposed the following rule:

> . . . that the investor does (or should) consider expected return a desirable thing *and* variance of return an undesirable thing. (p. 77)

That is all he said. If variance of return is a proxy for risk, Markowitz did not say so. He simply stated that it is an "undesirable thing." Only toward the end of the paper did he note:

> The concepts "yield" and "risk" appear frequently in financial writings. Usually if the term "yield" were replaced by "expected yield" or "expected return," and "risk" by "variance of return," little change of apparent meaning would result. (p. 89)

This statement suggests that variance of return might be a proxy for risk, but Markowitz is careful to distance himself from that association. As it is worded, his comment suggests that *other authors* treat risk as akin to variance of return. Markowitz's finesse may stem from his perspective on probability: He wrote as a subjectivist.

One of Markowitz's professors at the University of Chicago was Leonard Savage, a leading advocate of the subjectivist interpretation of probability. Savage's influence on Markowitz's 1952 paper is evident. Footnote 7 of that paper describes probabilities as "in part subjective." Commenting on how to construct means and covariances for use in optimization, Markowitz suggested that procedures "should combine statistical techniques and the judgment of practical men." Seven years later, in his 1959 book, Markowitz referred repeatedly to subjective probabilities—which he and Savage called "personal probabilities." Markowitz cited the works of both Ramsey and Savage, and he devoted a chapter to subjective probability.

In an autobiography Markowitz penned when he received the 1990 Nobel Prize in economics, Markowitz identified two thinkers who profoundly influenced him while he was a teenager. One was Charles Darwin. The other was David Hume.

Uncertainty

Based on common usage, uncertainty is a state of not knowing whether a proposition is true or false.

Suppose you are in a casino. A man is about to roll a die. If the result is a six, you are going to lose $100. What is your risk? What, in your subjective opinion, is the probability that you will lose $100? If you say it is one chance in six, you may want to reconsider. I neglected to mention that the die is 10-sided. This example illustrates how one can be uncertain but not realize it.

To clarify, an individual is uncertain of a proposition if she

- does not know it to be true or false or
- is oblivious to the proposition.

Probability is often used as a metric of uncertainty, but its usefulness is limited. At best, probability quantifies *perceived* uncertainty.

Exposure

It is one thing to not know if a proposition is true or false. It is another thing to care. A self-conscious being is exposed to a proposition if the being would care whether or not the proposition is true. The word *would* is critical to this definition. It is possible to be exposed to a proposition without knowing of or considering the proposition.

In general, we are exposed to those propositions that have material consequences for us. We are not exposed to those propositions that do not have material consequences for us. The litmus test for materiality is: *Would we care*? If we immediately considered the proposition, would we have a preference for it to be true or false?

Temporarily impaired consciousness does not affect exposure. If a person is asleep or intoxicated, he remains exposed to a proposition so long as he *would* care were his consciousness not impaired.

Like uncertainty, exposure is a personal condition, but it is entirely distinct from uncertainty. The degree to which you are uncertain of a proposition does not affect the degree to which you are exposed to that proposition. You may be convinced that a proposition is true but still prefer that it be false. In such a situation, you are exposed to the proposition. Suppose it is raining. You are outdoors without protective rain gear. You are exposed to the rain because you care whether or not the proposition *it is raining* is true—you would prefer it to be false. There are as many possible exposures as there are meaningful propositions. You can be exposed to violence, loss, wealth, illness, friendship . . .

The immediacy of exposure is critical. Your current exposure depends on what would be your current preferences. George Shackle (1949) said:

> We decide on one particular course of action out of a number of rival courses because this one gives us, as an immediate present experience, the most enjoyment *by anticipation* of its outcome. (p. 10)[2]

Metrics for exposure can be based on the utility of Daniel Bernoulli (1738), John von Neumann and Oskar Morgenstern (1944), and Leonard Savage or on the state preferences of Kenneth Arrow (1953) and Gerard Debreu (1954). Both approaches are of limited usefulness. At best, they characterize *perceived* exposure.

Risk

Having clarified essential concepts, I now attempt to define risk. In this article, I am not interested in some aspect of risk or some category of risk. I am seeking a general definition. To this end, consider some situations that involve risk:

- trading natural gas,
- launching a new business,
- military adventures,
- asking for a pay raise,
- sky diving, and
- romance.

Any general definition must encompass all of these. The situations may appear disparate, but they share certain common elements. First, people care about the outcomes. If someone has a personal interest in what transpires, that person is exposed. Second, people don't know what will happen. In each situation, the outcome is uncertain. It seems that risk entails two essential components:

- exposure and
- uncertainty.

Risk, then, is exposure to a proposition of which one is uncertain.

Suppose a man leaps from an airplane without a parachute. If he is certain to die, he faces no risk. Risk requires both exposure *and* uncertainty.

Risk is a condition of individuals—humans and animals—that are self-aware.[3] Organizations, companies, and governments are not self-aware, so they are incapable of being at risk. Rather, they are conduits through which individuals—members, investors, employees, voters, and such—take risk. This fact is rarely acknowledged in today's literature on financial risk management, which tends to treat companies as risk takers. Looking through a company to see who ultimately bears specific risks can be enlightening. For example, increasing the accountability of managers increases career risk for those managers but tends to reduce price risk for stockholders. A pregnant question for the field of financial risk management is: Whose risks are being managed?

My definition of risk clarifies common usage. It offers insights, but it is flawed.

Operational Definitions

Anatol Rapoport (1953) observed:

> It is not necessary to look into books on philosophy to find words without referents. Any commencement speech, sermon, newspaper editorial, or radio commercial follows the same philosophic tradition of reinforcing the delusion that anything which is talked about is real: success, charity, public opinion, and four-way indigestion relief. Indeed, nothing is easier than to "define" these noises so as to make it appear that they mean something.
>
> - What is success? Success is the attainment of one's ideals.
> - What is charity? Charity is the practice of Christian attitudes.
> - What is public opinion? Public opinion is the prime mover of public policy in a democracy.
> - What is four-way indigestion relief? Four-way indigestion relief is a gentle action which alkalizes the system, purifies the blood, activates the bile, and helps Nature to re-establish equilibrium. (pp. 18–19)

Rapoport was commenting in the context of operationalism, a philosophy introduced by Percy Bridgman in his 1927 landmark work *The Logic of Modern Physics*. Bridgman was troubled by a passage from Isaac Newton's (1686) *Principia*. Newton stated:

> I do not define Time, Space, Place or Motion, as being well known to all. Only I must observe that the vulgar conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which, it will be convenient to distinguish them into Absolute and Relative, True and Apparent, Mathematical and Common.
>
> (1) Absolute, True, and Mathematical Time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called Duration. (1848 translation, p. 13)

Newton speaks to us through the ages. He was interested in time, and we are interested in risk, but his observations are applicable for our needs. We want to understand the risks we face, but we can hope to understand only those aspects that are sensible. Newton proposed a distinction between that which is true and that which is apparent. To him, there is true time and apparent time.

With a single paper, Albert Einstein (1905) dispensed with Newton's notion of true time. He proposed that

- if two observers are moving relative to each other, their experiences of time will differ;
- all motion is relative; so
- all time is relative.

There is no true time.

Humans may speak of such notions as truth, virtue, time, distance, risk, or market value as if they were real notions endowed with an objective existence. Reacting to the revelations of Einstein's relativity, Bridgman proposed his alternative view, called "operationalism."[4] He surmised that if all knowledge of the world stems from our experiences, then definitions can be meaningful only if they refer to experiences. He suggested that we formally define a concept by specifying a set of operations through which that concept is experienced:

> In general, we mean by any concept nothing more than a set of operations: *The concept is synonymous with the corresponding set of operations*. (p. 5)

Suppose an analyst at an international wholesaler compiles closing coffee prices for the day by obtaining the spot price of the company's last transaction in each grade of coffee prior to 4:30 p.m. New York time. The reporting system records the time of each trade to within a minute, and on one particular day, the system reports two transactions in Colombian UGQ at 4:26 p.m. The two trades were at slightly different prices, and there were no subsequent trades. Shrugging, the analyst averages the two prices and records the result as the closing price for Colombian UGQ.

A traditional interpretation is that the analyst has recorded her "best estimate" of the closing price. According to operationalism, there was no true closing price for the analyst to estimate. The traditional view holds that we measure a quantity q to obtain a quantity m that satisfies the equation

$$m = q + e,$$

where e is the error in our measurement. According to operationalism, the quantities q and e do not exist. All that exists is the measurement m, which is defined by the operations by which it was obtained. In the example, the notion of "closing price" is defined by the set of operations with which the analyst obtained her value.

What do we mean by the correlation between three-month LIBOR and six-month LIBOR? What do we mean by the market value of spot copper? What do we mean by the leptokurtosis of spot cocoa prices? We mean only the set of operations by which we calculate such quantities.

Bridgman's operationalism has much in common with the (largely concurrent) logical positivism of the Vienna Circle.[5] Bridgman wrote for scientists, so operationalism found acceptance among

(primarily social) scientists. The Vienna Circle wrote primarily for philosophers. Their logical positivism had roots in the empiricism of David Hume.

An Operational Perspective on Risk

In finance, terms abound that are intuitively understood in the absence of operational definitions. Examples include the implied volatility of the S&P 500 Index and the correlation between platinum prices and palladium prices. Many such terms are easy to define operationally. Identifying a number on a specific Bloomberg screen is an operational definition, as is specifying a formula to be applied to data obtainable from a specified source. Accordingly, we often refer to concepts that are not operationally defined, but we do so with the understanding that they can be operationally defined should the need arise. Risk is an exception.

My previous definition of risk is inadequate from an operational standpoint because, although it communicates what is meant by "risk" according to common usage, it is intuitive. The definition depends on the notions of exposure and uncertainty, neither of which can be defined operationally.

In the case of exposure, one can be exposed without being aware of the exposure. The litmus test for exposure is not *do we care?* It is *would we care?* This is a hypothetical, unobservable test. Operational definitions, by construction, apply only to that which can be perceived. At best, we can hope to operationally define only our *perception* of exposure.

The situation is similar for uncertainty. As the example of the 10-sided die illustrates, one can be uncertain without realizing it. Uncertainty that is not perceived cannot be defined operationally. All we can hope to define operationally is our *perception* of uncertainty.

Consequently, it is impossible to operationally define risk. At best, we can operationally define our *perception* of risk. There is no true risk.

As practitioners of finance, we use subjective probabilities to operationally define perceived uncertainty. We use utility or state preferences to operationally define perceived exposure. It is not so easy to operationally define perceived risk because perceived risk takes many forms. To simplify the task, we may operationally define some *aspects of perceived risk*. Following Markowitz's lead, we adopt risk metrics—such as variance of return or maximum likely credit exposure—to define specific aspects of perceived risk.

Risk metrics are widely used in financial applications, including setting risk limits, trader performance-based compensation, portfolio optimization, and capital allocation. For each such application, how should we choose which risk metric to use? If we want to limit market risk, should we use delta, value-at-risk, or beta? Our choice will have an impact on financial activity. With real money and real compensation on the line, philosophical debates flare. Told that he has violated a market risk limit, a trader may challenge the risk metric as irrelevant. Assigned prohibitive economic capital charges for her business unit, a manager may complain that the risk metric does not capture true risk.

What is risk? How can we quantify risks that cannot be perceived? If a trader or business manager has knowledge that is not reflected in a risk metric, does the risk metric misrepresent risk? In the absence of true risk, these questions are empty. A more practical question is whether a risk metric is useful. Used in a given application, will it promote behavior that management considers desirable?

Conclusion

This article has explored the nature of risk, as the term is commonly used. Subjective probability, utility, and state preferences are tools for characterizing the uncertainty and exposure components of risk. Such tools are limited by the fact that they apply only to those aspects of risk that are perceived.

Operationalism suggests that this problem is insurmountable. Because operational definitions apply only to that which can be perceived, we can never operationally define risk. At best, we can operationally define only our perception of risk.

A more manageable task is to operationally define some aspects of perceived risk. Risk metrics, such as variance of return, are used for this purpose. It is meaningless to ask if a risk metric captures risk. Instead, ask if it is useful.

Notes

1. This event is described in Bernstein (1992), p. 60.
2. Shackle indicated that the notion of most enjoyment is equivalent to that of least anguish.
3. Because animals have limited knowledge, they are capable of uncertainty. Those that can anticipate pain or pleasure are capable of exposure.
4. Bridgman avoided attaching a name to his ideas. Competing names, operationalism and operationism, were adopted by subsequent authors.
5. The Vienna Circle was a group of philosophers and scientists who met periodically in Vienna from 1922 to 1938. Their logical positivism became one of the most influential philosophical movements of the 20th century. Carl Hempel wrote a critical article titled "A Logical Appraisal of Operationism" (1965) that identified differences between operationalism and logical positivism.

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