An Overview of Logic, Proofs, Set Theory, and Functions

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The Power of AND | University of Wisconsin Eau Claire

Contents

Preface: This handout is meant primarily for those students who are already familiar with most of the subject matter contained within (that is, those who have taken a proofs class before). Its purpose is to provide a foundation as a proofs refresher when taking classes like Real Analysis I or II, Abstract Algebra I or II, Number Theory, Discrete Mathematics, Linear Algebra, etc.

1 Numerical Sets and Other Preliminary Symbols

The following are numerical sets that you should familiarize yourself with:

natural numbers[1]	$\mathbb{N} = \{1, 2, 3, \ldots\}$
integers	$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
rational numbers	$\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$
real numbers	$\mathbb{R} = \{\text{rational and irrational numbers}\}$
complex numbers	$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$
Gaussian integers	$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1}\}$
Eisenstein integers	$\mathbb{Z}[\rho] = \{a + b\rho \mid a, b \in \mathbb{Z} \text{ and } \rho = e^{\frac{2\pi i}{3}}\}$
even integers	$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\}$
odd integers	$2\mathbb{Z} + 1 = \{2k + 1 \mid k \in \mathbb{Z}\}$
arithmetic progression	$a\mathbb{Z} + b = \{ak + b \mid k \in \mathbb{Z}\}$ where a, b fixed

CULTURAL QUESTION 1: Why are the integers denoted $\mathbb{Z}$?

ANSWER: The German word for "numbers" is *Zahlen*. Germans contributed much to Zahlentheorie (number theory).

CULTURAL QUESTION 2: Why are the rationals denoted $\mathbb{Q}$?

ANSWER: It was first denoted Q in 1895 by Giuseppe Peano after *quoziente*, Italian for "quotient".

[1] We do not consider the number 0 to be a natural number, though it is common in fields such as computer science where loops and array elements start with the 0^{th} counter.

Below is a list of symbols with their associated meanings that we will come across in this proofs refresher.

Symbol	***Meaning***
$\mathbb{N}$	set of natural numbers (we exclude 0)
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\emptyset$	the nullset or emptyset
$\mathcal{U}$	the universal set
$\cup$	union
$\cap$	intersection
$\coprod$	disjoint union
$\in$	is an element of
$A \subseteq B$	A is a subset of B
$A \supseteq B$	B is a subset of A
$A \nsubseteq B$	A is not a subset of B
$A \nsupseteq B$	B is not a subset of A
$A - B$ or $A \backslash B$	set difference "A without B"
$A \times B$	Cartesian product of sets A and B
$a \mid b$	a divides b
$\gcd(a, b)$	greatest common divisor of a and b
$\text{lcm}(a, b)$	least common multiple of a and b

We also have some common abbreviations used in proofs for the most part as follows.

Abbreviations	***Meaning***
$\forall$	for all
$\exists$	there exists
BWOC	by way of contradiction
WLOG	without loss of generality
TFAE	the following are equivalent
s.t.	such that
: or equivalently |	such that (used in set-theory notation)
$\Longrightarrow$	implies
WWTS	we want to show
Q.E.D.	quod erat demonstrandum [end of proof]

2 Statements and Truth Tables

Definition 2.1. A **statement** (or proposition) is a declarative sentence that is true or false but not both.

Exercise 2.2. Which of the following are statements? Write **NS** if it is not a statement and **S** if it is. Also give the truth value if it is a statement. [You Do!]

(a) January is the first month of the year.

(b) June is the first month of the year.

(c) The Packers is the best football team.

(d) $6x + 3 = 17$.

(e) The equation $6x + 3 = 17$ has more than one solution.

(f) This statement is false.

(g) This *Proofs Refresher* course will help prepare you to write proofs.

Definition 2.3. A **compound statement** is a statement which results from the application of one or more logical connectives (for example, "not" $\sim$, "and" $\wedge$, "or" $\vee$, or "if-then" $\implies$) to a collection of simple statements.

Definition 2.4. A **truth table** is a mechanism for determining the truth values of compound statements.

Example 2.5. Below are the truth tables for negations, conjunctions, disjunctions, and implications.

(i) NEGATION: "not" symbol $\sim$

A	$\sim A$
T	F
F	T

The rule is: "Not" reverses the truth value.

(ii) CONJUNCTION: "and" symbol $\wedge$

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

The rule is: An "and" is true ONLY WHEN both sides are true.

(iii) DISJUNCTION: "or" symbol $\vee$

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

The rule is: An "or" is false ONLY WHEN both side are false.

(iv) IMPLICATION: "if-then" symbol $\Rightarrow$

A	B	$A \Longrightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

The rule is: An "if-then" is false ONLY WHEN the hypothesis holds but the conclusion fails.

Definition 2.6. A **vacuously true statement** is an 'if-then" statement in which the "if"-part (i.e., hypothesis) is false. This statement is true regardless of the truth value of the "then"-part (i.e., the conclusion). For example,

- If $1 = 0$, then an apple is a banana.

Equivalently, a vacuous truth is a statement that asserts that all members of an empty set possess a certain property. For example,

- Every blue pig is an elephant.

Definition 2.7. (Tautology versus Contradiction)

- A **tautology** is a statement that is always true.
- A **contradiction** is a statement that is always false.

Exercise 2.8. Is the statement $(A \vee \sim B) \vee \sim A$ a tautology, contradiction, or neither? [You Do!]

A	B	$\sim B$	$A \vee \sim B$	$\sim A$	$(A \vee \sim B) \vee \sim A$
T	T				
T	F				
F	T				
F	F				

We conclude that $(A \vee \sim B) \vee \sim A$ **is a :**

Definition 2.9. Two statements are **logically equivalent** if they have the same truth tables. We use the symbol $\equiv$ to denote this.

Exercise 2.10. Show that $\sim (A \vee B) \equiv \sim A \wedge \sim B$. [You Do!]

A	B	$A \vee B$	$\sim (A \vee B)$	$\sim A$	$\sim B$	$\sim A \wedge \sim B$
T	T					
T	F					
F	T					
F	F					

We conclude that:

Definition 2.11. **De Morgan's Laws** assert the following logical equivalencies:

- $\sim (A \vee B)$ is logically equivalent to $\sim A \wedge \sim B$.
- $\sim (A \wedge B)$ is logically equivalent to $\sim A \vee \sim B$.

Converting Sentences Into Formal Logic

Exercise 2.12. Convert the following sentences into formal logic. [You Do!]

(i) If it rains, then I get wet.

- Let P equal "it is raining".
- Let Q equal "I get wet".

Then the formal logic statement is: ____________

(ii) I hate math, but I like this course.

- Let P equal "I hate math".
- Let Q equal "I like this course".

Then the formal logic statement is: ____________

CAREFUL! Sometimes the word "but" is used in place of "and" when the part of the independent clause that follows the "but" is unexpected.

(iii) Katie Sue does not like Jammo, or Zhang Li likes Edith.

- Let P equal "Katie Sue likes Jammo".
- Let Q equal "Zhang Li likes Edith".

Then the formal logic statement is: ____________

(iv) If I study hard, then I will rock this course!

- Let P equal "I study hard".
- Let Q equal "I will not rock this course",

Then the formal logic statement is: ____________

3 Implications

First recall the "If-Then" truth table below:

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise 3.1. Compute the truth table for $\sim B \implies \sim A$. [You Do!]

A	B	$\sim A$	$\sim B$	$\sim B \implies \sim A$
T	T			
T	F			
F	T			
F	F			

QUESTION: Is $A \implies B$ logically equivalent to $\sim B \implies \sim A$? ☐

Definition 3.2. The **contrapositive** of the statement $A \implies B$ is the statement $\sim B \implies \sim A$.

Exercise 3.3. Compute the truth table for $B \implies A$. [You Do!]

A	B	$B \implies A$
T	T	
T	F	
F	T	
F	F	

QUESTION: Is $A \implies B$ logically equivalent to $B \implies A$? ☐

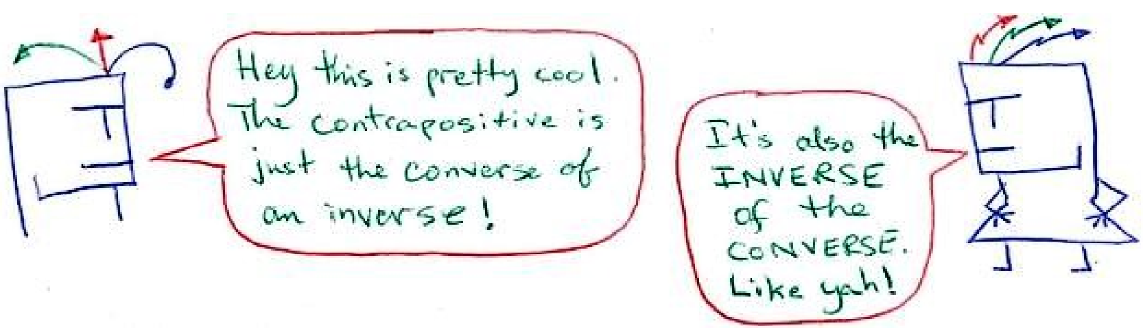
Definition 3.4. The **converse** of the statement $A \implies B$ is the statement $B \implies A$.

Exercise 3.5. Compute the truth table for $\sim A \Longrightarrow \sim B$. [You Do!]

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{\sim A}$	$\boldsymbol{\sim B}$	$\boldsymbol{\sim A \Longrightarrow \sim B}$
T	T			
T	F			
F	T			
F	F			

QUESTION: Is $A \Longrightarrow B$ logically equivalent to $\sim A \Longrightarrow \sim B$?

Definition 3.6. The **inverse** of the statement $A \Longrightarrow B$ is the statement $\sim A \Longrightarrow \sim B$.



Example 3.7. Consider the statement "If n is divisible by 10 or divisible by 12, then n is even."

Contrapositive and truth value?: If n is odd, then n is not divisible by 10 and not divisible by 12. **NOTE:** It is more colloquial in English to say "If n is odd, then n is neither divisible by 10 nor 12". - TRUE.

Converse and truth value?: If n is even, then n is divisible by 10 or divisible by 12. - FALSE, let $n = 4$.

Inverse and truth value?: If n is neither divisible by 10 nor 12, then n is odd. - False, let $n = 8$.

The Or-Form of an Implication

Exercise 3.8. Compute the truth table for $\sim A \vee B$ and compare it to $A \implies B$. [You Do!]

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{\sim A}$	$\boldsymbol{\sim A \vee B}$	$\boldsymbol{A \implies B}$
T	T			
T	F			
F	T			
F	F			

QUESTION: Is $A \implies B$ logically equivalent to $\sim A \vee B$? [ ]

Example 3.9. Consider the statement "If I don't study, then I fail."

The Or-Form of the implication: I study or I fail.

NOTE: This is a very true statement, so STUDY DILIGENTLY everyone!

The Negation of an Implication

Since we know that $A \implies B$ logically equivalent to $\sim A \vee B$, then it is simple to find the negation of $A \implies B$. We simply take the negation of its equivalent form $\sim A \vee B$.

Exercise 3.10. What is the negation of $\sim A \vee B$? [ ] Verify this is indeed $\sim (A \implies B)$ via the truth table. [You Do!]

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{\sim B}$	$\boldsymbol{A \wedge \sim B}$	$\boldsymbol{A \implies B}$	$\boldsymbol{\sim (A \implies B)}$
T	T				
T	F				
F	T				
F	F				

Exercise 3.11. Consider the statement "If I have a baby, then my life will suck."

Write the negation: [You Do!]

Exercise 3.12. Write a sequence of logical equivalencies to prove again (but without truth tables that the negation of $A \implies B$ is indeed $A \wedge \sim B$. [You Do!]

Example 3.13. Consider the statement

"Today is not Easter, or tomorrow is Monday."

What is the truth value of this statement?
NOTE: Do not assume today is this very day.

HINT: Consider all possible cases of the day of the week it can be, and whether today is Easter or tomorrow is Monday holds in each case.

If today is Monday, then it is DEFINITELY not Easter. The same goes for the cases of Tuesday through Saturday. Now on the final case when it is Sunday, then we don't know if it is Easter Sunday; however, it is certain that "tomorrow is Monday" will definitely hold. So the OR-STATEMENT is true.

Exercise 3.14. Write down the If-Then form of the Or-Statement in the example above. [You Do!]

4 Predicates and Quantifiers

Definition 4.1. A **predicate** (or propositional function) is a sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for the variables.

The **domain of a predicate** is the set of all values that may be substituted in place of the variable.

Example 4.2. Let D be the set of all animals. Let $P(x)$ be the predicate

"x is a mammal".

Exercise 4.3. For which x in the subset $\{\text{sharks}, \text{whales}, \text{kangaroos}\} \subseteq D$ is the predicate $P(x)$ from the example above true? [You Do!]

Definition 4.4. A **quantifier** is a logical symbol which makes an assertion about the set of values which make one or more predicates true.

QUESTION: What are two VERY WELL-KNOWN quantifiers that you use a lot in mathematics?

ANSWER: "for all" $\forall$ and "there exists" $\exists$
Note: "there exists" is usually followed by a "such that". We sometimes abbreviate "such that" by "s.t.".

Exercise 4.5. Let $P(x, y)$ be the predicate $xy \in \mathbb{Z}$. [Answer "Yes" or "No"!] If "No", then provide x, y-values that give a counterexample.

- If $D = \mathbb{N}$, is $P(x, y)$ true $\forall x, y \in D$?
- If $D = \mathbb{Z}$, is $P(x, y)$ true $\forall x, y \in D$?
- If $D = \mathbb{Q}$, is $P(x, y)$ true $\forall x, y \in D$?
- If $D = \mathbb{R}$, is $P(x, y)$ true $\forall x, y \in D$?

Universal and Existential Statements

REMARK: In the following, let $Q(x)$ be a predicate with domain D.

Definition 4.6. A **universal statement** is a statement using a **universal quantifier** $\forall$ as follows:

$$\forall\, x \in D,\ Q(x).$$

For example, let $D = \{\text{mammals}\}$ and $Q(x) =$ "x has mammary glands".

- "$\forall\, x \in D,\ Q(x)$" in words means "Every mammal has mammary glands."

Definition 4.7. An **existential statement** is a statement using an **existential quantifier** $\exists$ as follows:

$$\exists\, x \in D \text{ s.t. } Q(x).$$

For example, let $D = \{\text{mammals}\}$ and $Q(x) =$ "x is a primate".

- "$\exists\, x \in D$ s.t. $Q(x)$" in words means "There is a mammal that is a primate."

How to (Dis)Prove a Universal and Existential Statements:

- **To Prove:** $\forall\, x \in D,\ Q(x)$

  Let $x \in D$. Then show that $Q(x)$ holds.

- **To Disprove:** $\forall\, x \in D,\ Q(x)$

  Find a counterexample (i.e., show $\exists\, x \in D$ s.t. $Q(x)$ fails).

- **To Prove:** $\exists\, x \in D$ **s.t.** $Q(x)$

  Find at least one $x \in D$ such that $Q(x)$ holds.

- **To Disprove:** $\exists\, x \in D$ **s.t.** $Q(x)$

  Show that for all $x \in D$, the statement $Q(x)$ fails.

Exercise 4.8. Let $D = \mathbb{Z}$ and consider the universal statement:

$$\forall\, x \in D, \text{ we have } \frac{x}{x^2+1} < \frac{1}{2}.$$

Is the predicate $P(x)$ is "$\frac{x}{x^2+1} < \frac{1}{2}$" true for all $x \in D$?

ANSWER: [You Do!]

Exercise 4.9. Let $D = \{2, 5, 9, 10\}$ and consider the universal statement:

$$\forall\, x \in D, \text{ we have } x^2 > x + 1.$$

Is the predicate $P(x)$ is "$x^2 > x + 1$" true for all $x \in D$?

ANSWER: [You Do!]

Remark 4.10. To prove the above you can use a technique called a **proof by exhaustion** for obvious reasons. ☺

Exercise 4.11. Let $D = \mathbb{R}$ and consider the universal conditional[2]

$$x < y \text{ and } w < z \implies xw < yz.$$

Is the predicate $P(x, y, w, z)$ is "$x < y$ and $w < z \implies xw < yz$" true for all $x \in D$?

ANSWER: [You Do!]

[2]This term "universal conditional" will be explained on the next page in Remark 4.12.

Remark 4.12. The statement in the previous example is called a **universal conditional statement** since it is understood to mean

$$\forall\, x, y, w, z \in \mathbb{R}, \quad P_1(x, y, w, z) \implies P_2(x, y, w, z)$$

where $P_1(x, y, w, z)$ is $x < y$ and $w < z$, and $P_2(x, y, w, z)$ is $xw < yz$.

Example 4.13. Universal conditional statements in informal English are quite often implied rather than specific. Let us write the following statements formally.

(i) All real numbers are positive when squared.

For all numbers x, if $x \in \mathbb{R}$ then $x^2 > 0$.

(ii) No odd numbers are divisible by 2.

For all numbers n, if $n \in 2\mathbb{Z} + 1$ then n is not divisible by 2.

A Helpful Proof Tip (using negations)

To prove a statement is true, it is often helpful (or easier) to instead consider proving the NEGATION of the statement is false.

Exercise 4.14. We will formally learn how to take the negations on the next page, but for now attempt to write an "informal" negation for each of the two universal conditional statements above in Example 4.13. [You Do!]

(i)

(ii)

Negations of Universal and Existential Statements

How to Negate Universal and Existential Statements:

- **To negate a universal statement:**

$$\sim (\forall x \in D,\ Q(x)) \equiv \boxed{\exists x \in D \text{ s.t.} \sim Q(x)}$$

- **To negate an existential statement:**

$$\sim (\exists x \in D \text{ s.t. } Q(x)) \equiv \boxed{\forall x \in D,\ \sim Q(x)}$$

Exercise 4.15. Negate "there is a natural number that is even and prime".

ANSWER: [You Do!]

Exercise 4.16. Negate "all dogs go to heaven".

ANSWER: [You Do!]

NOTE: This negation is false because it is well known that it is indeed true that ALL dogs go to heaven as can be seen in the image below[3]. And hence the universal statement is true.

[3]This painting was done by artist Jim Warren and appears with his permission. You can see more of his work at https://jimwarren.com/.

Statements with Multiple Quantifiers

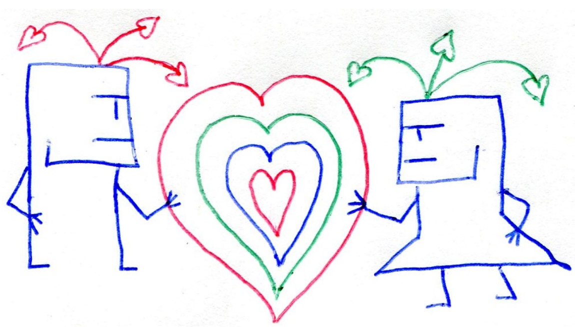
Many statements mix $\forall$ and $\exists$, It important to note the following;

- Order of the symbols $\forall$ and $\exists$ matters!!
- Order does not matter if there are two $\forall$'s or two $\exists$'s.

Example 4.17. Let the domain D be the set of all people. And let $P(x, y)$ be the predicate "x loves y".

- **Interpret the following:** $\forall x \in D,\ \exists y \in D$ **s.t.** $P(x, y)$

Everyone loves at least one person.



- **Interpret the following:** $\exists x \in D$ **s.t.** $\forall y \in D,\ P(x, y)$

There exists a person who loves everyone.

How to Negate Multiply-Quantified Statements:

- **To negate a universal-existential statement:**

$\sim (\forall x \in D,\ \exists y \in E$ s.t. $P(x, y))$ is logically equivalent to

$$\boxed{\exists x \in D \text{ s.t. } \forall y \in E,\ \sim P(x, y)}$$

- **To negate an existential-universal statement:**

$\sim (\exists x \in D$ s.t. $\forall y \in E,\ P(x, y),)$ is logically equivalent to

$$\boxed{\forall x \in D,\ \exists y \in E \text{ s.t. } \sim P(x, y)}$$

How to Prove a Universal-Existential Statement:

- **To Prove:** $\forall x \in D, \exists y \in E$ **s.t.** $P(x, y)$

Let $x \in D$. And find a $y \in E$ such that $P(x, y)$ holds.

How to Prove an Existential-Universal Statement:

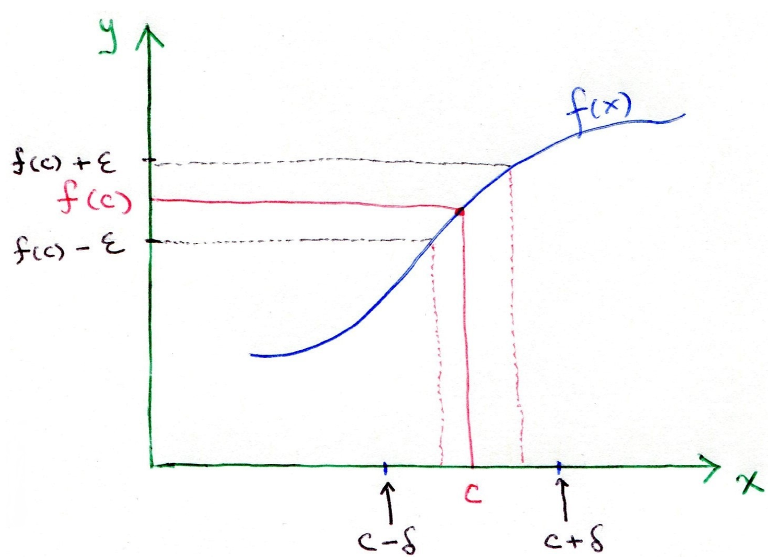
- **To Prove:** $\exists x \in D$ **s.t.** $\forall y \in E, P(x, y)$

Find an $x \in D$ for which no matter what choice of $y \in E$,
we have $P(x, y)$ holds.

The Calculus I definition of continuity is a multiply-quantified statement!

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at point $c \in \mathbb{R}$ if

$$\forall \epsilon > 0, \ \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$



Exercise 4.18. Negate the continuity definition above (using English and not ∀-∃ symbols).

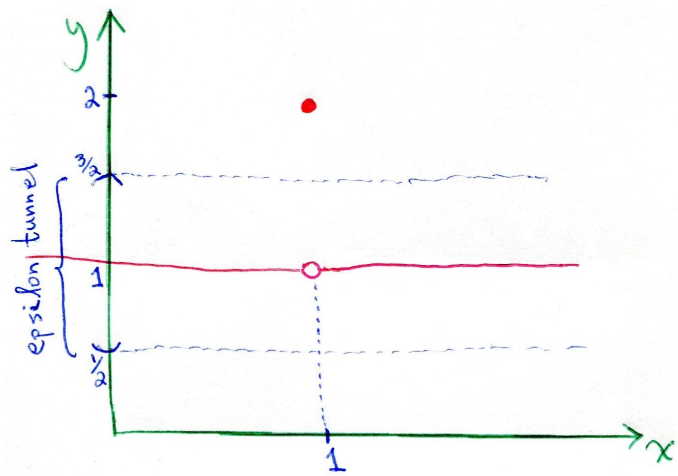
ANSWER: [You Do!]

Example 4.19. Prove that the function

$$f(x) = \begin{cases} 2 & \text{if } x = 1 \\ 1 & \text{if } x \neq 1 \end{cases}.$$

is discontinuous at $x = 1$.

HINT: Let us draw the function and consider the "epsilon tunnel" around $f(1)$ with $\epsilon = \frac{1}{2}$.



Proof. Set $\epsilon := \frac{1}{2}$.

WWTS: $\forall\, \delta > 0,$ we have $0 < |x - 1| < \delta$, but $|f(x) - f(1)| \geq \frac{1}{2}$.

Observe that no matter what δ is, for any $x \neq 1$ with $|x - 1| < \delta$, we claim that $|f(x) - f(1)| \geq \frac{1}{2}$. This follows since $f(x) = 1$ whenever $x \neq 1$ and $f(1) = 2$, so we have

$$\begin{aligned} |f(x) - f(1)| &= |1 - 2| \\ &= 1 \\ &\geq \frac{1}{2}. \end{aligned}$$

Thus f is discontinuous at $x = 1$.

Q.E.D.

More Multiply-Quantified Statements

Exercise 4.20. Which of the following statements are true? We are not asking for proofs or counterexamples (in this exercise). [You Do!]

(1) $\forall\, y \in \mathbb{R},\ \exists\, x \in \mathbb{R}$ such that $x^2 < y + 1$.

(2) $\exists\, x \in \mathbb{R}$ such that $\forall\, y \in \mathbb{R}$, we have $x^2 + y^2 = 9$.

(3) $\forall\, x \in \mathbb{R}$ and $\forall\, y \in \mathbb{R}$, we have $x^2 + y^2 \geq 0$.

(4) $\forall\, x \in \mathbb{R},\ \exists\, y \in \mathbb{R}$ such that $x^2 + y^2 \leq 0$.

(5) $\exists\, x \in \mathbb{R}$ such that $\forall\, y \in \mathbb{R}$, we have $x^2 + y^2 < 0$.

(6) $\exists\, x \in \mathbb{R}$ and $\exists\, y \in \mathbb{R}$ such that $x^2 + y^2 \leq 0$.

(7) $\exists\, x \in \mathbb{Z}$ such that $\forall\, y \in \mathbb{Z}$, we have $x + y = 0$.

(8) $\forall\, x$ with $x \neq 0$, $\exists\, y \in \mathbb{R}$ such that $x^y > y^x$.

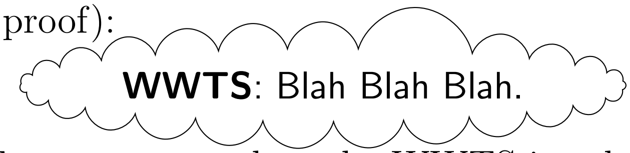
5 Writing Formal Proofs

The Four Types of Proofs:

- (I) Direct Proofs
- (II) Proof by Cases
- (III) Proof by Contradiction
- (IV) Proof by Contrapositive

Remark 5.1. **Tips and best starting points for a proof:**

1. Start by writing what hypotheses (i.e., assumptions) are given (e.g., $x \in \mathbb{Q}$).
2. Write down your WWTS in the proof RIGHT AFTER stating the assumptions, and perhaps put it in a box or a bubble like so (to remind you that is NOT part of your proof):

   NOTE: It is NOT necessary to place the WWTS in a bubble/box/etc., but it IS important to remember that you cannot use this WWTS in your proof.
3. Mathematically/Symbolically say what that means. (e.g., $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$). This may be done at times right before the WWTS.
4. **Some things NOT to do!!!!!!!!:**
   (a) NEVER replace English words with math blackboard slang.
   (b) Do not put mathmode symbols both before [**AND**] after a comma.

   Do not write garbage like: "Choose a $\mathbb{R}$ x s.t. $\forall y$, $y > x$."

EXAMPLE: Rewriting the garbage in the box above so it looks less "trashy":

Choose a real number x such that for all y, we have $y > x$.[4]

[4] Of course, this statement though written very pretty, is not true. [WHY?]

How to Prove the Four Types of Proofs

(I) DIRECT PROOF:

How to Prove the Universal Conditional Statement:

For all $x \in D$, we have $P(x)$ implies $Q(x)$.

- If the statement is not already in the form above, then write it in this form.
- *FIRST SENTENCE OF PROOF*

  Assume $x \in D$ and suppose $P(x)$ holds.

- *SECOND SENTENCE OF PROOF*

  WWTS: $Q(x)$ holds.

  NOTE: The exact words of $Q(x)$ need not be in the bubble. Sometimes you put there only what it suffices to show.

- *BODY OF PROOF*
- *LAST LINE OF PROOF* We conclude that $Q(x)$ holds.
- Put a Q.E.D. or just a $\square$.

(II) PROOF BY CASES:

How to Prove the Conditional Statement:

If A_1 or A_2 or A_3 or . . . or A_n, then B.

- **(CASE 1)** Assume A_1 holds. **WWTS**: B holds. Then prove this.
- **(CASE 2)** Assume A_2 holds. **WWTS**: B holds. Then prove this.
- $\vdots$ etc...
- **(CASE n)** Assume A_n holds. **WWTS**: B holds. Then prove this.
- *LAST LINE OF PROOF* We conclude that B holds.
- Put a Q.E.D. or just a $\square$.

(III) PROOF BY CONTRADICTION:

How to Prove the Conditional Statement:

If A, then B.

- Assume A holds.
- Suppose by way of contradiction (BWOC) that B is false.
- **THERE IS NO WWTS IN THIS TYPE OF PROOF!!!**
- Derive any contradiction.
- Hence it cannot be that B is false.
- *LAST LINE OF PROOF* We conclude that B holds.
- Put a Q.E.D. or just a $\square$.

(IV) PROOF BY CONTRAPOSITIVE:

How to Prove the Conditional Statement:

If A, then B.

- Restated it in its contrapositive form, we have

  If $\sim B$, then $\sim A$.

  NOTE: You don't write this contrapositive form in your formal proof.
- *FIRST SENTENCE OF PROOF*

  Assume $\sim B$ holds.
- *SECOND SENTENCE OF PROOF*

  WWTS: $\sim A$ holds.

  NOTE: The exact words of $\sim A$ need not be in the bubble. Sometimes you put there only what it suffices to show.
- *BODY OF PROOF*
- *LAST LINE OF PROOF* We conclude that $\sim A$ holds.
- Put a Q.E.D. or just a $\square$.

An Example of a Direct Proof

Example 5.2. Prove the following multiply-quantified statement:

$$\forall\, x \in \mathbb{R},\ \exists\, y \in \mathbb{R} \text{ such that } x + y = 0.$$

Proof. Let $x \in \mathbb{R}$.

WWTS: $\exists\, y \in \mathbb{R}$ s.t. $x + y = 0$.

Set $y := -x$. Observe that $y \in \mathbb{R}$ and it follows that

$$x + y = x + (-x) = 0$$

as desired. Thus the claim holds.

Q.E.D.

Exercise 5.3. Prove that the following statement is false:

$$\exists\, x \in \mathbb{R} \text{ such that } \forall\, y \in \mathbb{R},\ \text{ we have } x + y = 0. \tag{1}$$

RECALL: To prove a statement is false either find a counterexample, OR prove that the statement's negation is true. Statement (1) does not lend itself to finding a counterexample, so let's prove the negation is true. WHAT IS THE NEGATION OF STATEMENT (1)? [You Do!]

Now prove the negation.

Proof.

Q.E.D.

An Example of a Proof by Contrapositive

This lemma below literally CANNOT be proven directly!!

Think about it. Give it a shot! Your proof would start as follows:

Proof. Assume n^2 is even.

WWTS: n is even.

Since n^2 is even, then $n^2 = 2k$ for some $k \in \mathbb{Z}$.

THERE IS LITERALLY NOWHERE TO GO FROM HERE! THINK ABOUT IT!

Lemma 5.4. *If n^2 is even, then n is even.*

Proof. Let n be an odd integer.

WWTS: n^2 is odd.

Since n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Squaring n we get the following sequence of equalities

$$\begin{aligned} n^2 = (2k+1)(2k+1) &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Set $M := 2k^2 + 2k$. Observe that $M \in \mathbb{Z}$ by closure of multiplication and addition in $\mathbb{Z}$. Hence $n^2 = 2M + 1$ is odd. Therefore we conclude

If n^2 is even, then n is even.

Q.E.D.

NOTE: It is nice to end a proof by contrapositive by stating the statement as originally written. But it is not necessary too; ending with "n^2 is odd" is fine too.

An Example of a Proof by Contradiction

The so-called FIRST CRISIS IN MATH!!

Long ago, in the 6th century BC, the followers of the school of **Pythagoras**, the Pythagoreans, came to a crisis in math. It was a long held view that all numbers are the ratio of two integers. However, the Pythagorean member **Hippasus** is thought to be the one to discover that there is a length that cannot be the ratio of two integers – a Greek Crisis! This length was the $\sqrt{2}$.

It is rumored that the drowning of Hippasus was the punishment from the gods for divulging this secret[5].

Theorem 5.5. *The value $\sqrt{2}$ is irrational.*

Proof. Assume by way of contradiction that $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ with $b \neq 0$, and we may assert that the greatest common divisor of a and b is 1. Then we have the following sequence of implications

$$\begin{aligned} \sqrt{2} = \frac{a}{b} &\implies \sqrt{2}b = a \\ &\implies 2b^2 = a^2 && \text{by squaring both sides} \\ &\implies a^2 \text{ is even} \\ &\implies a \text{ is even} && \text{by Lemma 5.4.} \end{aligned}$$

Hence $a = 2k$ for some $k \in \mathbb{Z}$. Substituting this value a in the equality $\sqrt{2}b = a$, we have $\sqrt{2}b = 2k$ and the following sequence of implications

$$\begin{aligned} \sqrt{2}b = 2k &\implies 2b^2 = 4k^2 && \text{by squaring both sides} \\ &\implies b^2 = 2k^2 \\ &\implies b^2 \text{ is even} \\ &\implies b \text{ is even} && \text{by Lemma 5.4.} \end{aligned}$$

However, both a and b being even contradicts the fact that 1 is their greatest common divisor. Therefore we conclude that $\sqrt{2}$ is irrational.

Q.E.D.

[5]We would provide a reference for this, but there is a lot of documented debate with historians about the first Greek discoverer of the irrationality of $\sqrt{2}$. But for sure, the drowning of Hippasus is certain.

An Example of a Proof by Cases

An Important Consequence of the Division Algorithm

Theorem 5.6 (Division Algorithm)**.** *Given integers a and b, with $b > 0$, there exist unique integers q and r satisfying*

$$a = qb + r \quad \textit{such that} \quad 0 \le r < b.$$

A consequence of this theorem is one super well-known result that every integer is even (i.e., in $2\mathbb{Z}$) or odd (i.e. in $2\mathbb{Z}+1$). The 2 is replaceable here. For instance, every integer lies in ONE AND ONLY one of the sets $3\mathbb{Z}$ or $3\mathbb{Z}+1$ or $3\mathbb{Z}+2$.

Theorem 5.7. *If n is an integer not divisible by 3, then $n^2 \in 3\mathbb{Z}+1$.*

Proof. Assume n is an integer not divisible by 3. Then either n lies in $3\mathbb{Z}+1$ or $3\mathbb{Z}+2$.

(CASE 1:) Assume $n \in 3\mathbb{Z}+1$.

WWTS: $n^2 \in 3\mathbb{Z}+1$.

Since $n \in 3\mathbb{Z}+1$ then $n = 3m+1$ for some $m \in \mathbb{Z}$. Thus we have

$$n^2 = (3m+1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1.$$

Set $M := 3m^2 + 2m$. Clearly $M \in \mathbb{Z}$, and hence $n^2 = 3M + 1 \in 3\mathbb{Z}+1$.

(CASE 2:) Assume $n \in 3\mathbb{Z}+2$.

WWTS: $n^2 \in 3\mathbb{Z}+1$.

Since $n \in 3\mathbb{Z}+2$ then $n = 3m+2$ for some $m \in \mathbb{Z}$. Thus we have

$$n^2 = (3m+2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1.$$

Set $M := 3m^2+4m+1$. Clearly $M \in \mathbb{Z}$, and hence $n^2 = 3M+1 \in 3\mathbb{Z}+1$.

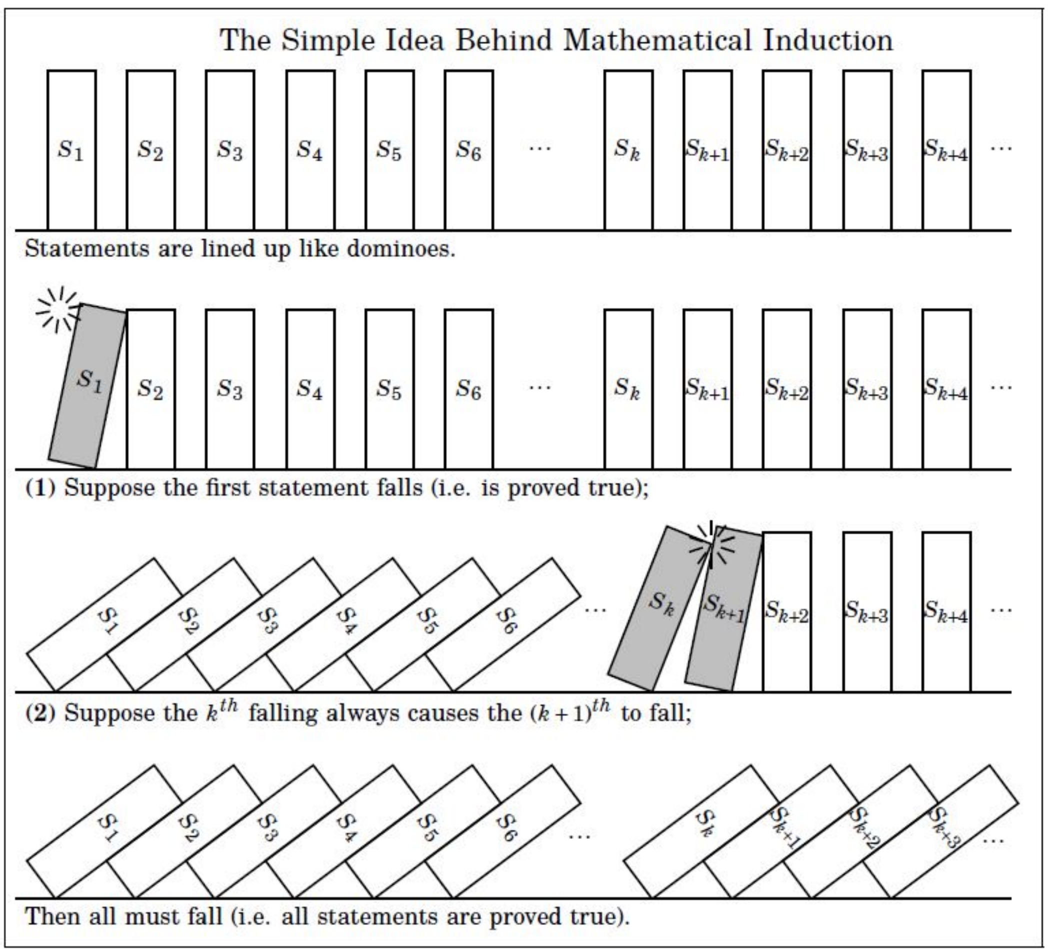
Q.E.D.

6 Mathematical Induction

The Falling Dominoes Analogy

Imagine that we want to prove that an infinite number of statements S_1, S_2, S_3, etc. are all true. The *dominoes analogy* of math induction is the following:

- Visualize each statement as a domino. And visualize the act of the n^{th} domino falling to mean that statement S_n is proven true.
- Suppose you can knock over the first domino (i.e., you can prove S_1).
- Suppose you can show that if any k^{th} domino falling will definitely force the $(k+1)^{\text{th}}$ domino to fall.
- We can conclude that S_1 falls and knocks down S_2. Next S_2 falls and knocks down S_3. Then S_3 knocks down S_4, etc.
- Hence all dominoes fall!!! **Q.E.D.**



The Principle of Mathematical Induction

Let $P(n)$ be a property defined on the integers n. Let $a \in \mathbb{N}$ be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true
2. For every $k \geq a$, we have $P(k)$ implies $P(k+1)$.

Then $P(n)$ is true for all $n \geq a$.

The Method of Mathematical Induction

PROVE: For all $n \geq a$, it follow that $P(n)$ is true.

(Proof)

- **Step 1:** (Base Case) Show that $P(a)$ is true.
- **Step 2:** (Induction Hypothesis) Assume $P(k)$ holds for some $k \geq a$. **WARNING!:** The word "some" is VERY IMPORTANT! This existential quantifying word asserts that the k is a PARTICULAR (i.e., fixed) but arbitrarily chosen integer.
- **Step 3:** Show that $P(k+1)$ holds. **WARNING!:** If you manage to prove $P(k+1)$ without ever using the fact that $P(k)$ holds, then your proof is DEFINITELY wrong.
- **Step 4:** Conclude that $P(n)$ is true for all $n \geq a$.

Exercise 6.1. Suppose that you were to prove that $1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}$ for all $n \geq 1$. State the induction hypothesis and the WWTS. [You Do!]

Example 6.2. Prove that $2n + 1 < 2^n$ for all $n \geq 3$.

Base Case: $(n = 3)$ Observe that $2(3) + 1 = 6 < 8 = 2^3$, and hence the base case holds.

Induction Hypothesis: Assume that $2k + 1 < 2^k$ for some $k \geq 3$.

WWTS: $2(k + 1) + 1 < 2^{k+1}$.

It suffices to show that $2k + 3 < 2^{k+1}$. Consider the following sequence of equalities and inequalities:

$$\begin{aligned} 2k + 3 &= (2k + 1) + 2 \\ &< 2^k + 2 && \text{by Induction Hypothesis} \\ &< 2^k + 2^k && \text{since } 2 < 2^k \text{ for all } k \geq 2 \\ &= 2 \cdot 2^k \\ &= 2^{k+1}, \end{aligned}$$

as desired. Hence $2n + 1 < 2^n$ for all $n \geq 3$.

Q.E.D.

Exercise 6.3. Prove by induction that $n^2 < 2^n$ for all $n \geq 5$. [You Do!]

> **HINT:** In your attempt to prove the $k+1$ case holds, you may find it useful at some point to use the result from Example 6.2.

Obviously the proof will not fit in this small space. Place your solution as always on the solution pages.

For the next exercise, it is useful to know the definition of divisibility in the integers.

Definition 6.4. A nonzero integer d divides n (symbolically $d \mid n$) if and only if there exists a $k \in \mathbb{Z}$ such that $n = d \cdot k$. Equivalently,

$$\begin{aligned} d \mid n &\iff n \text{ is a } \underline{\text{multiple}} \text{ of } d \\ &\iff d \text{ is a } \underline{\text{factor}} \text{ of } n \\ &\iff d \text{ is a } \underline{\text{divisor}} \text{ of } n \end{aligned}$$

We say n is **divisible** by d.

Exercise 6.5. Prove that 3 divides $n^3 - n$ for all $n \geq 1$. [You Do!]

Q.E.D.

7 Quick Review of Set Theory & Set Theory Proofs

Definition 7.1. A **set** is any collection of objects.

- Repetition of objects is ignored. For instance $\{x, x, y\} = \{x, y\}$.
- Order does not matter. For instance $\{x, y\} = \{y, x\}$.

A member of a set is called an **element**. It is customary to use a capital letter to denote a set.

Definition 7.2. The **cardinality** of a set is the number of elements in the set. We use the symbols $N(A)$, $n(A)$, or $|A|$ to denote this "size" of a set A.

Two types of infinity ∞ that will arise in this class are $\aleph_0$ (aleph-naught) and $\aleph_1$ (aleph-one). The former is the cardinality of the natural numbers, while the latter is the cardinality of the real numbers.

Definition 7.3. The **nullset**, or **empty set**, is the set that contains no elements. It is denoted $\emptyset$ or equivalently $\{\}$.

Exercise 7.4. Compute the following cardinalities. [You Do!]

$$\begin{aligned}
A = \{a, b, c\} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \{\{a, b\}, a, b\} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \{\{\{a\}, a\}, a\} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \mathbb{Z} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = 2\mathbb{Z} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \mathbb{Q} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \mathbb{C} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \{\emptyset\} \quad &\Longrightarrow \quad |A| = \boxed{}
\end{aligned}$$

WARNING! Two Easily Confused Symbols: $\in$ versus $\subseteq$

Definition 7.5. The notation $x \in A$ means "x **is an element of** A".

Definition 7.6. The notation $A \subseteq B$ means "A **is a subset of** B".

$$A \subseteq B \quad \text{if and only if} \quad (\text{For every } x \in A, \text{ we have } x \in B).$$

Exercise 7.7. Let $A = \{\{a, b\}, a, c\}$. Answer the following. [True or False?]

$a \in A$	
$b \in A$	
$\{a, c\} \subseteq A$	
$\{a, b\} \subseteq A$	
$\{a, b\} \in A$	
$\{\{a, b\}\} \subseteq A$	
$\emptyset \subseteq A$	

QUESTION: Why is the nullset a subset of any set A?

ANSWER: Consider the statement $\emptyset \subseteq A$. Since there are no elements in $\emptyset$, it is clear that ALL of the elements in $\emptyset$ are contained in A, regardless of what set A is. This is called a vacuously true statement.

Definition 7.8. The **power set** of a given set A is the set of all subsets of A. We denote the power set of A by $\mathcal{P}(A)$.

For example if $A = \{a, b, c\}$, then the power set of A is

$$\mathcal{P}(A) = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\Big\}.$$

Exercise 7.9. Let $|A| = 10$. What is the cardinality of the power set of A? In general if $|B| = n$, then what is $|\mathcal{P}(B)|$ equal to? [You Do!]

Definition 7.10. The **universal set** is a larger known set wherein a given set lives. We denote it $\mathcal{U}$. This set is generally understood in context or deliberately stated.

EXAMPLE: Find the universal sets.

$$A = \{\text{dogs}\} \implies \mathcal{U} = \boxed{\text{animals}}$$

$$A = \{\text{even integers}\} \implies \mathcal{U} = \boxed{\text{integers}}$$

Definition 7.11. Let A and B be sets. Then the **intersection** of A and B (denoted $A \cap B$) is the set of elements in $\mathcal{U}$ that lie in both A and B. Set-theoretically,

$$A \cap B := \{x \in \mathcal{U} \mid x \in A \text{ and } x \in B\}.$$

Definition 7.12. Let A and B be sets. Then the **union** of A and B (denoted $A \cup B$) is the set of elements in $\mathcal{U}$ that lie in A or B (or both). Set-theoretically,

$$A \cup B := \{x \in \mathcal{U} \mid x \in A \text{ or } x \in B\}.$$

Definition 7.13. Let A be a set. The **complement** of the set A (denoted A^c or A') is the set of elements in $\mathcal{U}$ that do not lie in A. Set-theoretically,

$$A^c := \{x \in \mathcal{U} \mid x \notin A\}.$$

Definition 7.14. Let A and B be sets. The **difference** $A \backslash B$ of A and B is defined as $\{x \in A \mid x \notin B\}$.

NOTE: $A \backslash B$ is sometimes written $A - B$. Moreover $A \backslash B = A \cap B^c$.

Just as we have De Morgan's Laws for logic symbols $\vee$ and $\wedge$ (recall Definition 2.11), we have De Morgan's Law for $\cup$ and $\cap$, respectively, as follows:

Definition 7.15. Let A and B be sets. Then **De Morgan's Laws** assert the following set equalities:

- $(A \cup B)^c$ is equal to $A^c \cap B^c$.
- $(A \cap B)^c$ is equal to $A^c \cup B^c$.

Set Theory on Collections of Sets

Let J be any indexing set of any size (finite or infinite). Suppose for each $j \in J$, we have a set A_j. Consider the collection of sets $\{A_j \mid j \in J\}$.

Definition 7.16. We have the following:

- **"Big Union"**: $\bigcup_{j \in J} A_j = \{x \in \mathcal{U} \mid x \in A_j \text{ for some } j \in J\}$
- **"Big Intersection"**: $\bigcap_{j \in J} A_j = \{x \in \mathcal{U} \mid x \in A_j \text{ for every } j \in J\}$
- Let $A_1, A_2, \ldots, A_n$ be a finite collection of sets. The **Cartesian product** of the collection is denoted $A_1 \times A_2 \times \cdots A_n$ and is defined as
$$\{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$
- Two sets A and B are **disjoint** if $A \cap B = \emptyset$. Moreover the collection of sets $\{A_j \mid j \in J\}$ is **pairwise disjoint** if $A_i \cap A_j = \emptyset$ for $i, j \in J$ such that $i \neq j$.
- A collection of sets $\{A_j \mid j \in J\}$ is a **partition of a set** X if and only if
  1. $X = \bigcup_{j \in J} A_j$, and
  2. the collection is pairwise disjoint.

  We use the notation $X = \coprod_{j \in J} A_j$ to denote this pairwise **disjoint union**.

Example 7.17. Prove that the sets $3\mathbb{Z}$, $3\mathbb{Z}+1$, and $3\mathbb{Z}+2$ form a partition of $\mathbb{Z}$.

Proof. Consider the sets $3\mathbb{Z}$, $3\mathbb{Z}+1$, and $3\mathbb{Z}+2$.

WWTS: $\mathbb{Z} = \coprod_{j=0,1,2} 3\mathbb{Z}+j.$

By the division algorithm (see Theorem 5.6), every integer n can be represented in exactly one of three forms:

$$n = 3k \quad \text{or} \quad n = 3k+1 \quad \text{or} \quad n = 3k+2$$

for some integer k, and hence $\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}{+}1 \cup 3\mathbb{Z}{+}2$. This also implies that the three sets are pairwise disjoint and hence $\mathbb{Z} = \coprod_{j=0,1,2} 3\mathbb{Z}+j$.

Q.E.D.

How to prove two sets are equal?

Claim 7.18. *Let A and B be sets. Then*

$$A = B \quad \text{if and only if} \quad A \subseteq B \text{ and } B \subseteq A.$$

Exercise 7.19. Prove that $10\mathbb{Z}+7 \subseteq 5\mathbb{Z}+2$ but $10\mathbb{Z}+7 \not\supseteq 5\mathbb{Z}+2$. [You Do!]

Proof.

Q.E.D.

8 Functions, Bijections, Compositions, Etc.

The following is a variety of important definitions related to functions defined on arbitrary sets. For the following definitions, let X and Y be sets and consider a function $f : X \to Y$.

Definition 8.1. The **domain** of f is the set X, and the **codomain** of f is the set Y.

Definition 8.2. The **range** of f is the set

$$\{y \in Y \mid f(x) = y \text{ for some } x \in X\}.$$

Equivalently, this set is called the **image** of f.

Definition 8.3. Let $A \subseteq X$ and $B \subseteq Y$. Then we have the following sets:

- The **image of A under f** is denoted $f(A)$ and defined as

$$f(A) = \{y \in Y \mid f(a) = y \text{ for some } a \in A\}.$$

- The **preimage of B under f** is denoted $f^{-1}(B)$ and defined as

$$f^{-1}(B) = \{x \in X \mid f(x) = b \text{ for some } b \in B\}.$$

WARNING!!!!!: By using the symbol $f^{-1}(B)$, we do not mean that the function f has an inverse. The symbol $f^{-1}(B)$ simply denotes a set. For every $B \subseteq Y$, the set $f^{-1}(B)$ (which may possibly be empty) is well-defined REGARDLESS if the function f has an inverse.

Exercise 8.4. Let $X = \{a, b, c, d, e\}$ and $Y = \{1, 2, 3\}$. Define $f : X \to Y$ such that $f(a) = 1, f(b) = 2$, and $f(c) = f(d) = f(e) = 3$. Then compute the following three sets: image $f(\{a, b\})$, preimage $f^{-1}(\{3\})$, and range of f. [You Do!]

Injective, Surjective, and Bijective Functions

Definition 8.5. Consider a function $f : X \to Y$. The function f is called **injective** if the following implication holds for all elements $x_1, x_2 \in X$:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

We sometimes refer to such a function as being **one-to-one**.

Definition 8.6. Consider a function $f : X \to Y$. The function f is called **surjective** if the following universal-existential statement holds:

$$\text{For each } y \in Y, \text{ there exists an } x \in X \text{ such that } f(x) = y.$$

We sometimes refer to such a function as being **onto**.

Exercise 8.7. For each function denote in the corresponding box (Yes/No) if the function is injective and/or surjective. The first two are done for you. [You Do!]

the function	**injective?**	**surjective?**
$f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$	No	No
$f : \mathbb{R} \to [0, \infty)$ by $f(x) = x^2$	No	Yes
$f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 4n - 5$		
$f : \mathcal{P}(\{a, b, c\}) \to \mathbb{Z}$ by $f(X) = \lvert X \rvert$ for each subset $X \in \mathcal{P}(\{a, b, c\})$. (See footnote[6].)		
$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f(x, y) = (x + 1, 2 - y)$		
$f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by $f(r, s) = r + \sqrt{2}s$		

Definition 8.8. Consider a function $f : X \to Y$. The function f is called **bijective** if it is both injective and surjective. If so, we say that f gives a **one-to-one correspondence** between A and B.

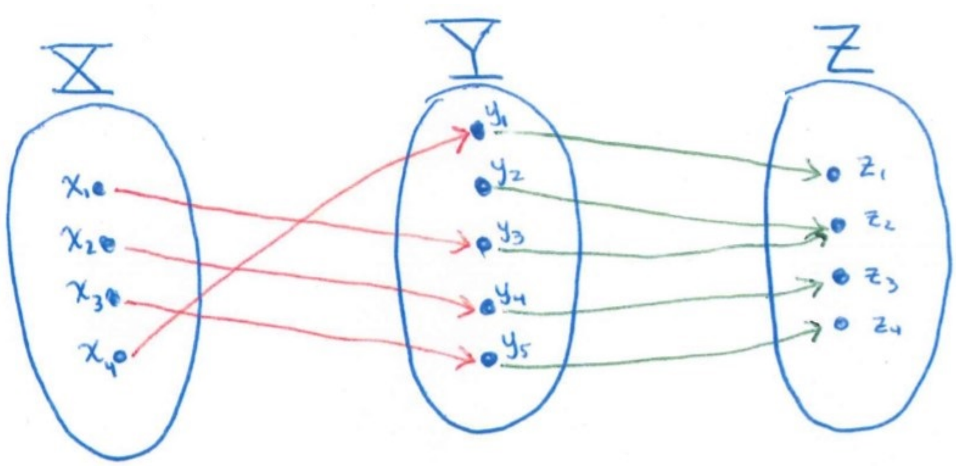
[6]Recall the definition of "power set" given in Definition 7.8.

Composition of Functions

Definition 8.9. Consider functions f and g with $f : X \to Y$ and $g : Y \to Z$. We define the **composition of functions** f and g by $g \circ f : X \to Z$ by

$$(g \circ f)(x) := g(f(x)) \text{ for all } x \in X.$$

Exercise 8.10. Let f and g be functions with $f : X \to Y$ and $g : Y \to Z$ given by the following arrow diagram:



Answer the following questions: [You Do!]

- What is $(g \circ f)(x_1)$, $(g \circ f)(x_2)$, $(g \circ f)(x_3)$, and $(g \circ f)(x_4)$?

- Fill in the table below with answers "Yes" or "No".

the function	**injective?**	**surjective?**
$g \circ f$		
f		
g		

Theorem 8.11. *Suppose that* $f : X \to Y$ *and* $g : Y \to Z$. *Then*

(i) If $g \circ f$ *is injective, then* f *is injective.*

(ii) If $g \circ f$ *is surjective, then* g *is surjective.*

NOTE: We will prove part (i) and leave it to you to prove part (ii).

Proof of (i). Assume $g \circ f$ is injective.

WWTS: f is injective.

Suppose we have $f(a) = f(b)$ for $a, b \in X$. It suffices to show that $a = b$. Since $f(a) = f(b)$, then $g(f(a)) = g(f(b))$. It follows that $(g \circ f)(a) = (g \circ f)(b)$. Since $g \circ f$ is injective, then $a = b$ as desired. Hence f is injective.

Q.E.D.

Exercise 8.12. Prove part (ii) of Theorem 8.11. [You Do!]

Proof of (ii).

Q.E.D.

9 Solutions to all exercises

Exercise 2.2:

Exercise 2.8: Is the statement $(A \vee \sim B) \vee \sim A$ a tautology, contradiction, or neither?

A	B	$\sim B$	$A \vee \sim B$	$\sim A$	$(A \vee \sim B) \vee \sim A$
T	T				
T	F				
F	T				
F	F				

We conclude that $(A \vee \sim B) \vee \sim A$ **is a :**

Exercise 2.10: Show that $\sim (A \vee B) \equiv \sim A \wedge \sim B$.

A	B	$A \vee B$	$\sim (A \vee B)$	$\sim A$	$\sim B$	$\sim A \wedge \sim B$
T	T					
T	F					
F	T					
F	F					

We conclude that:

Exercise 2.12:

Exercise 3.1: Compute the truth table for $\sim B \Longrightarrow \sim A$.

A	B	$\sim A$	$\sim B$	$\sim B \Longrightarrow \sim A$
T	T			
T	F			
F	T			
F	F			

QUESTION: Is $A \Longrightarrow B$ logically equivalent to $\sim B \Longrightarrow \sim A$? ☐

Exercise 3.3: Compute the truth table for $B \implies A$.

A	B	$B \implies A$
T	T	
T	F	
F	T	
F	F	

QUESTION: Is $A \implies B$ logically equivalent to $B \implies A$? [ ]

Exercise 3.5: Compute the truth table for $\sim A \implies \sim B$.

A	B	$\sim A$	$\sim B$	$\sim A \implies \sim B$
T	T			
T	F			
F	T			
F	F			

QUESTION: Is $A \implies B$ logically equivalent to $\sim A \implies \sim B$? [ ]

Exercise 3.8: Compute the truth table for $\sim A \vee B$ and compare it to $A \implies B$.

A	B	$\sim A$	$\sim A \vee B$	$A \implies B$
T	T			
T	F			
F	T			
F	F			

QUESTION: Is $A \implies B$ logically equivalent to $\sim A \vee B$? [ ]

Exercise 3.10: What is the negation of $\sim A \vee B$? [ ] Verify this is indeed $\sim (A \implies B)$ via the truth table.

A	B	$\sim B$	$A \wedge \sim B$	$A \implies B$	$\sim (A \implies B)$
T	T				
T	F				
F	T				
F	F				

Exercise 3.11: The negation of the statement "If I have a baby, then my life will suck." is the following

Exercise 3.12: Write a sequence of logical equivalencies to prove again (but without truth tables that the negation of $A \Longrightarrow B$ is indeed $A \wedge \sim B$.

Exercise 3.14: Write down the If-Then form of the Or-Statement "Today is not Easter, or tomorrow is Monday."

Exercise 4.3:

Exercise 4.5: Let $P(x, y)$ be the predicate $xy \in \mathbb{Z}$.

- If $D = \mathbb{N}$, is $P(x, y)$ true $\forall x, y \in D$?
- If $D = \mathbb{Z}$, is $P(x, y)$ true $\forall x, y \in D$?
- If $D = \mathbb{Q}$, is $P(x, y)$ true $\forall x, y \in D$?
- If $D = \mathbb{R}$, is $P(x, y)$ true $\forall x, y \in D$?

Exercise 4.8:

Exercise 4.9:

Exercise 4.11:

Exercise 4.14: 'An ''informal" negation for each of the two universal conditional statements from Example 4.13 are:

(i)

(ii)

Exercise 4.15: Negation of "there is a natural number that is even and prime" is:

Exercise 4.16: Negation of "all dogs go to heaven" is:

Exercise 4.18: A negation of the continuity definition (using English and not ∀-∃ symbols is the following

Exercise 4.20: Statements (3), (5), and (8) are true. The rest are false.

Exercise 5.3 (Part 1 of 2) This question had two parts. First give the negation of the Statement (1):

Exercise 5.3 (Part 2 of 2) And here is a proof for the negation of Statement (1):
Proof.

Q.E.D.

Exercise 6.1: Place the induction hypothesis and WWTS below.

Exercise 6.3: Prove by induction that $n^2 < 2^n$ for all $n \geq 5$.

Q.E.D.

Exercise 6.5: Prove that 3 divides $n^3 - n$ for all $n \geq 1$.

Q.E.D.

Exercise 7.4:

$$
\begin{aligned}
A = \{a, b, c\} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \{\{a, b\}, a, b\} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \{\{\{a\}, a\}, a\} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \mathbb{Z} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = 2\mathbb{Z} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \mathbb{Q} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \mathbb{C} \quad &\Longrightarrow \quad |A| = \boxed{} \\
A = \{\emptyset\} \quad &\Longrightarrow \quad |A| = \boxed{}
\end{aligned}
$$

Exercise 7.7: Let $A = \{\{a, b\}, a, c\}$. Answer the following.

$a \in A$	
$b \in A$	
$\{a, c\} \subseteq A$	
$\{a, b\} \subseteq A$	
$\{a, b\} \in A$	
$\{\{a, b\}\} \subseteq A$	
$\emptyset \subseteq A$	

Exercise 7.9: The corresponding sizes of the power sets of A and B are respectively:

Exercise 7.19: Prove that $10\mathbb{Z} + 7 \subseteq 5\mathbb{Z} + 2$ but $10\mathbb{Z} + 7 \not\supseteq 5\mathbb{Z} + 2$.
Proof.

Q.E.D.

Exercise 8.4: The image $f(\{a, b\})$, preimage $f^{-1}(\{3\})$, and range of f are:

Exercise 8.7:

the function	**injective?**	**surjective?**
$f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$	No	No
$f : \mathbb{R} \to [0, \infty)$ by $f(x) = x^2$	No	Yes
$f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 4n - 5$		
$f : \mathcal{P}(\{a, b, c\}) \to \mathbb{Z}$ by $f(X) = \lvert X\rvert$ for each subset $X \in \mathcal{P}(\{a, b, c\})$.		
$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f(x, y) = (x + 1, 2 - y)$		
$f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by $f(r, s) = r + \sqrt{2}s$		

Exercise 8.10: Refer to the diagram given in this exercise.

- What is $\big(g \circ f\big)(x_1)$, $\big(g \circ f\big)(x_2)$, $\big(g \circ f\big)(x_3)$, and $\big(g \circ f\big)(x_4)$?

- Fill in the table below with answers "Yes" or "No".

the function	**injective?**	**surjective?**
$g \circ f$		
f		
g		

Exercise 8.12: Prove part (ii) of Theorem 8.11.

Proof.

Q.E.D.

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