## 2.3 Bounds of sets of real numbers

### 2.3.1 Upper bounds of a set; the least upper bound (supremum)

Consider $S$ a set of real numbers.

$S$ is called **bounded above** if there is a number $M$ so that any $x \in S$ is less than, or equal to, $M$: $x \leq M$. The number $M$ is called *an upper bound* for the set $S$.

Note that if $M$ is an upper bound for $S$ then any bigger number is also an upper bound.

Not all sets have an upper bound. For example, the set of natural numbers does not.

*A number $B$ is called* **the least upper bound** *(or* **supremum***) of the set $S$ if:*

*1) $B$ is an upper bound: any $x \in S$ satisfies $x \leq B$, and*

*2) $B$ is* the smallest *upper bound. In other words, any smaller number is not an upper bound:*

$$\textit{if } \; t < B \; \textit{ then there is } x \in S \; \textit{ with } \; t < x$$

*Notation:*

$$B = \sup S = \sup_{x \in S} x$$

Upper bounds of $S$ may, or may not belong to $S$.

For example, the interval $(-2, 3)$ is bounded above by 100, 15, 4, 3.55, 3. In fact 3 is its least upper bound.

The interval $(-2, 3]$ also has 3 as its least upper bound.

When the supremum of $S$ is a number that belongs to $S$ then it is also called the **maximum** of $S$.

Examples:

1) The interval $(-2, 3)$ has supremum equal to 3 and no maximum; $(-2, 3]$ has supremum, and maximum, equal to 3.

2) The function $f(x) = x^2$ with domain $[0, 4)$ has a supremum (equals $4^2$), but not a maximum. The function $g(x) = x^2$ with domain $[0, 4]$ has (not only a supremum, but also) a maximum; it equals $g(4) = 4^2$.

The interval $(-2, +\infty)$ is not bounded above.

If the set $S$ is not bounded above (also called *unbounded above*) we write (conventionally)

$$\sup S = +\infty$$

### 2.3.2 Bounded sets do have a least upper bound.

This is a fundamental property of real numbers, as it allows us to talk about limits.

**Theorem** *Any nonempty set of real numbers which is* bounded above *has a supremum.*

*Proof.*

We need a good notation for a real number given by its decimal representation. A real number has the form

$$a = a_0.a_1a_2a_3a_4... \qquad \text{where} \;\; a_0 \text{ is an integer and } a_1, a_2, a_3, ... \in \{0, 1, 2, ...9\}$$

To eliminate ambiguity in defining real numbers by their decimal representation, let us decide that if the sequence of decimals ends up with nines: $a = a_0.a_1a_2...a_n9999...$ (where $a_n < 9$) then we choose this number's decimal representation as $a = a_0.a_1a_2...(a_n + 1)0000....$ (For example, instead of 0.4999999.. we write 0.5.)

Let $S$ be a nonempty set of real numbers, bounded above.

Let us construct the least upper bound of $S$.

Consider first all the approximations by integers of the numbers $a$ of $S$: if $a = a_0.a_1a_2...$ collect the $a_0$'s. This is a collection of integer numbers. It is bounded above (by assumption). Then there is a largest one among them, call it $B_0$.

Next collect only the numbers in $S$ which begin with $B_0$. (There are some!) Call their collection $S_0$.

Any number in $S \setminus S_0$ (number of $S$ not in $S_0$) is smaller than any number in $S_0$.

Look at the first decimal $a_1$ of the numbers in $S_0$. Let $B_1$ be the largest among them. Let $S_1$ be the set of all numbers in $S_0$ whose first decimal is $B_1$.

Note that the numbers in $S_1$ begin with $B_0.B_1$

Also note that any number in $S \setminus S_1$ is smaller than any number in $S_1$.

Next look at the second decimal of the numbers in $S_1$. Find the largest, $B_2$ etc.

Repeating the procedure we construct a sequence of smaller and smaller sets $S_0, S_1, S_2, ...S_n, ...$

$$S \supset S_0 \supset S_1 \supset S_2 \supset ... \supset S_n \supset ...$$

Note that every set $S_n$ contains al least one element (it is not empty).

At each step $n$ we have constructed the set $S_n$ of numbers of $S$ which start with $B_0.B_1B_2...B_n$; the rest of the decimals can be anything. Also all numbers in $S \setminus S_n$ are smaller than all numbers of $S_n$. (The construction is by induction!)

We end up with the number $B = B_0.B_1B_2...B_nB_{n+1}....$

We need to show that $B$ is the least upper bound.

To show it is an upper bound, let $a \in S$. If $a_0 < B_0$ then $a < B$. Otherwise $a_0 = B_0$ and we go on to compare the first decimals. Either $a_1 < B_1$ therefore $a < B$ or, otherwise, $a_1 = B_1$. Etc. So either $a < B$ or $a = B$. So $B$ is an upper bound.

To show it is the least (upper bound), take any smaller number $t < B$. Then $t$ differs from $B$ at some first decimal, say at the $n$th decimal: $t = B_0.B_1B_2...B_{n-1}t_nt_{n+1}...$ and $t_n < B_n$. But then $t$ is not in $S_n$ and $S_n$ contains numbers bigger than $t$. QED

### 2.3.3 Lower bounds

By exchanging "less than" $<$ with "greater than" $>$ throughout the section §2.3.1 we can similarly talk about lower bounds.

Here it is.

$S$ is called *bounded below* if there is a number $m$ so that any $x \in S$ is bigger than, or equal to $m$: $x \geq m$. The number $m$ is called *a lower bound* for the set $S$.

Note that if $m$ is a lower bound for $S$ then any smaller number is also a lower bound.

*A number $b$ is called* **the greatest lower bound** *(or* **infimum***) of the set $S$ if:*

*1) $b$ is a lower bound: any $x \in S$ satisfies $x \geq b$, and*

*2) $b$ is* the greatest *lower bound. In other words, any greater number is not a lower bound:*

$$\textit{if } \; b < t \; \textit{ then there is } x \in S \; \textit{ with } \; x < t$$

*Notation:*

$$b = \inf S = \inf_{x \in S} x$$

Greatest lower bounds of $S$ may, or may not belong to $S$. For example, the interval $(-2, 3)$ is bounded below by -100, -15, -4, -2. In fact $-2$ is its infimum (greatest lower bound). The interval $[-2, 3)$ also has $-2$ as its infimum.

When the infimum of $S$ belongs to $S$ then it is called the **minimum** of $S$.

The interval $(-\infty, -2)$ is not bounded below.
If the set $S$ is not bounded below we write (conventionally)

$$\inf S = -\infty$$

**Theorem** ***Any nonempty set of real numbers which is bounded below has an infimum.***

*Proof.*

No, we need not repeat the proof of §2.3.2. We do as follows.

Let $S$ be a nonempty set which is bounded below. Construct the set $T$ which contains all the opposites $-a$ of the numbers $a$ of $S$:

$$T = \{-a\,;\ \text{where } a \in S\}$$

The set $T$ is nonempty and is bounded above. By the Theorem of §2.3.2, $T$ has a least upper bound, call it $B$. Then its opposite, $-B$, is the greatest lower bound for $S$. Q.E.D.

### 2.3.4 Bounded sets

A set which is bounded above and bounded below is called **bounded**.

So if $S$ is a bounded set then there are two numbers, $m$ and $M$ so that $m \leq x \leq M$ for any $x \in S$. It sometimes convenient to lower $m$ and/or increase $M$ (if need be) and write $|x| < C$ for all $x \in S$.

A set which is not bounded is called *unbounded*.

For example the interval $(-2, 3)$ is bounded.

Examples of unbounded sets: $(-2, +\infty), (-\infty, 3)$, the set of all real numbers $(-\infty, +\infty)$, the set of all natural numbers.

## 2.4 What are the Real Numbers?

In practice we do not use the whole infinite sequence of decimals of an irrational number. What we do use are the properties of the given number.

Some of the general properties of real numbers were listed in §2.2. There are more, of course, but they can *all* be deduced from the listed five.

The modern approach is to define the set of real numbers through its properties:

**Definition** *A set with properties* **I-V** *is called the set of real numbers.*

This is an axiomatic definition: properties **I-V** are taken to be axioms - statements considered to be true. All other properties of real numbers are deduced from these five, using logic.

When an axiom system is established there are two major questions:

1) Are there enough axioms to match our intuition on the concept we want to define?

In our case if we omit axiom **V**, the first four are also satisfied by the rational numbers!

(**Note:** by axiom **V** the real numbers are a completion of the rationals.)

2) Are they consistent? Is there a set for which the specified axioms are true?

Yes, there is, since we do have our model with decimal representation of real numbers. They satisfy **I-III** by the way operations are defined, **IV** is very easy to show. Only **V** needs a proof.

Other models for axioms **I-V**: the number line, many physical quantities (temperature, velocity (on a straight line), time, etc.)

**Remarks** (easy to see when one thinks in decimal representations):

**1.** between any two rational numbers there is another rational number (can you imagine how the set of all rationals looks like when plotted on the number line?);

**2.** between any two rational numbers there also are irrational numbers;

**3.** similarly, between any two irrationals there are rationals, and irrationals.

## Exercises

**Definition** We call the supremum of the function $f(x)$ for $x \in S$ the number $\sup_{x\in S} f(x)$ (read: the supremum of the set of all $y$'s).

**2.3.1** Consider the function $f(x) = x^4$. Does the function have a supremum and/or maximum for
**a)** $x \in [-2, 2]$
**b)** $x \in [-2, 1]$
**c)** $x \in [-2, 0]$
**d)** $x \in [-2, 0)$?
(As always, *explain* your answers.)

**2.3.2** Does the function in problem **2.3.1** have an infimum, and/or minimum of the domains **a)-d)**?

**2.3.3** Denote by $S$ the set of all the rational numbers between 0 and $\pi$:

$$S = \{\, x\, ;\, x \text{ rational and } 0 \leq x < \pi\}$$

**a)** Explain why this set $S$ necessarily has a supremum.
**b)** Guess what this supremum is.
**c) Bonus problem!** Explain why (or, prove that) the number you guessed is indeed the supremum of $S$.
**d)** Explain why this set $S$ has an infimum.
**e)** Guess what this infimum is.
**f)** True or false: $\inf S = \min S$?

**2.3.4** Consider the set

$$T = \{0.2\, ,\, 0.23\, .\, 0.234\, ,\, 0.2343\, ,\, 0.23434\, ,\, 0.234343\, ,\, 0.2343434\, , \ldots\}$$

Explain why this set has a supremum and find it. Is is number rational? If so, write it as a fraction.

**2.3.5 Bonus!** Prove that

$$\sup\{2, 2.2, 2.22, 2.222, 2.2222, \ldots\} = 2.\overline{2}$$