The Method of Substitution

You know how to use a graph to find the point of intersection of two linear equations. However, graphing is not always the most efficient or accurate method.

If you are graphing by hand, the point of intersection must be on the grid lines to give an exact answer.

If you use a graphing calculator or *The Geometer's Sketchpad*®, you can find the point of intersection to a chosen number of decimal places. However, the equations must be expressed in the form $y = mx + b$ first to enter them into the calculator or computer. Rearranging some equations is not easy.

There are other ways to find the point of intersection of two linear relations. One of these is an algebraic method called the **method of substitution**.

method of substitution

■ solving a linear system by substituting for one variable from one equation into the other equation

■ Investigate

How can you use substitution to solve a linear system?

Sometimes, at the beginning of geography class, Mrs. Thomson gives her students a puzzle to solve. One morning the puzzle is as follows.

The sum of the number of cantons in Switzerland and the states in Austria is 35. One less than triple the number of Austrian states is the same as the number of Swiss cantons. How many states are there in Austria and how many cantons are there in Switzerland?

Wesam wrote two equations to represent the information:

$S + A = 32$ ①
$3A - 4 = S$ ②
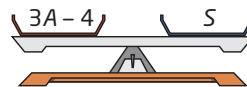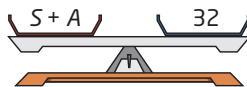
1. **a)** What does the S represent in the first equation?
   b) What does the S represent in the second equation?
   c) Do the S's in both equations represent the same value or different values?

2. a) What equation results if you substitute $3A - 4$ from the second equation into the first equation in place of S?
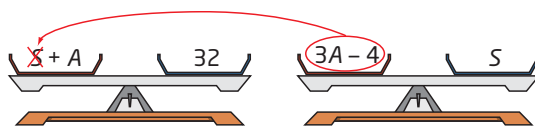
b) Solve the resulting equation for A.

c) What does this mean in the context of this question?

d) How can you find the value for S?

e) Find that value.

f) What did you do to find the values for A and S?

3. a) Solve the first equation for A.

b) Substitute that value for A into the second equation.

c) Solve for S.

d) Did you get the same answer as you found in step 2 part e)?

4. Reflect

a) Do you think that you have found the point of intersection of the linear system that Wesam wrote? Use a graph to check.

b) What is the answer to the geography puzzle?

Example 1 Solve Using the Method of Substitution

The lines $y = -x + 8$ and $x - y = 4$ intersect at right angles. Find the coordinates of the point of intersection.

Solution

Label the equations of the lines ① and ②.

$y = -x + 8$ ①
$x - y = 4$ ②

Step 1: Equation ① is $y = -x + 8$, so you can substitute $-x + 8$ in equation ② for y.

$$x - y = 4$$
$$x - (-x + 8) = 4$$
$$x + x - 8 = 4$$
$$2x - 8 = 4$$
$$2x = 4 + 8$$
$$2x = 12$$
$$x = 6$$

> The phrase "intersect at right angles" is extraneous information. I don't need this fact to find the point of intersection of the two lines.

> Now I have one equation in one variable. I can solve for x.

> I still need to find the y-coordinate.

Step 2: Substitute $x = 6$ in equation ① to find the corresponding value for y.

$y = -x + 8$
$y = -(6) + 8$
$y = 2$

Making Connections

If lines intersect at right angles, they are perpendicular. You can check that these two lines are perpendicular using their slopes. In grade 9, you learned that the product of the slopes of perpendicular lines is –1. The line $y = -x + 8$ has slope –1. The line $x - y = 4$ can be rearranged to give $y = x - 4$; its slope is 1. The product of the two slopes, $(-1) \times 1$, is –1.

Step 3: Check by substituting $x = 6$ and $y = 2$ into both original equations.

In $y = -x + 8$:

L.S. $= y$ **R.S.** $= -x + 8$
$\quad = 2$ $\qquad\quad = -(6) + 8$
$\qquad\qquad\qquad\quad = 2$

$\qquad$ **L.S.** $=$ **R.S.**

In $x - y = 4$:

L.S. $= x - y$ **R.S.** $= 4$
$\quad = 6 - 2$
$\quad = 4$

$\qquad$ **L.S.** $=$ **R.S.**

The solution checks in both equations. This means that the point (6, 2) lies on both lines.

Step 4: Write a conclusion.
The point of intersection is (6, 2).

Example 2 Solve Using the Method of Substitution

Find the solution to the linear system

$x + y = 5$
$3x - y = 7$

Solution

Label the equations of the lines ① and ②.
$x + y = 5$ ①
$3x - y = 7$ ②

Step 1: Rearrange equation ① to obtain an expression for y.
Note: Here you could just as easily solve equation ① for x or equation ② for y.
$x + y = 5$
$\quad y = 5 - x$

Now substitute $5 - x$ into equation ② in place of y.
$3x - (5 - x) = 7$
$\quad 3x - 5 + x = 7$
$\qquad\quad 4x - 5 = 7$
$\qquad\qquad\quad 4x = 7 + 5$
$\qquad\qquad\quad 4x = 12$
$\qquad\qquad\quad\; x = 3$

Step 2: Substitute $x = 3$ into equation ① to find the corresponding value for y.
$x + y = 5$
$3 + y = 5$
$\quad y = 5 - 3$
$\quad y = 2$

Step 3: Check by substituting $x = 3$ and $y = 2$ into both original equations.

In $x + y = 5$:

L.S. $= x + y$ **R.S.** $= 5$
$= 3 + 2$
$= 5$

L.S. $=$ **R.S.**

In $3x - y = 7$:

L.S. $= 3x - y$ **R.S.** $= 7$
$= 3(3) - 2$
$= 9 - 2$
$= 7$

L.S. $=$ **R.S.**

The solution checks in both equations.

Step 4: Write a conclusion.
The solution is $x = 3$, $y = 2$.

Example 3 Solve Using the Method of Substitution

Where do the lines $2x - y = 4$ and $4x + y = 9$ intersect?

Solution

Method 1: Solve Algebraically by Hand

Label the equations of the lines ① and ②.

$2x - y = 4$ ①
$4x + y = 9$ ②

$4x + y = 9$ ②
$y = 9 - 4x$

> I can choose to isolate either of the variables. I'll look to see which will be less work. In equation ①, the x is multiplied by 2 and the y is negative. In equation ②, the x is multiplied by 4 and the y is positive. Isolating the y in equation ② will take fewer steps.

Next, substitute $9 - 4x$ in place of y in equation ①.

$2x - y = 4$
$2x - (9 - 4x) = 4$
$2x - 9 + 4x = 4$
$6x - 9 = 4$
$6x = 4 + 9$
$6x = 13$
$x = \dfrac{13}{6}$

> I am substituting $9 - 4x$ for y.

> Now I have only one variable, so I can solve for x.

Then, substitute back into equation ② to find the value for y.

$4x + y = 9$
$4\left(\dfrac{13}{6}\right) + y = 9$
$\dfrac{26}{3} + y = 9$
$y = \dfrac{27}{3} - \dfrac{26}{3}$
$y = \dfrac{1}{3}$

The lines intersect at $\left(\dfrac{13}{6}, \dfrac{1}{3}\right)$.

Technology Tip

There are several ways to solve linear systems using a CAS. The steps shown here follow the steps used by hand.

### Method 2: Use a Computer Algebra System (CAS)

When a solution involves fractions, a CAS is helpful for checking your work.

Turn on the TI-89 calculator. If the CAS does not start, press (HOME).
- Press (2nd) (F1) to access the **F6** menu.
- Select **2:NewProb** to clear the CAS.
- Press (ENTER).

Solve equation ② for y:
- Type in the equation
  $4x + y = 9$.
- Press (ENTER).
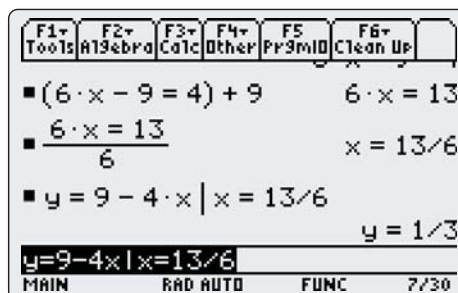- Place brackets around the equation.
- Type (−)4 (X). Press (ENTER).
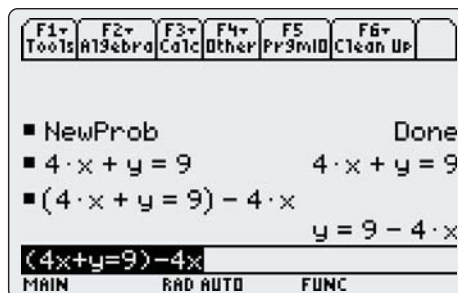
Substitute $9 - 4x$ in place of y in equation ①. Press (ENTER). **Copy** the simplified form, and **Paste** it into the command line. Put brackets around the equation, and add 9. Press (ENTER).
Copy the new form of the equation, and **Paste** it into the command line. Put brackets around the equation, and divide by 6. Press (ENTER).
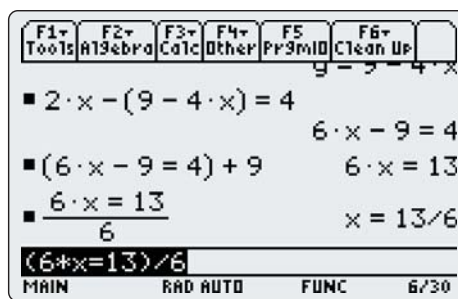
To find the corresponding value for y:
- **Copy** $y = 9 - 4x$ and **Paste** it into the command line (or retype it).
- Type (|) (X) (=) 13 (÷) 6.
- Press (ENTER).

Technology Tip

The (|) key means "such that." It is used to evaluate an expression for a given value.

The lines intersect at $\left(\dfrac{13}{6}, \dfrac{1}{3} \right)$.

Note that the solution to Example 3 involves fractions. This is an example that cannot be solved accurately by graphing, unless a graphing calculator is used.
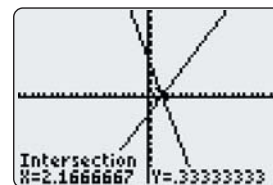
Example 4 A Fish Tale

Stephanie has five more fish in her aquarium than Brett has. The two have a total of 31 fish. How many fish does Stephanie have? How many fish does Brett have?

Solution

Model the information using equations.
Let S represent the number of fish that Stephanie has.
Let B represent the number of fish that Brett has.

From the first sentence,
$S = 5 + B$ ①

From the second sentence,
$S + B = 31$ ②

> I have two linear equations in two unknowns, so this is a linear system.

Substitute $5 + B$ for S in ②.

$$S + B = 31$$
$$5 + B + B = 31$$
$$5 + 2B = 31$$
$$2B = 31 - 5$$
$$2B = 26$$
$$B = 13$$

Substitute 13 for B in ①.
$$S = 5 + B$$
$$S = 5 + 13$$
$$S = 18$$

Look back: Verify that this solution works in the original problem statements. Stephanie has 5 more fish than Brett has. 18 is 5 more than 13. The two have 31 fish altogether. $18 + 13 = 31$.
Make a final statement: Stephanie has 18 fish and Brett has 13 fish.

Key Concepts

- To solve a linear system by substitution, follow these steps:
  Step 1: Solve one of the equations for one variable in terms of the other variable.
  Step 2: Substitute the expression from step 1 into the other equation and solve for the remaining variable.
  Step 3: Substitute back into one of the original equations to find the value of the other variable.
  Step 4: Check your solution by substituting into both original equations, or into the statements of a word problem.

- When given a question in words, begin by defining how variables are assigned. Remember to answer in words.

Communicate Your Understanding

C1 Describe the steps you would take to solve this linear system using the method of substitution.

$y = 3x + 1$ ①
$x + y = 3$ ②

C2 Your friend was absent today. He calls to find out what he missed. Explain to him the idea of solving by substitution.

C3 Compare solving by graphing and solving by substitution. How are the two methods similar? How are they different?

C4 When is it an advantage to be able to solve by substitution? Give an example.

■ Practise

You may wish to check your work using a CAS.

For help with question 1, see Example 1.

1. Solve each linear system using the method of substitution. Check your answers.

a) $y = 3x - 4$
$x + y = 8$

b) $x = -4y + 5$
$x + 2y = 7$

c) $y = -2x + 3$
$4x - 3y = 1$

d) $2x + 3y = -1$
$x = 1 - y$

For help with questions 2 to 5, see Examples 2 and 3.

2. In each pair, decide which equation you will use first to solve for one variable in terms of the other variable. Do that step. Do not solve the linear system.

a) $x + 2y = 5$
$3x + 2y = 6$

b) $2x + y = 6$
$3x + 2y = 10$

c) $2x + 5y = 7$
$x - 3y = -2$

d) $3x - y = 5$
$7x + 2y = 9$

e) $2x - y = 2$
$4x + y = 16$

3. Is $(3, -5)$ the solution for the following linear system? Explain how you can tell.

$2x + 5y = -19$
$6y - 8x = 54$

4. Solve by substitution. Check your solution.

a) $x + 2y = 3$
$5x + 4y = 8$

b) $6x + 5y = 7$
$x - y = 3$

c) $2m + n = 2$
$3m - 2n = 3$

d) $3a + 2b = 4$
$2a + b = 6$

e) $2x + y = 4$
$4x - y = 2$

5. Find the point of intersection of each pair of lines.

a) $2x = y + 5$
$3x + y = -9$

b) $4x + 2y = 7$
$-x - y = 6$

c) $p + 4q = 3$
$5p = -2q + 3$

d) $a + b + 6 = 0$
$2a - b - 3 = 0$

e) $x - 2y - 2 = 0$
$3x + 4y - 16 = 0$

Connect and Apply

For help with questions 6 to 11, see Example 4.

6. Samantha works twice as many hours per week as Adriana. Together they work a total of 39 h one week.

   a) State how you will assign variables.

   b) Write an equation to represent the information in the first sentence.

   c) Write an equation to represent the information in the second sentence.

   d) Use the method of substitution to find the number of hours worked by each person.

7. Jeff and Stephen go to the mall. The two boys buy a total of 15 T-shirts. Stephen gets three less than twice as many T-shirts as Jeff.

   a) Write an equation to represent the information in the second sentence.

   b) Write an equation to represent the information in the third sentence.

   c) Solve the linear system by substitution to find the number of T-shirts each boy bought.

   d) If the T-shirts cost $8.99 each, how much did each boy spend before taxes?

8. Ugo plays hockey and is awarded 2 points for each goal and 1 point for each assist. Last season he had a total of 86 points. He scored 17 fewer goals than assists.

   a) Write a linear system to represent the information.

   b) Solve the system using the method of substitution.

   c) What does the solution represent in the context of this question?

9. Joanne's family decides to rent a hall for her retirement party. Pin Hall charges $500 for the hall and $15 per meal. Bloom Place charges $350 for the hall and $18 per meal.

   a) Write two equations to represent the information.

   b) Solve the linear system to find the number of guests for which the charges are the same at both halls.

10. Charlene makes two types of quilts. For the first type, she charges $25 for material and $50/h for hand quilting. For the second type, she charges $100 for material and $20/h for machine quilting. For what number of hours are the costs the same?

11. Pietro needs to rent a truck for 1 day. He calls two rental companies to compare costs. Joe's Garage charges $80 for the day plus $0.22/km. Ace Trucks charges $100/day and $0.12/km. Under what circumstances do the two companies charge the same amount? When would it be better for Pietro to rent from Joe's Garage?

12. Explain why the following linear system is not easy to solve by substitution.

$$3x + 4y = 10$$
$$2x - 5y = 9$$

13. Explain why it would be appropriate to solve the following linear system either by substitution or by graphing.

$$x + y = 4$$
$$y = 2x + 4$$

14. The following three lines intersect to form a triangle.

$$y = x + 1$$
$$2x + y = 4$$
$$x + y = 5$$

   a) Find the coordinates of each vertex.

   b) Is this a right triangle? Explain how you know.

15. Sensei's Judo Club has a competition for the students. If you win a grappling match, you are awarded 5 points. If you tie, you are awarded 2 points. Jeremy grappled 15 times and his score was 48 points. How many grapples did Jeremy win?

📔 **Did You Know?** _____

Grappling is the term used for wrestling in both judo and ju jitso. In judo you throw your opponent and grapple him or her on the ground.

16. Chapter Problem The Clarke family considers the option of renting a car for 1 day, rather than the full week. One agent recommends a full-size car for a flat fee of $90/day with unlimited kilometres. Another agent recommends a mid-size car that costs $40/day plus 25¢/km driven.

a) Write an equation to represent the cost for the full-size car.

b) Write an equation to represent the cost for the mid-size car.

c) Solve to find when the costs of the two car are the same.

d) In what circumstances will the mid-size car cost less?

e) If the Clarkes want to drive to visit relatives in Parksville, about 120 km away, which option will cost less? Explain. Remember that they plan to return the car the same day.
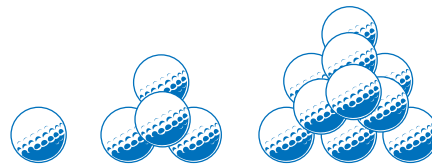
Achievement Check

17. a) Solve this linear system using the method of substitution.

$$2y - x = -10$$
$$y = -\frac{3}{2}x - 1$$

b) Verify your solution graphically.

c) A blue spruce tree grows an average of 15 cm per year. An eastern hemlock grows an average of 10 cm per year. When they were planted, a blue spruce was 120 cm tall and an eastern hemlock was 180 cm tall. How many years after planting will the trees reach the same height? How tall will they be?

Extend

18. The Tragically Hip held a concert to help raise funds for local charities in their hometown of Kingston. A total of 15 000 people attended. The tickets were $8.50 per student and $12.50 per adult. The concert took in a total of $162 500. How many adults came to the concert?

19. a) What happens when you try to solve the following system by substitution?

$$4x - 2y = 9$$
$$y = 2x + 1$$

b) Solve by graphing and explain how this is related to the solution when solving by substitution.

20. Simplify each equation, and then solve the linear system by substitution.

a) $2(x - 4) + y = 6$
$3x - 2(y - 3) = 13$

b) $2(x - 1) - 4(2y + 1) = -1$
$x + 3(3y + 2) - 2 = 0$

21. The following three lines all intersect at one point. Find the coordinates of the point of intersection and the value of k.

$$2x + 3y = 7$$
$$x + 4y = 16$$
$$4x - ky = 9$$

22. Math Contest Toni and her friends are building triangular pyramids with golf balls. Write a formula for the number of golf balls in a pyramid with n layers.

23. Math Contest In a magic square, the sum of the numbers in any row, column, or diagonal is the same. What is the sum of any row of this magic square?

A −6   **B** 0   **C** 10   **D** 15   **E** 18

x	−2	−3x
0		4