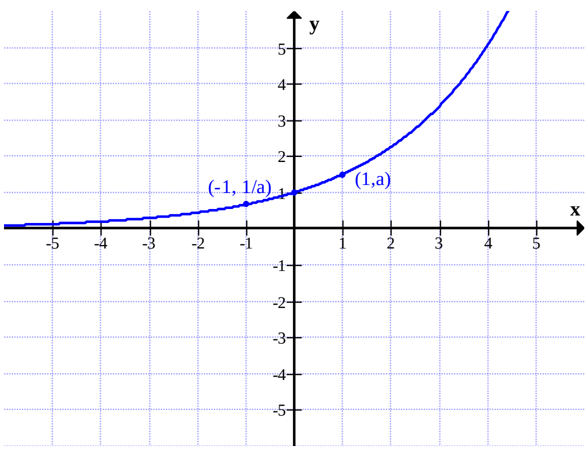
4. Exponential and logarithmic functions

4.1 Exponential Functions

A function of the form $f(x) = a^x$, $a > 0$, $a \neq 1$ is called an **exponential** function. Its domain is the set of all real numbers. For an exponential function f we have $\frac{f(x+1)}{f(x)} = a$. The graph of an exponential function depends on the value of a.

a> 1



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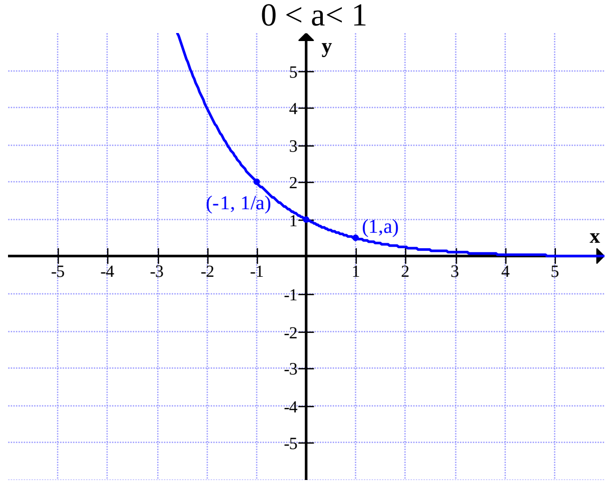
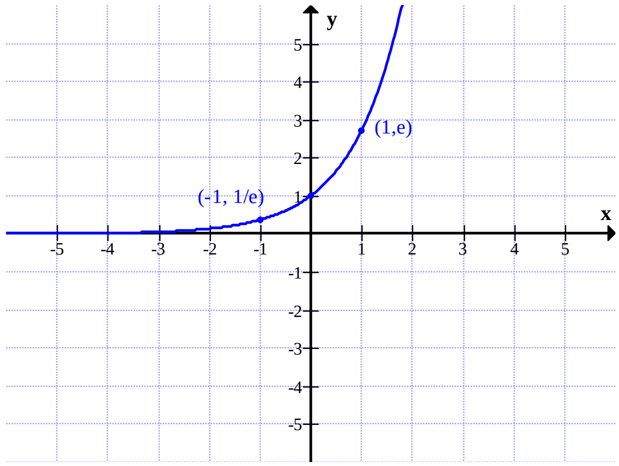


Points on the graph: (-1, 1/a), (0,1), (1, a)

Properties of exponential functions

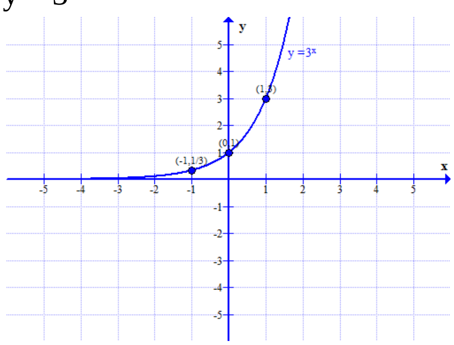
1. The domain is the set of all real numbers: Df = R
2. The range is the set of positive numbers: Rf = $(0, +\infty)$.
   (This means that **a^x is always positive**, that is $a^x > 0$ for all x. The equation a^x = negative number has no solution)
3. There are no x-intercepts
4. The y-intercept is (0, 1)
5. The x-axis (line y = 0) is a **horizontal asymptote**
6. An exponential function is **increasing** when $a > 1$ and **decreasing** when $0 < a < 1$
7. An exponential function is one to one, and therefore has the inverse. The inverse of the exponential function $f(x) = a^x$ is a logarithmic function $g(x) = \log_a(x)$
8. Since an exponential function is **one to one** we have the following property:
   If $a^u = a^v$, then $u = v$.
   (This property is used when solving exponential equations that could be rewritten in the form $a^u = a^v$.)

Natural exponential function is the function $f(x) = e^x$, where e is an irrational number, $e \approx 2.718281\ldots$.
The number e is defined as the number to which the expression $(1+\frac{1}{n})^n$ approaches as n becomes larger and larger. Since e > 1, the graph of the natural exponential function is as below

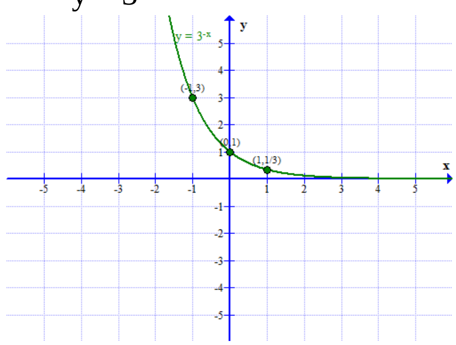


Example: Use transformations to graph $f(x) = 3^{-x} - 2$. Start with a basic function and use one transformation at a time. Show **all** intermediate graphs.
This function is obtained from the graph of $y = 3^x$ by first reflecting it about y-axis (obtaining $y = 3^{-x}$) and then shifting the graph down by 2 units. Make sure to plot the three points on the graph of the basic function!
Remark: Function $y = 3^x$ has a horizontal asymptote, so remember to shift it too when performing shift up/down

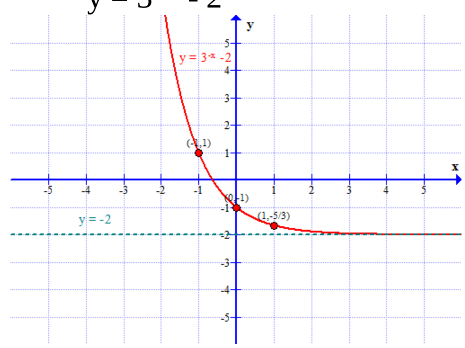
$y = 3^x$



$y = 3^{-x}$

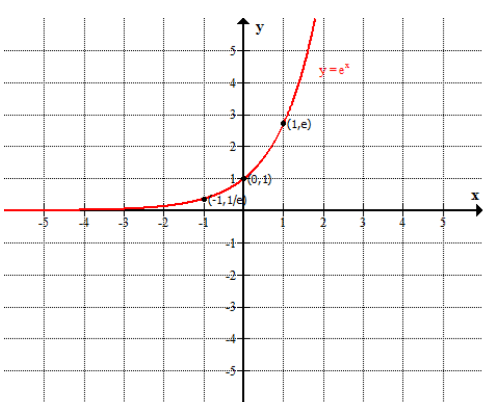


$y = 3^{-x} - 2$

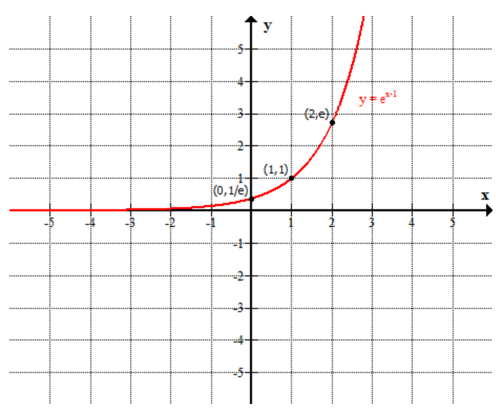


Example: Use transformations to graph $f(x) = 3e^{2x-1}$. Start with a basic function and use one transformation at a time. Show **all** intermediate graphs.

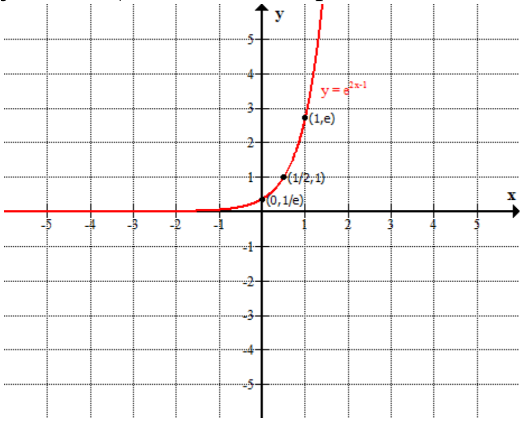
Basic function: $y = e^x$



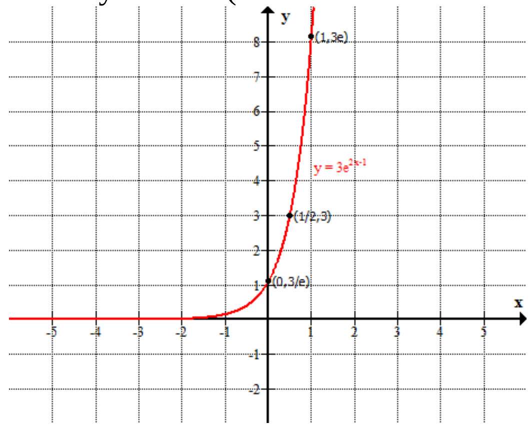
$y = e^{x-1}$ (shift to the right by1)



$y = e^{2x-1}$ (horizontal compression 2 times)



$y = 3e^{2x-1}$ (vertical stretch 3 times)



Example: Solve $4^{x^2} = 2^x$

(i) Rewrite the equation in the form $a^u = a^v$

Since $4 = 2^2$, we can rewrite the equation as

$$\left(2^2\right)^{x^2} = 2^x$$

Using properties of exponents we get $2^{2x^2} = 2^x$.

(ii) Use property 8 of exponential functions to conclude that u = v

Since $2^{2x^2} = 2^x$ we have $2x^2 = x$.

(iii) Solve the equation u = v

$$2x^2 = x$$

$$2x^2 - x = 0$$

$$x(2x-1) = 0$$

$$x = 0 \qquad 2x - 1 = 0$$

$$x = 1/2$$

Solution set = {0, ½ }

4.2 Logarithmic functions

A **logarithmic** function $f(x) = \log_a(x)$, $a > 0$, $a \neq 1$, $x > 0$ (logarithm to the base a of x) is the inverse of the exponential function $y = a^x$.

Therefore, we have the following properties for this function (as the inverse function)

(I) $y = \log_a(x)$ if and only if $a^y = x$

This relationship gives the definition of $\log_a(x)$: **$\log_a(x)$ is an exponent to which the base a must be raised to obtain x**

Example:

a) $\log_2(8)$ is an exponent to which 2 must be raised to obtain 8 (we can write this as $2^x = 8$) Clearly this exponent is 3, thus $\log_2(8) = 3$

b) $\log_{1/3}(9)$ is an exponent to which 1/3 must be raised to obtain 9: $(1/3)^x = 9$. Solving this equation for x, we get $3^{-x} = 3^2$, and $-x = 2$ or $x = -2$. Thus $\log_{1/3}(9) = -2$.

c) $\log_2(3)$ is an exponent to which 2 must be raised to obtain 3: $2^x = 3$. We know that such a number x exists, since 3 is in the range of the exponential function $y = 2^x$ (there is a point with y-coordinate 3 on the graph of this function) but we are not able to find it using traditional methods. If we want to refer to this number, we use $\log_2(3)$.

The relationship in (I) allows us to move from exponent to logarithm and vice versa

Example:

- Change the given logarithmic expression into exponential form: $\log_2 x = 4$
  The exponential form is: $2^4 = x$.
  Notice that this process allowed us to find value of x, or to solve the equation $\log_2(x) = 4$
- Change the given exponential form to the logarithmic one: $2^x = 3$. Since x is the exponent to which 2 is raised to get 3, we have $x = \log_2(3)$.
  Note that the base of the exponent is always the same as the base of the logarithm.

Common logarithm is the logarithm with the base 10. Customarily, the base 10 is omitted when writing this logarithm:

$$\log_{10}(x) = \log(x)$$

Natural logarithm is the logarithm with the base e (the inverse of $y = e^x$): $\ln(x) = \log_e(x)$

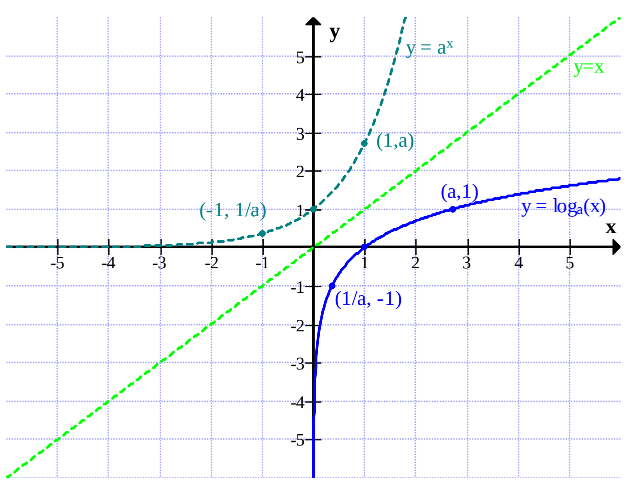
(II) Domain of a logarithmic function $= (0, \infty)$
(We can take a logarithm of a positive number only.)
Range of a logarithmic function $= (-\infty, +\infty)$

(III) $\log_a(a^x) = x$, for all real numbers
$a^{\log_a(x)} = x$, for all $x > 0$

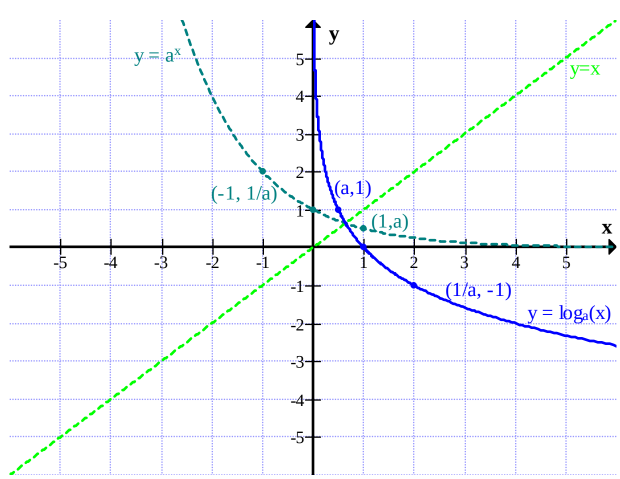
Example $\log_2 2^5 = 5$, $\ln e^3 = 3$, $3^{\log_3(2)} = 2$, $e^{\ln 7} = 7$

(IV) Graph of $f(x) = \log_a(x)$ is symmetric to the graph of $y = a^x$ about the line $y = x$

a > 1



0 < a < 1



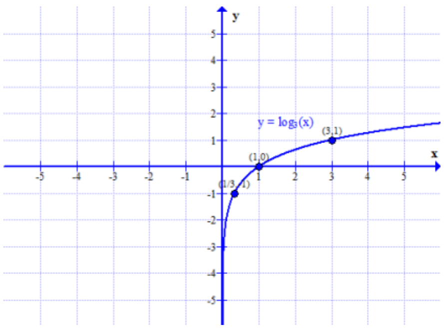
Points on the graph of $y = \log_a(x)$: $(1/a, -1)$, $(1,0)$, $(a, 1)$

(V) The x-intercept is (1, 0).
(VI) There is no y-intercept
(VII) The y-axis (the line x = 0) is the vertical asymptote
(VIII) A logarithmic function is increasing when $a > 1$ and decreasing when $0 < a < 1$
(IX) A logarithmic function is one to one. Its inverse is the exponential function

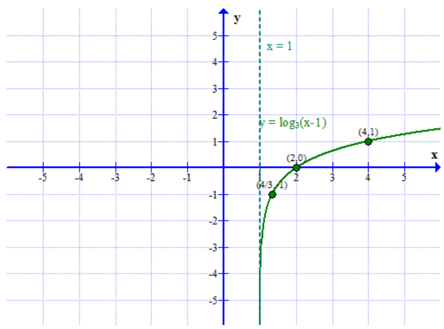
(X) Because a logarithmic function is one to one we have the following property:
If $\log_a(u) = \log_a(v)$, then $u = v$
(This property is used to solve logarithmic equations that can be rewritten in the form $\log_a(u) = \log_a(v)$.)

Example: Use transformations to graph $f(x) = -2\log_3(x-1) + 3$. Start with a basic function and use one transformation at a time. Show **all** intermediate graphs. Plot the three points on the graph of the basic function

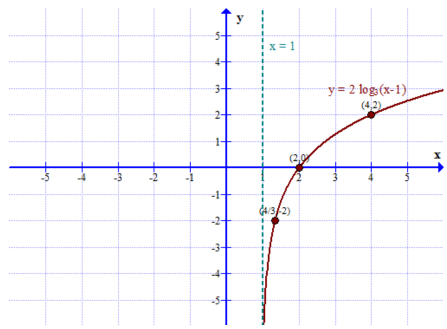
a) $y = \log_3(x)$



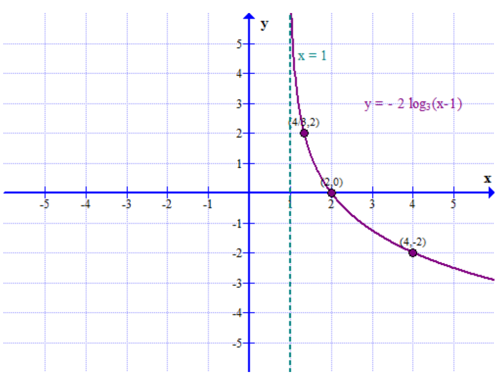
b) $y = \log_3(x-1)$



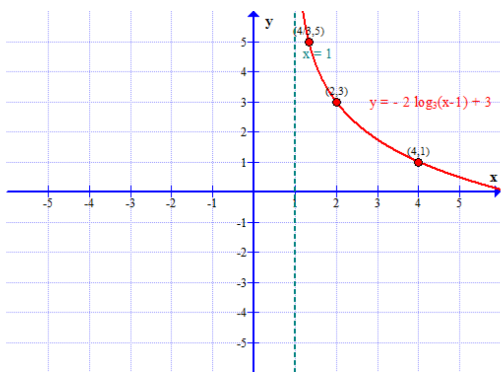
c) $y = 2\log_3(x-1)$



d) $y =-2\log_3(x-1)$



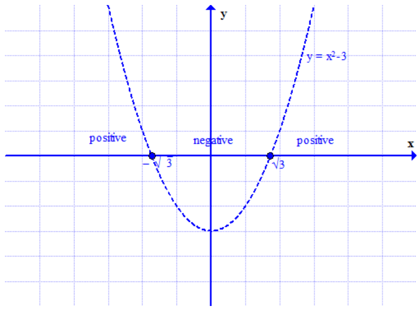
e) $y = -2\log_3(x-1) + 3$



Remark: Since a logarithmic function has a vertical asymptote, do not forget to shift it when shifting left/right

Example: Find the domain of the following functions (A logarithm is defined only for positive (> 0) values)

a) $f(x) = \log_{1/2}(x^2 - 3)$
Df: $x^2 - 3 > 0$
$x^2 - 3 = 0$
$x^2 = 3$
$x = \pm\sqrt{3}$



Df = $(-\infty,-\sqrt{3})\cup(\sqrt{3},+\infty)$

b) g(x)= $\ln\left(\dfrac{2x+3}{x^2-9}\right)$

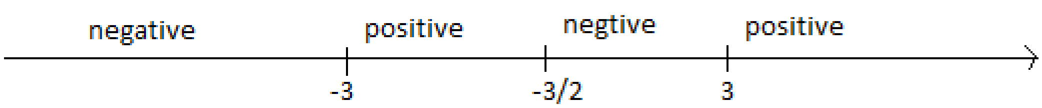
Dg: $\dfrac{2x+3}{x^2-9} > 0$

$2x+3 = 0$ $\qquad$ $x^2 - 9 = 0$

$2x = -3$ $\qquad$ $x^2 = 9$

$x = -3/2$ $\qquad$ $x = \pm 3$

use the test points to determine the sign in each interval



Dg= $(-3,-3/2)\cup(3,+\infty)$

Example: Solve the following equations

a) $\log_5(x^2 + x + 4) = 2$

(i) Find the domain of the logarithm(s)

$x^2 + x + 4 > 0$

$x^2 + x + 4 = 0$

$x = \dfrac{-1\pm\sqrt{1-4(1)(4)}}{2} = \dfrac{-1\pm\sqrt{-15}}{2}$ not a real number

Since $y = x^2 + x + 4$ has no x-intercepts and the graph is a parabola that opens up, the graph must always stay above x-axis. Therefore, $x^2 + x + 4 > 0$ for all x

(ii) Change the equation to the exponential form and solve

$x^2 + x + 4 = 5^2$

$x^2 + x + 4 = 25$

$x^2 + x - 21 = 0$

$x = \dfrac{-1\pm\sqrt{1-4(1)(-21)}}{2} = \dfrac{-1\pm\sqrt{85}}{2}$

since there are no restrictions on x, above numbers are solutions of the equation.

b) $e^{-2x+1} = 13$

This is an exponential equation that can be solved by changing it to the logarithmic form

$-2x + 1 = \log_e(13)$

$-2x+1 = \ln(13)$

$-2x = -1 + \ln 13$

$x = \frac{-1+\ln 13}{-2} = \frac{1-\ln 13}{2}$

Since this is an exponential equations, there are no restrictions on x. Solution is $x = \frac{1-\ln 13}{2}$

4.3 Properties of logarithms

Properties of logarithms:

Suppose $a > 0$, $a \neq 1$ and M, N > 0

(i)	$\log_a(1) = 0$ $\log_a(a) = 1$	*Example:* $\log_2(1) = 0$ $\log_{15}(15) = 1$ $\ln(1) = 0$ $\ln(e) = 1$
(ii)	$a^{\log_a(M)} = M$	*Example:* $6^{\log_6(7)} = 7$ $e^{\ln(4)} = 4$
(iii)	$\log_a(a^r) = r$	*Example:* $\log_3(3^4) = 4$ $\ln(e^{2x}) = 2x$
(iv)	$\log_a(M \cdot N) = \log_a(M) + \log_a(N)$ $\log_a(M) + \log_a(N) = \log_a(M \cdot N)$	*Example :* $\log_5(10) = \log_5(5) + \log_5(2)$ $\ln(x+1) + \ln(x-1) = \ln[(x+1)(x-1)]$
(v)	$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$ $\log_a(M) - \log_a(N) = \log_a\left(\frac{M}{N}\right)$	*Example:* $\log_4\left(\frac{15}{2}\right) = \log_4(15) - \log_4(2)$ $\log_4(12) - \log_4(3) = \log_4\left(\frac{12}{3}\right)$
(vi)	$\log_a(M^r) = r \cdot \log_a(M)$ $r \cdot \log_a(M) = \log_a(M^r)$	*Example:* $\log(3^x) = x\log(3)$ $5\log_3(x+1) = \log_3[(x+1)^5]$
(vii)	If $M = N$, then $\log_a(M) = \log_a(N)$ If $\log_a(M) = \log_a(N)$, then $M = N$	*Example:* if $x = 4$, then $\log_a(x) = \log_a(4)$ if $\log_4(x-1) = \log_4(2x-5)$, then $x-1 = 2x-5$

(viii) Change of the base formula

$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$, where b is any positive number different than 1

In particular,

$\log_a(M) = \frac{\log(M)}{\log(a)}$ and $\log_a(M) = \frac{\ln(M)}{\ln(a)}$

This formula is used to find values of logarithms using a calculator.

Example: Evaluate $\log_2(3)$

$$\log_2(3) = \frac{\ln(3)}{\ln(2)} \approx 1.5849$$

Example : Write $\log_3\left(\frac{x(x+2)^3}{\sqrt{x^2+1}}\right)$ as a sum/difference of logarithms. Express powers as product.

$$\log_3\left(\frac{x(x+2)^3}{\sqrt{x^2+1}}\right) = \log_3[x(x+2)^3] - \log_3\left(\sqrt{x^2+1}\right) =$$

$$\log_3(x) + \log_3[(x+2)^3] - \log_3\left(x^2+1\right)^{1/2} =$$

$$\log_3(x) + 3\log_3(x+2) - \tfrac{1}{2}\log_3(x^2+1)$$

Example: Write as a single logarithm

$3\log_4(3x+1) - 2\log_4(2x-1) - \log_4(x) =$

$= \log_4[(3x+1)^3] - \log_4[(2x-1)^2] - \log_4(x) =$

$$= \log_4\left(\frac{(3x+1)^3}{(2x-1)^2}\right) - \log_4(x) = \log_4\left[\frac{\frac{(3x+1)^3}{(2x-1)^2}}{x}\right] = \log_4\left[\frac{(3x+1)^3}{x(2x-1)^2}\right]$$

4.4 Exponential and logarithmic equations

A **logarithmic equation** is an equation that contains a variable " inside " a logarithm.
Since a logarithm is defined only for positive numbers, before solving a logarithmic equation you must find its domain (alternatively, you can check the apparent solutions by plugging them into the original equation and checking whether all logarithms are well defined).

There are two types of logarithmic equations:

(A) **Equations reducible to the form $\log_a(u) = r$,** where u is an expression that contains a variable and r is a real number

To solve such equation change it to the exponential form $a^r = u$ and solve.

Example: Solve $3\log_2(x-1) + \log_2(3) = 5$

(i) Determine the domain of the equation. (What is "inside" of any logarithm must be positive)
$x-1 > 0$
$x > 1$
(Only numbers greater than 1 can be solutions of this equation)

(ii) Use properties of logarithms to write the left hand side as a single logarithm
$\log_2(x-1)^3 + \log_2(3) = 5$
$\log_2(3(x-1)^3) = 5$

(iii) Change to the exponential form
$2^5 = 3(x-1)^3$

(iv) Solve
$32 = 3\,(x-1)^3$
$32/3 = (x-1)^3$
$x-1 = \sqrt[3]{32/3}$
$x = 1 + \sqrt[3]{32/3}$

(v) Since $x = 1 + \sqrt[3]{32/3}$ is greater than 1, it is the solution

(B) Equations reducible to the form $\log_a(u) = \log_a(v)$.

To solve such equation use the (vii) property of logarithms to get the equation u = v. Solve the equation.
Example: Solve $\log_5(x) + \log_5(x-2) = \log_5(x+4)$.

(i) Determine the domain of the equation. (What is "inside" of any logarithm must be positive)
$x > 0$ and $x - 2 > 0$ and $x + 4 > 0$
$x > 0$ and $x > 2$ and $x > -4$
If x is to satisfy all these inequalities, then $x > 2$
(Only numbers greater than 2 can be solutions of this equation)

(ii) Use properties of logarithms to write each side of the equation as a single logarithm

$\log_5(x(x-2)) = \log_5(x + 4)$

(iii) Since the logarithms are equal ($\log_a(M) = \log_a(N)$), we must have (M = N)
$x(x-2) = x + 4$
(iv) Solve
$x(x-2) = x + 4$
$x^2 - 2x = x + 4$
$x^2 - 3x - 4 = 0$
$(x-4)(x+1) = 0$
$x = 4$ or $x = -1$
(v) Since any solution must be greater than 2, only $x = 4$ is the solution

Exponential equations

These are equations in which a variable appears in the exponent. Since exponential functions are defined for all real numbers, there are no restrictions on a variable and we do not have to check the solutions.

There are three types of exponential equations:

(A) **Equations that can be reduced to the form $a^u = r$**, where u is an expression that contains a variable and r is a positive real number. If r is negative or 0, the equation has no solution.

To solve such equation, change into logarithmic form and solve

Example: Solve $3 \cdot 4^{2x-1} = 5$

(i) Write the equation in the desired form (exponent = a number)
$4^{2x-1} = 5/3$

(ii) Change to the logarithmic form
$2x-1 = \log_4(5/3)$

(iii) Solve
$2x = 1 + \log_4(5/3)$
$x = \frac{1+\log_4(5/3)}{2}$
To find an approximate value, use the change of the base formula to rewrite $\log_4(5/3)$ as $\log(5/3)/\log 4$

(B) Equations that can be reduced to the form $a^u = a^v$.

To solve such an equation use the property of exponential functions that says that if $a^u = a^v$, then $u = v$ and solve it.

Example Solve $(16)^x \cdot 2^{x^2} = 4^6$

(i) Use the properties of exponents to write the equation in the desired form. Notice that all bases (16, 2, 4) are powers of 2, $16 = 2^4$, $2 = 2^1$, $4 = 2^2$.

$$(16)^x \cdot 2^{x^2} = 4^6$$
$$(2^4)^x \cdot 2^{x^2} = (2^2)^6$$
$$2^{4x} \cdot 2^{x^2} = 2^{12}$$
$$2^{4x+x^2} = 2^{12}$$

(ii) Use the property (7)
$4x + x^2 = 12$

(iii) Solve
$x^2 + 4x - 12 = 0$
$(x+6)(x-2) = 0$
$x = -6$ or $x = 2$

Solutions: -6, 2

(C) Equations that can be reduced to the form $a^u = b^v$

To solve such equation apply the log (or ln) to both sides of the equation (property (vii) of logarithms), use the property of logarithms to bring the u and v outside of the logarithms and solve for the variable. Keep in mind that log(a) and log(b) are just numbers (like 1.34 or 3)

Example: Solve $2^{x+1} = 5^{1-2x}$

(i) Apply log to both sides
$\log(2^{x+1}) = \log(5^{1-2x})$

(ii) Use properties of logarithms. (Enclose the powers into the parentheses)
$(x+1)\log(2) = (1-2x)\log(5)$

(iii) Solve

Eliminate parentheses $x\log(2)+\log(2) = \log(5) - 2x\log(5)$

Bring the terms with x to the left hand side $x\log(2) + 2x\log(5) = \log(5) - \log(2)$

Factor out x $x(\log(2)+2\log(5)) = \log(5) - \log(2)$

Divide, to find x $x = \dfrac{\log(5)-\log(2)}{\log(2)+2\log(5)}$

You could use properties of logarithms to write the solution as $x = \dfrac{\log(5/2)}{\log(2\cdot 5^2)} = \dfrac{\log(5/2)}{\log(50)}$

If an exponential equation cannot be transformed to one of the types above, try to substitute by *u* an exponential expression within the equation. This might reduce the equation to an algebraic one, like quadratic or rational.

Example: Solve $2^{2x} + 2^{x+2} - 12 = 0$

(i) Rewrite the equation so that 2^x appears explicitly
$(2^x)^2 + 2^x \cdot 2^2 - 12 = 0$
$(2^x)^2 + 4\cdot(2^x) - 12 = 0$

(ii) Substitute $u = 2^x$
$u^2 + 4u - 12 = 0$

(iii) Solve the equation for u
$(u+6)(u-2) = 0$
$u = -6$ or $u = 2$

(iv) Back- substitute and solve for x
$2^x = -6$ or $2^x = 2$
No solution $x = 1$

Solution: $x = 1$