

# Mortgage Math

→ What is PV of \$1000 per month for 15 months plus \$10,000 paid 15 months from now at 10% nominal annual interest?

$$\begin{aligned} \$22,875 &= \frac{\$1,000}{1 + .10/12} + \dots + \frac{\$1,000}{(1 + .10/12)^{15}} + \frac{\$10,000}{(1 + .10/12)^{15}} \\ &= (14.045)1000 + (0.8830)10000 \\ &= \$14,045 + \$8,830 \\ &= (PVIFA_{.00833,15}) * PMT + (PVIF_{.00833,15}) * FV \end{aligned}$$

(With calculator set to pmts at “END” of periods,  
and P/YR=12...)

**Mortgage Math Keys:**

15----> N key  
10----> I/YR key  
1000 ----> PMT key  
10000----> FV key  
PV ----> -22,875

**DCF Keys:**

10----> I/YR key  
0 ----> CFj key  
1000----> CFj key  
14 ----> Nj key  
11000---->CFj key  
NPV ----> 22,875

# How the Calculator "Mortgage Math" Keys Work...

The five "mortgage math" keys on your calculator (N,I,PV,PMT,FV) solve:

$$0 = -PV + \frac{PMT}{1+r} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} + \frac{FV}{(1+r)^N}$$

$$\text{or: } 0 = -\text{PV} + (\text{PVIFA}_{r,N}) * \text{PMT} + (\text{PVIF}_{r,N}) * \text{FV}$$

where:  $r = i / m$ ,

where:  $i$  = Nominal annual interest rate  
 $m$  = Number of payment periods per year ( $m \rightarrow P/YR$ ).

## Example:

10%, 20-yr fully-amortizing mortgage with payments of \$1000/month.

The calculator solves the following equation for PV:

$$0 = -PV + \frac{1000}{1.00833} + \frac{1000}{(1.00833)^2} + \dots + \frac{1000}{(1.00833)^{240}} + \frac{0}{(1.00833)^{240}}$$

The result is:  $PV = 103625$ .

# THE BASIC RULES OF CALCULATING LOAN PAYMENTS & BALANCES

Let:

$P$  = Initial Contract Principal (Loan Balance at time zero, when money is borrowed)

$r_t$  = Contract Interest rate (per payment period, e.g.,  $=i/m$ ) applicable for payment in Period " $t$ "

$IE_t$  = Interest portion of payment in Period " $t$ "

$PP_t$  = Principal paid down ("amortized") in the Period " $t$ " payment

$OLB_t$  = Outstanding loan balance after the Period " $t$ " payment has been made

$PMT_t$  = Amount of the loan payment in Period " $t$ "

# THE FOUR BASIC RULES:

1)  $IE_t = r_t(OLB_{t-1})$

2)  $PP_t = PMT_t - IE_t$

3)  $OLB_t = OLB_{t-1} - PP_t$

**Equivalent to PV of remaining loan payments**

4)  $OLB_0 = P$

*Know how to use these rules so that you can calculate payment schedule, interest, principal, and outstanding balance after each payment, for any type of loan that can be dreamed up!*

# APPLICATION OF THE FOUR RULES TO SPECIFIC LOAN TYPES

## 1) Fixed-Rate loans (FRMs):

The contract interest rate is constant throughout the life of the loan:

$$r_t = r, \text{ all } t.$$

## 2) Constant-Payment loans (CPMs):

The payment is constant throughout the life of the loan:

$$PMT_t = PMT, \text{ all } t.$$



### 3) Constant-Amortization loans (CAMs):

The principal amortization is constant throughout the life of the loan:

$$PP_t = PP, \text{ all } t.$$

### 4) Fully-Amortizing loans:

Initial contract principal is fully paid off by maturity of loan:

$$\sum PP_t = P \text{ over all } t=1, \dots, N.$$

### 5) Partially-Amortizing loans:

Loan principal not fully paid down by due date of loan:

$$\sum PP_t < P, \text{ so } OLB_N \text{ must be paid as "balloon" at maturity.}$$

## 6) Interest-Only loans:

The principal is not paid down until the end:

$$PMT_t = IE_t, \text{ all } t$$

(equivalently:  $OLB_t = P$ , all  $t$ , and in calculator equation:  $FV = -PV$ ).

## 7) Graduated Payment loans (GPMs):

The initial payment is low, usually initial  $PMT_1 < IE_1$ , so  $OLB$  at first grows over time (“negative amortization”), followed by higher payments scheduled later in the life of the loan.

## 8) Adjustable-Rate loans (ARMs):

The contract interest rate varies over time ( $r_t$  not constant, not known for certain in advance, loan payment schedules & expected yields must be based on assumptions about future interest rates).

# Classical Fixed-Rate Mortgage

The “classical” mortgage is both FRM &

CPM:

$$\text{PMT} = P / (\text{PVIFA}_{r,N}) = P / [(1 - 1/(1+r)^N) / r]$$

# \$60,000, 12%, 30-year CPM...

MONTH	BEG. BAL.	INTEREST	PMT	PRIN	END BAL.
1	\$60,000.00	\$600.00	\$617.17	\$17.17	\$59,982.83
2	\$59,982.83	\$599.83	\$617.17	\$17.34	\$59,965.49
3	\$59,965.49	\$599.65	\$617.17	\$17.51	\$59,947.98

# Using Your Calculator

## 1) Calculate Loan Payments:

**Example:** \$100,000 30-year 10% mortgage  
with monthly payments:

360----> N

10----> I/YR

100000 ----> PV

0 ----> FV

PMT----> - 877.57

## 2) Calculate Loan Amount (Affordability):

**Example:** You can afford \$500/month payments on 30-year, 10% mortgage:

360----> N

10----> I/YR

500----> PMT

0----> FV

PV----> - 56,975.41 = Amt you can borrow.

- If you can borrow 80% of house value, how much can you afford to purchase?
  - Purchase Price = \$56,975 / 0.80
  - Purchase Price = \$71,218



### 3) Calculate Outstanding Loan Balance:

**Example:** What is the remaining balance on \$100,000, 10%, 30-year, monthly-payment loan after 5 years (after 60 payments have been made)?

First get loan terms in the registers:

```
360----> N
10----> I/YR
100000----> PV
0----> FV
PMT----> - 877.57
```

Then calculate remaining balance either way below:

```
N ----> 60          N----> 300
FV ----> - 96,574.32  PV----> 96,574.32
```

#### **4) Calculate payments & balloon on partially amortizing loan:**

Same as (3) above.

#### **5) Calculate the payments on an interest-only loan:**

**Example:** A \$100,000 interest-only 10% loan with monthly payments:

N can be anything,

10 ----> I/YR,

100000 ----> PV,

-100000----> FV,

PMT ----> -833.33

## **6) Meet affordability constraint by trading off payment amount with amortization rate:**

**Example:** Go back to example #2 on the previous page. The affordability constraint was a \$500/mo payment limit. Suppose the \$56,975 which can be borrowed at 10% with a 30-year amortization schedule falls short of what the borrower needs.

How much slower amortization rate would enable the borrower to obtain \$58,000?

Enter:

I/YR = 10, PV = -58000, PMT = 500, FV = 0,

Compute: N = 410.

Thus, the amortization rate would have to be 410 months, or 34 years.

**Note:** This does not mean loan would have to have a 34-year maturity, it could still be a 30-year **partially-amortizing** loan, with balloon of \$20,325 due after 30 years.

## 7) Determining principal & interest components of payments:

**Example:** For the \$100,000, 30-year, 10% mortgage in problem #1 on the previous page, break out the components of the 12 payments numbering 50 through 61.

In the HP-10B, after entering the loan as in problem #1, enter:

50, INPUT, 61, AMORT, = \$9,696.06 int, = \$834.80 prin,  
=\$96,501 OLB<sub>61</sub>.

To get the corresponding values for the subsequent calendar year, press AMORT again, to get: = \$9,608.65 int, = \$922.21 prin, =\$95,579 OLB<sub>73</sub>.

*(Other business calculators can do this too.)*

# Loan Yields and Mortgage Valuation

Loan Yield = Effective Interest Rate

Yield = IRR of loan

Recall: IRR based on cash flows.

*Using calculator equation:*

$$0 = -PV + \frac{PMT}{1+r} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} + \frac{FV}{(1+r)^N}$$

*Let:*

$$PV = CF_0$$

$$PMT = CF_t, \quad t=1, 2, \dots, N-1$$

$$PMT + FV = CF_N$$

*N = Holding Period*

where:  $CF_j$  represents actual cash flow at end of period "j".



Then, by the definition of "r" in the equation above, we have:

$$0 = NPV = -CF_0 + \frac{CF_1}{1+r} + \frac{CF_2}{(1+r)^2} + \dots + \frac{CF_N}{(1+r)^N}$$

(bearing in mind that:

$$\frac{PMT}{(1+r)^N} + \frac{FV}{(1+r)^N} = \frac{PMT + FV}{(1+r)^N} = \frac{CF_N}{(1+r)^N}$$

Expressed in nominal per annum terms ( $i=mr$ , where  $m=P/YR$ ), we can thus find the yield by computing the I/YR, provided the values in the N, PV, PMT, and FV registers equal the appropriate actual cash flow and holding period values.

*In 2ndary mkt, loans are priced so their yields equal the “mkt’s required yield” (like expected total return,  $E(r)=r_f+RP$ , from before).*

*At the time when a loan is originated (primary market), the loan yield is usually approximately equal to its contract interest rate. (But not exactly...)*

## The tricky part in loan yield calculation:

- (a) The holding period over which we wish to calculate the yield may not equal the maturity of the loan (e.g., if the loan will be paid off early, so  $N$  may not be the original maturity of the loan):  $N \neq \text{maturity}$  ;
- (b) The actual time-zero present cash flow of the loan may not equal the initial contract principal on the loan (e.g., if there are "points" or other closing costs that cause the cash flow disbursed by the lender and/or the cash flow received by the borrower to not equal the contract principal on the loan,  $P$ ):  $CF_0 \neq P$  ;

(c) The actual liquidating payment that pays off the loan at the end of the presumed holding period may not exactly equal the outstanding loan balance at that time (e.g., if there is a "prepayment penalty" for paying off the loan early, then the borrower must pay more than the loan balance, so FV is then different from OLB):  $CF_N \neq PMT + OLB_N$

So we must make sure that the amounts in the N, PV, and FV registers reflect the actual cash flows...

# Example

- \$200,000 mortgage, 30-year maturity, monthly payments
- 10% annual interest
- The loan has “2 points”
  - (‘discount points’ or prepaid interest)
- Also a 3 point prepayment penalty through end of 5<sup>th</sup> year.

- What is yield (“effective interest rate”) assuming holding period of 4 years (i.e., borrower will pay loan off after 48 months)?
- Break this problem into 3 steps:
  - (1) Compute the loan cash flows using the contract values of the parameters  
( $N=360$ ,  $I=10\%$ ,  $PV=200000$ ,  $FV=0$ , Compute  $PMT=\$1755.14$ );
  - (2) Alter the amounts in the registers to reflect the actual cash flows;
  - (3) Compute yield.
- *(You must do these steps in this order.)*

### Step 1)

360-----> N

10-----> I/YR

200000 -----> PV

0 -----> FV

PMT-----> - 1755.14

### Step 2)

48-----> N

FV-----> - 194804 X 1.03 = - 200,649 -----> FV

196000 -----> PV

### Step 3)

I/YR-----> 11.22%



Expected yield (like  $E(r)$  or “going-in IRR”) is 11.22%, even though “contractual interest rate” on the loan is only 10%.

(When closing costs and prepayment penalties are quoted in “points”, you do not need to know the amount of the loan to find its yield.)

***General rule to calculate yield:***

***Change the amount in the PV Register***  
***last,***

***(just prior to computing the yield).***

*Equivalent solution to previous problem:  
Use CF keys instead of mortgage math keys...*

196000 ----> CFj key

- 1755.14 ----> CFj key

47 ----> Nj key

- 202404 ----> CFj key

IRR ----> 11.22%

# *Using Market Yields to Value Mortgages*

*(Note: This is performing a DCF NPV analysis of the loan as an investment, finding what price can be paid for the loan so the deal is  $NPV=0$ . Market's required yield is "r", the opportunity cost of capital for the loan.)*

# Example

- \$100,000 mortgage, 30-year, 10%, 3 points prepayment penalty before 5 years.
- Expected time until borrower prepays loan = 4 years.
- How much is the loan worth today if the market yield is 11.00%?

Step 1)

360--->N,

10--->I/YR,

100000--->PV,

0--->FV,

Compute PMT---> -877.57.

Step 2)

48---->N,

FV----> -97,402 \* 1.03 = -100,324 ---->FV.

Step 3)

I/YR---->11.00%.

Step 4)

PV----> 98,697.

The loan is worth \$98,697.

*(Watch out for order of steps. Cash flows first, then input the market yield, then compute the loan value as the PV.)*

## **Determining required “discount points” (or “origination fee”):**

To avoid lender doing NPV < 0 deal in making loan, we need:

$$(100,000 - 98,697) / 100,000 = 1.30\% = 1.30 \text{ points}$$