Sequences and their limits

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The limit idea

For the purposes of calculus, a *sequence* is simply a list of numbers

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$

that goes on indefinitely. The numbers in the sequence are usually called *terms*, so that x_1 is the first term, x_2 is the second term, and the entry x_n in the general nth position is the nth *term*, naturally. The subscript $n = 1, 2, 3, \ldots$ that marks the position of the terms will sometimes be called the *index*. We shall deal only with *real* sequences, namely those whose terms are real numbers. Here are some examples of sequences.

- the sequence of positive integers: $1, 2, 3, \ldots, n, \ldots$
- the sequence of primes in their natural order: $2, 3, 5, 7, 11, \ldots$
- the decimal sequence that estimates $1/3$: .3, .33, .333, .3333, .33333, $\ldots$
- a binary sequence: $0, 1, 0, 1, 0, 1, \ldots$
- the zero sequence: $0, 0, 0, 0, \ldots$
- a geometric sequence: $1, r, r^2, r^3, \ldots, r^n, \ldots$
- a sequence that alternates in sign: $\frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots, \frac{(-1)^n}{n}, \ldots$
- a constant sequence: $-5, -5, -5, -5, -5, \ldots$
- an increasing sequence: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots, \frac{n}{n+1}, \ldots$
- a decreasing sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$
- a sequence used to estimate e: $(\frac{3}{2})^2, (\frac{4}{3})^3, (\frac{5}{4})^4 \ldots, (\frac{n+1}{n})^n \ldots$

- a seemingly random sequence: $\sin 1, \sin 2, \sin 3, \ldots, \sin n, \ldots$
- the sequence of decimals that approximates π:

  $$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, 3.1415926, 3.14159265, \ldots$$

- a sequence that lists all fractions between 0 and 1, written in their lowest form, in groups of increasing denominator with increasing numerator in each group:

  $$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \ldots$$

It is plain to see that the possibilities for sequences are endless.

Ways to prescribe a sequence

A sequence is prescribed by making clear what its nth term is supposed to be. We can use a long list to indicate a pattern, but shorter notations such as

$$\{x_n\}_{n=1}^{\infty}\text{, or more briefly } \{x_n\}\text{, or even the unadorned } x_n$$

are suitable as well. For some sequences it is possible to give a simple formula for the nth term as a function of the index n. For example, the nth term of the sequence $1, 1/2, 1/3, 1/4, \ldots$ is $x_n = 1/n$. For other sequences, such as the sequence of primes or the sequence for the decimal expansion of π, a clean formula for the nth term is not available. Nevertheless, the entry in the nth position remains uniquely specified.

At times the sequence $\{x_n\}$ is given, not by a direct formula for the nth term, but rather *recursively*. To specify a sequence recursively, you state explicitly what one or more of the beginning terms are, and then you give a formula for the general entry in terms of its preceding terms. Here is an example of a famous sequence that is defined recursively. Let

$$f_0 = 1, f_1 = 1,\ \text{and for indices } n \geq 2,\ \text{let } f_n = f_{n-2} + f_{n-1}.$$

According to this specification, the first few terms of this sequence $\{f_n\}$ go as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

This is the celebrated Fibonacci sequence.

Notice that the index n need not always start at $n = 1$. In the Fibonacci sequence it has been traditional to have the index start at $n = 0$. For the sequence given by $x_n = \sqrt{n-2}$ it is suitable to have the index start at $n = 2$. For our purposes, it is not so important where the index starts. What matters most in discussing limits is the behaviour of the sequence in the long run. That is, for large values of the index.

Exercises

1. In the displayed sequence above, that lists all rational numbers between 0 and 1 in order of increasing denominator, write the next 10 terms after the displayed term of $5/9$.

2. If the sequence $\{x_n\}$ is defined recursively by

$$x_1 = 2, x_2 = -1, x_{n+2} = x_n/x_{n+1} \text{ for } n = 1, 2, 3, \ldots,$$

   write the first 6 terms of the sequence.

3. Write the first 12 terms of the sequence given by $x_n = \sin(n\pi/6)$.

4. Write the first 8 terms of the sequence $x_n = \arctan(\sin(n\pi/2))$.

The limit of a sequence

We could say that a given sequence $\{x_n\}$ has a limiting value of p as n tends to ∞ when the terms x_n eventually get microscopically close to the number p. For instance, the sequence $\{1/n^5\}$ seems to have a limiting value of 0. The sequence $.3, .33, .333, .3333, \ldots$ seems to have a limiting value of $1/3$. Simple as this may seem, an approach to limits based on such hopeful impressions is only the beginning.

To go further we must ask **quantitative** questions. For example, how far do you have to take $1/n^5$ to be sure that it approximates $p = 0$ with 8 decimal places of accuracy? Let's see what the answer could be. We need to know how far to go with n before we hit $1/n^5 < 1/10^8$. In other words, how far should we go before we obtain $10^{8/5} < n$? Since $10^{8/5} \approx 39.8$, it seems pretty clear that we have to wait until $n > 39.8$. Once $n = 40$ and beyond, we can be sure that $1/n^5$ approximates 0 with 8 decimal places of accuracy. If we wanted 16 decimal places of accuracy we would wait until n had gone beyond $10^{16/5} \approx 1584.9$, in other words until n hit 1585. If we wanted still more accuracy, say 80 decimal places we would wait quite a bit more, until in fact n got past $10^{80/5} = 10^{16}$. No matter how much accuracy we specify, the limit can be approximated to satisfy that accuracy if we wait long enough.

This quantitative approach brings us to a central idea in calculus. The idea is that a sequence $\{x_n\}$ has a limit p provided x_n can be brought **as close to p as we like** by simply going far enough out in the sequence. In the tradition of calculus the symbol used to specify an arbitrary amount of closeness is the Greek letter ϵ, called epsilon. You should get used to thinking of the letter ϵ to represent an arbitrary, yet very small positive number.

Here is the formal and very important definition of limit of a sequence.

Definition of limit of a sequence
A sequence $\{x_n\}$ has a limit p provided that for any tolerance $\epsilon > 0$, we can obtain a real number K such that

$$|x_n - p| < \epsilon \text{ whenever the index } n > K.$$

To establish that p is the limit of x_n a kind of *challenge-response* game has to be played. The challenge is an arbitrary, small, positive number ϵ. The response is a number K that specifies how far out one should go in the sequence in order to ensure that $|x_n - p| < \epsilon$. In other words K is a *cut-off* point which guarantees that for indices n beyond that point the sequence x_n estimates p with the desired accuracy ϵ. Typically, the smaller the tolerance ϵ, the farther out you will have to go with a cut-off point in order to achieve $|x_n - p| < \epsilon$. Thus, we expect that the choice of a cut-off K will have to take ϵ into consideration.

When a sequence $\{x_n\}$ has a limit p we often say that the sequence *tends to p as n tends to ∞*. Alternately we can say that the sequence *converges to p*. Sequences that have a limit are thereby known as *convergent sequences*.

Shorthand notations for limits are available. We could write the equation

$$\lim_{n\to\infty} x_n = p,$$

to suggest that the limit as n tends to ∞ of x_n is p. We can also write

$$x_n \to p \text{ as } n \to \infty$$

to suggest that x_n converges to p as n tends to ∞. Even more simply we can write

$$x_n \to p$$

to indicate that x_n approaches or tends to p. All of the above notations and terminologies are interchangeable and commonly used.

Let us work out a few examples in order to get used to this limit idea.

Example 1.

The sequence given by $x_n = \dfrac{n}{n+1}$ seems to have the limit $p = 1$, as n tends to ∞. Let us pursue this observation in a quantitative manner as prescribed by the definition of limit. Thus take an arbitrary tolerance $\epsilon > 0$. Now we must find a cut-off number K such that

$$\left| \frac{n}{n+1} - 1 \right| < \epsilon \ \text{ whenever } n > K.$$

Using a common denominator, this boils down to finding a number K such that

$$\left| \frac{-1}{n+1} \right| < \epsilon \ \text{ whenever } n > K.$$

With a little more algebra it becomes apparent that we need to find a number K such that

$$\frac{1}{\epsilon} < n + 1 \ \text{ whenever } n > K.$$

In other words we need a number K such that

$$\frac{1}{\epsilon} - 1 < n \ \text{ whenever } n > K.$$

Such a cut-off K is now apparent, namely take $K = \dfrac{1}{\epsilon} - 1$. We can be sure that

$$\text{if } \ n > \frac{1}{\epsilon} - 1, \text{ then } \left| \frac{n}{n+1} - 1 \right| < \epsilon.$$

For instance, suppose $\epsilon = .002$ is given. We have decided above that a suitable cut-off point is

$$K = \frac{1}{.002} - 1 = 499.$$

In other words, starting with the 500th term of the sequence, you know that from then on the sequence will be less than the distance .002 away from the limit 1. Try checking with your calculator that the distance between the limit 1 and the terms

$$\frac{500}{501}, \frac{501}{502}, \frac{502}{503}. \frac{503}{504}, \ldots$$

is indeed less than .002.

Example 2.

Take the sequence $x_n = 1/\sqrt{n}$ whose limit appears to be $p = 0$. Say we want

$$|x_n - 0| < \frac{1}{13}.$$

What is a good cut-off K which will ensure that that $|x_n - 0| < 1/13$ will indeed happen once $n > K$? Well, we want $1/\sqrt{n} < 1/13$. By squaring we see that this will happen when $1/n < 1/169$, which happens when $n > 169$. A suitable cut-off we are looking for is $K = 169$.

Next say we wanted

$$|x_n - 0| < \frac{1}{100}.$$

By the same argument as above we can see that once $n > 100^2 = 10000$, then we will have $|x_n - 0| < 1/100$. Again a suitable cut-off is available.

More generally if we had any $\epsilon > 0$ and we wanted $|x_n - 0| < \epsilon$, how far should we take n to be sure that this accuracy of estimation kicks in? Since we want $1/\sqrt{n} < \epsilon$ we had better have $1/n < \epsilon^2$. In other words we had better have $n > 1/\epsilon^2$. A suitable cut-off is $K = 1/\epsilon^2$.

At this point somebody might ask:

> *Regarding the inequality* $|x_n - 0| < 1/13$ *up above, I can see that* $K = 169$ *is a good cut-off, while* $K = 168$ *is not quite good enough. So* 169 *seems to be the best possible cut-off that lets us achieve an accuracy of* $1/13$ *in this example. Do I always have to find the best possible cut-off as we did in this example?*

As far as the definition of limits is concerned, the answer is **no**. For instance, cut-offs such as $K = 170$ or $K = 500$ are just as suitable. Once a suitable K is found, any larger K is just as suitable in fulfilling the limit definition. Thus there is not a "one and only" answer for a suitable cut-off point K. Depending on the problem, it may be too difficult to determine the *best possible* cut-off value K. On the other hand a suitable cut-off may well be obtainable. The limit concept can tolerate such a compromise. In the next example we show how a suitable cut-off K can be found, without having to worry about the best possible K.

Example 3.

A calculator sampling for several large n would seem to indicate that

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0.$$

To prove that 0 is the limit, take any $\epsilon > 0$. We need to find a cut-off K such that

$$\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \epsilon \text{ when } n > K.$$

With a bit of algebra we see that

$$\begin{aligned}
\left|\sqrt{n+1} - \sqrt{n} - 0\right| &= \sqrt{n+1} - \sqrt{n} \\
&= (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}}.
\end{aligned}$$

We have to decide how far to go with n in order to be sure that

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \epsilon.$$

It is not so clear how we are to go about isolating n in this inequality. Rather than confront that difficulty we **do something clever**. We notice that

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}}, \text{ since the denominator on the left is bigger.}$$

From this it should be clear that

$$\text{if we have } 1/\sqrt{n} < \epsilon, \text{ then we also get } \left|\sqrt{n+1} - \sqrt{n} - 0\right| < \epsilon.$$

There is no doubt that $1/\sqrt{n} < \epsilon$ happens when $n > 1/\epsilon^2$. Thus a suitable cut-off is $K = 1/\epsilon^2$. If $n > 1/\epsilon^2$, we can be sure that $\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \epsilon$.

Notice that the cut-off $K = 1/\epsilon^2$ might not be the best possible, but we have proven it is good enough.

Exercises

5. Take the constant sequence $2, 2, 2, 2, \ldots$. The limit had better be 2. How far do you have to go with the terms $x_n = 2$ of the sequence in order to be sure that $|x_n - 2| < 10^{-10}$? The answer should be obvious.

6. Show that if $n > \dfrac{1000}{9}$, then $\left|\dfrac{n}{3n+1} - \dfrac{1}{3}\right| < \dfrac{1}{1000}$.
   You are given an $\epsilon > 0$. If $n > \dfrac{1}{9\epsilon}$, show that $\left|\dfrac{n}{3n+1} - \dfrac{1}{3}\right| < \epsilon$.

7. Find a cut-off K that ensures $1/2^n < \epsilon$ when $n > K$.

8. We think that $e^{-n} \to 0$. Given $\epsilon > 0$ find a cut-off K that ensures $e^{-n} < \epsilon$ when $n > K$.

9. Prove from the definition of limits that $\lim_{n\to\infty} 1/n^2 = 0$.

10. Prove using the limit definition that $1/(2n-1) \to 0$ as $n \to \infty$.

11. We can sense that $\left(3 + \dfrac{1}{n}\right)^2 \to 9$ as $n \to \infty$. If $n > 70$, show that $\left|(3 + \dfrac{1}{n})^2 - 9\right| < \dfrac{1}{10}$. Hint: you know that $\dfrac{1}{n^2} \leq \dfrac{1}{n}$ always.

    More generally take any $\epsilon > 0$. If $n > \dfrac{7}{\epsilon}$, show that $\left|(3 + \dfrac{1}{n})^2 - 9\right| < \epsilon$.

    This proves that $\left(3 + \dfrac{1}{n}\right)^2 \to 9$.

12. Apply the limit definition to prove that $\displaystyle\lim_{n\to\infty} \frac{n^2 - 1}{2n^2 + 3} = \frac{1}{2}$.

A slightly less formal language for the limit idea

Suppose that $x_n \to p$. This means that for any $\epsilon > 0$ we will have

$$|x_n - p| < \epsilon \ \text{ whenever } n \text{ is beyond some cut-off number } K.$$

We can say this more succinctly as follows:

$$\text{given any } \epsilon > 0, \text{ then } \ |x_n - p| < \epsilon \ \text{ eventually.}$$

Here the word "eventually" captures the idea that there is a cut-off point K that is suitable for the given ϵ, but we would prefer not to name a specific K at this time. Here come some examples illustrating this less formal language.

Example 4.

Let us demonstrate that $(-2/3)^n \to 0$ as $n \to \infty$. As usual take any positive tolerance $\epsilon > 0$. We must show that

$$\left|\left(\frac{-2}{3}\right)^n - 0\right| < \epsilon \ \text{ eventually.}$$

Of course, this is the same as getting

$$\left(\frac{2}{3}\right)^n < \epsilon \text{ eventually.}$$

By taking logarithms we reduce the problem to showing that

$$\ln\left(\left(\frac{2}{3}\right)^n\right) < \ln\epsilon \text{ eventually.}$$

From properties of logarithms the above will be true provided that

$$n\ln\left(\frac{2}{3}\right) < \ln\epsilon \text{ eventually.}$$

After dividing by the negative number $\ln(2/3)$, we come down to showing that

$$n > \frac{\ln\epsilon}{\ln(2/3)} \text{ eventually.}$$

Now there is no doubt that n will get past the fixed number $\ln\epsilon/\ln(2/3)$ eventually. After all, n goes beyond any fixed number eventually. Thus we have given the proof that $(-2/3)^n \to 0$.

Observe in Example 4 that if we wanted a suitable cut-off, we could take it to be $K = \ln\epsilon/\ln(2/3)$. For instance, when $\epsilon = 1/10^6$ is the given tolerance, a suitable cut-off number is

$$K = \ln(1/10^6)/\ln(2/3) \approx 34.1$$

So we learn that once our sequence hits the 35th term, from then on it will estimate 0 with an accuracy that is better than $1/10^6$.

Example 4 can readily be modified to prove that

$$r^n \to 0 \text{ for any given base } r \text{ such that } -1 < r < 1.$$

There was nothing special about the base $r = -2/3$, other than the fact it lies between -1 and 1.

Example 5.

We all believe that

$$\frac{1}{3} = 0.333333\ldots \text{ as an infinite decimal expansion.}$$

What this really means is that

$$\text{the limit of the sequence } \frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \frac{3333}{10000}, \ldots \text{ is } \frac{1}{3}.$$

If $x_n = \dfrac{333\cdots3}{1000\cdots0}$ where each 3 and each 0 is repeated n times, let us show that indeed $x_n \to 1/3$. Given a small tolerance $\epsilon > 0$ we must prove that

$$\left|x_n - \frac{1}{3}\right| < \epsilon \text{ eventually.}$$

With some little calculations we can see that

$$\left|x_1 - \frac{1}{3}\right| = \left|\frac{3}{10} - \frac{1}{3}\right| = \left|\frac{9}{30} - \frac{10}{30}\right| = \frac{1}{30},$$

and next

$$\left|x_2 - \frac{1}{3}\right| = \left|\frac{33}{100} - \frac{1}{3}\right| = \left|\frac{99}{300} - \frac{100}{300}\right| = \frac{1}{300},$$

and following this pattern

$$\left|x_n - \frac{1}{3}\right| = \frac{1}{3 \cdot 10^n}.$$

So we need to know that

$$\frac{1}{3 \cdot 10^n} < \epsilon \text{ eventually.}$$

In other words we need to know that

$$\frac{1}{3\epsilon} < 10^n \text{ eventually.}$$

It should now be quite clear that that 10^n goes past any number eventually, so we are done. If we wanted to actually find a suitable cut-off K, we can go one step further and observe that the last inequality amounts to having

$$\log_{10}\left(\frac{1}{3\epsilon}\right) < n \text{ eventually.}$$

We can see that once n has gone beyond the cut-off number $K = \log_{10}(1/3\epsilon)$, then $|x_n - 1/3| < \epsilon$ will kick in.

Example 6.

This example is a bit more subtle. By testing the values of $\cos(1/n)$ on a calculator (using radian mode!) it becomes evident that this sequence tends to 1. Let us verify that

$$\cos\left(\frac{1}{n}\right) \to 1 \text{ as } n \to \infty.$$

Take any tolerance $\epsilon > 0$. Now we need to show that

$$\left|\cos\left(\frac{1}{n}\right) - 1\right| < \epsilon \text{ eventually.}$$

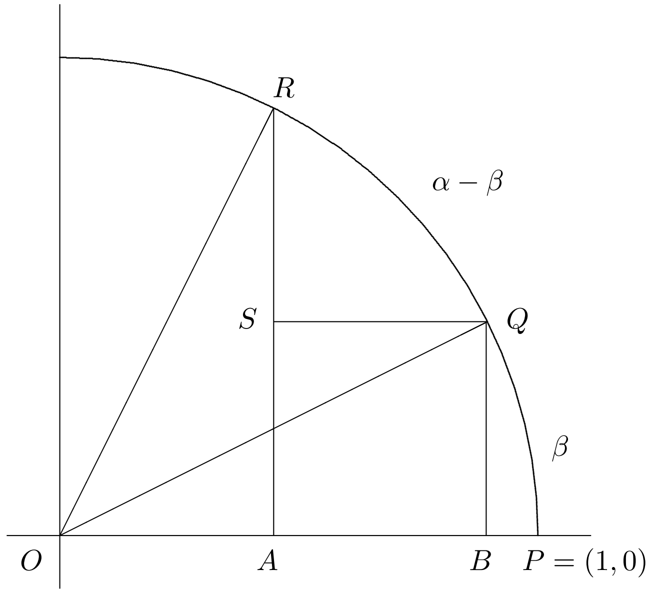
After we remember that $1 = \cos(0)$ the problem comes down to showing that

$$\left|\cos\left(\frac{1}{n}\right) - \cos(0)\right| < \epsilon \text{ eventually.}$$

We need to do a bit of trigonometry on the side. Namely, observe that for any two angles α and β (in radian mode) we have the inequality

$$|\cos\alpha - \cos\beta| \leq |\alpha - \beta|.$$

This significant little fact can be seen from the diagram that follows.



$$
\begin{aligned}
\beta &= \text{arclength}\, PQ & & OB = \cos\beta \\
\alpha &= \text{arclength}\, PR & & OA = \cos\alpha \\
\alpha - \beta &= \text{arclength}\, QR & \geq\ & SQ = AB = \cos\beta - \cos\alpha
\end{aligned}
$$

In particular for any index n we have

$$\left|\cos\left(\frac{1}{n}\right) - 1\right| = \left|\cos\left(\frac{1}{n}\right) - \cos(0)\right| \leq \left|\frac{1}{n} - 0\right| = \frac{1}{n}.$$

From this clever estimate we see that

once we have $\dfrac{1}{n} < \epsilon$, then we will also have $\left|\cos\left(\dfrac{1}{n}\right) - 1\right| < \epsilon$.

Quite obviously we will have $1/n < \epsilon$ eventually, and thereby

$$\left|\cos\left(\frac{1}{n}\right) - 1\right| < \epsilon \text{ eventually.}$$

Exercises

13. Show that $\dfrac{n}{n^2+1} < .001$ eventually.
    For any $\epsilon > 0$ show that $\dfrac{n}{n^2+1} < \epsilon$ eventually.

14. Given $\epsilon > 0$, show that $\sin(1/n) < \epsilon$ eventually.
    Hint: $|\sin x| \leq |x|$ for all numbers x.

15. Prove that $\sin(n^2)/\sqrt{n} < 10^{-9}$ eventually. Does it still work if we replace 10^{-9} by any $\epsilon > 0$?

16. Prove that $1/10^n \to 0$ by showing that for any $\epsilon > 0$ we have $1/10^n < \epsilon$ eventually.

17. Show that $\left|\ln\left(e + \dfrac{1}{n}\right) - 1\right| < \dfrac{1}{100}$ eventually. Repeat the problem with $\epsilon > 0$ replacing $\dfrac{1}{100}$. Hint: $1 = \ln e$.

18. Given $\epsilon > 0$, show that $2^n/n! < \epsilon$ eventually. This is a bit harder. First look at and explain the hint:

$$\frac{2^n}{n!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdot \cdots \cdot \frac{2}{n-1} \cdot \frac{2}{n} \leq \frac{4}{n}.$$

    You have just proven that $2^n/n! \to 0$. Next prove that $3^n/n! \to 0$ by the method of imitation of what you just did with $2^n/n!$. Imitate some more and prove $10^n/n! \to 0$. Given any constant a, what do you think $a^n/n!$ tends to?

19. Show that for any $\epsilon > 0$ we have $n!/n^n < \epsilon$ eventually.
    Hint: first show that $n!/n^n \leq 1/n$.

20. Prove that $n/e^n \to 0$ by showing that for any $\epsilon > 0$ you will eventually get $n/e^n < \epsilon$.
    Hint: You can see that $n/e^n = (n/2^n)(2/e)^n \leq (2/e)^n$ since $n/2^n \leq 1$.

21. This is a little bit of theory. If a sequence tends to a positive limit, show that the sequence eventually must itself become positive. More precisely suppose that

$$x_n \to p \text{ and that } p > 0.$$

    Prove that $x_n > 0$ eventually.

Sequences that do not converge

Of course not all sequences have a limit. Let us examine a couple of situations of this sort.

Example 7.

Take the sequence given by $x_n = (-1)^n$. This is the sequence

$$-1, 1, -1, 1, -1, \ldots$$

that bounces back and forth between -1 and 1. It would appear that this sequence has no limit. How could we prove such a fact on the basis of the definition of limits? Well, to show that the limit is not there, why not suppose it is there and from that deduce a contradiction? Taking this approach, suppose that for some number p we have

$$x_n \to p \text{ as } n \to \infty.$$

This means that

$$\text{for any } \epsilon > 0 \text{ we get } |x_n - p| < \epsilon \text{ eventually.}$$

This is so even for a small ϵ such as $\epsilon = 1/10$. We are being told that our sequence of -1's and 1's is eventually within $1/10$ of some number p. This tells us that both -1 and 1 are within $1/10$ of the same number p. It follows that -1 and 1 are at most $1/5$ from each other. That's impossible, since the distance between -1 and 1 is 2.

The above example can easily be modified to show that any sequence $\{x_n\}$ that repeats at least two values infinitely often will never converge.

The next example illustrates another way that a sequence can fail to have a limit.

Example 8.

Take the sequence of perfect squares $x_n = n^2$. This sequence blows up when n is big. It has no limit. How can we use the limit definition to explain this? Well, suppose that it had a limit, say p. So for any $\epsilon > 0$ we would get

$$|n^2 - p| < \epsilon \text{ eventually.}$$

Thus, even for $\epsilon = 1$ we would have

$$|n^2 - p| < 1 \text{ eventually.}$$

So eventually every squared integer n^2 would be stuck betwen the numbers $p - 1$ and $p + 1$, clearly a ridiculous outcome.

Bounded sequences

The sequence in Example 8 has the property that it is not bounded. A sequence $\{x_n\}$ is called *bounded* provided that there are two constants a, b which satisfy

$$a \leq x_n \leq b \text{ for all indices } n.$$

The constants a, b are called *lower and upper bounds* of the sequence, respectively. A simple example of a bounded sequence is

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

having suitable lower and upper bounds $a = 0, b = 1$, respectively. Notice by this example that a bounded sequence need not converge. On the other hand,

every convergent sequence must be bounded.

Here is the explanation of this fact in terms of the limit definition. Let the sequence $\{x_n\}$ converge to the limit p. Taking $\epsilon = 1$ we know that $|x_n - p| < 1$ eventually. Unfolding this inequality we get that $p - 1 < x_n < p + 1$ eventually. So the terms are eventully stuck between two bounds. In other words, there are only a finite number of terms not stuck between two bounds. By lowering the lower bound and raising the upper bound to take in the few terms that are missing, we can make sure that all the terms are stuck between two bounds. Thus every sequence with a limit is bounded.

Knowing that every convergent sequence is bounded, we thereby know that an unbounded sequence has no limit. This little fact can be used to rule out the presence of limits in some cases. Thus, for example, sequences such as $x_n = \log n$ or $x_n = 2^n$ or $x_n = \sqrt{n}$ have no limit because they are not bounded. However be careful! Whereas unbounded sequences never converge, a bounded sequence may or may not converge.

Exercises

22. Show that the sequence $x_n = \ln n$ is not bounded by proving that every number b fails to be a bound.
    Note: in order that b not be a bound for the sequence, you have demonstrate that there is some n so that $\ln n > b$.
    Does the sequence $\ln n$ converge?

23. Show that $\dfrac{n^3+1}{n^2-5}$ is eventually greater than any given number b. Why does this sequence have no limit?

    Hint: first explain why $\dfrac{n^3+1}{n^2-5} > \dfrac{n^3+1}{n^2} = n + \dfrac{1}{n^2}$, then use this fact.

24. Is the sequence $\cos(n\pi/4)$ is bounded? Does this sequence have a limit? Explain your answers.

Some limits to remember

Of course, you should not have to confirm every limit from scratch using the definition. Once the limit concept has been understood it is time to move on and simply remember the limits of a few sequences by heart. Some limits are instinctively clear, while others are more subtle. Here are some limits which you should remember. Every limit below can be proven by paying careful attention to the limit definition.

- If $\{x_n\}$ is the constant sequence $a, a, a, \ldots$, then $x_n \to a$.
- If $a > 0$, then $\dfrac{1}{n^a} \to 0$ as $n \to \infty$.
- If $a > 0$, then $\sqrt[n]{a} \to 1$ as $n \to \infty$.
- If $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.
- If a is any real number, then $\dfrac{a^n}{n!} \to 0$.
- If $a > 1$, then $\dfrac{n}{a^n} \to 0$.
- $\dfrac{\ln n}{n} \to 0$ as $n \to \infty$
- $\left(1 + \dfrac{1}{n}\right)^n \to e$.

Limit properties for everyday use

In conjuction with our knowledge of a few basic limits, some natural properties of limits will permit us to compute the limit of a wide assortment of sequences, virtually by inspection. We use the following theorem constantly, sometimes without even realizing it.

Theorem 9. *If* $x_n \to p$ *and* $y_n \to q$ *as* $n \to \infty$*, then*

- $x_n + y_n \to p + q$ *(addition formula)*
- $x_n - y_n \to p - q$ *(subtraction formula)*
- $x_n y_n \to pq$ *(multiplication formula)*
- $\dfrac{x_n}{y_n} \to \dfrac{p}{q}$*, provided that* $y_n \neq 0$ *and* $q \neq 0$*. (division formula)*

Proof. Although these properties may seem obvious, we can only be certain that they are correct by putting them to the test of the limit definition. What is difficult to appreciate about this theorem is, not so much the proof, but rather the fact something needs to be proved. Let us prove just the addition formula, that the limit of a sum is the sum of the limits, and take the other three properties on faith. We are given that

$$x_n \to p \text{ and } y_n \to q \text{ as } n \to \infty.$$

We must prove that

$$x_n + y_n \to p + q \text{ as } n \to \infty.$$

Take $\epsilon > 0$. According to the limit definition we must show that

$$|(x_n + y_n) - (p + q)| < \epsilon \text{ eventually.}$$

Well, we can use the triangle inequality to obtain

$$|(x_n + y_n) - (p + q)| = |(x_n - p) + (y_n - q)| \leq |x_n - p| + |y_n - q| \,.$$

We know that x_n gets as close as we like to p eventually, while y_n gets as close as we like to q eventually. Thus in particular,

$$|x_n - p| < \frac{\epsilon}{2} \text{ and } |y_n - q| < \frac{\epsilon}{2} \text{ eventually.}$$

It follows that

$$|x_n - p| + |y_n - q| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ eventually.}$$

Consequently

$$|(x_n + y_n) - (p + q)| < \epsilon \text{ eventually,}$$

and the proof is done. □

The next example illustrates how Theorem 9 typically gets used.

Example 10.

Let us find the limit of $\dfrac{2n^3 - 5n + 7}{8n^3 + 9n^2 - 4}$. We are tempted to use the division formula for limits. Unfortunately both the numerator $2n^3 - 5n + 7$ and the denominator $8n^3 + 9n^2 - 4$ blow up for large values of n, and thereby do not have a limits to which the division formula can be applied. We need a trick! Divide both numerator and denominator by the highest power of n, namely n^3, and you get

$$\frac{2n^3 - 5n + 7}{8n^3 + 9n^2 + 4} = \frac{2 - \frac{5}{n^2} + \frac{7}{n^3}}{8 + \frac{9}{n} - \frac{4}{n^3}}.$$

Now use Theorem 9 liberally and repeatedly. We know the constant sequence 7 tends to the limit 7, while the sequence $1/n^3$ tends to 0. By the multiplication formula $7/n^3 \to 7 \cdot 0 = 0$. Likewise $5/n^2 \to 0$. Clearly the constant sequence 2 has limit of 2. Using the addition and subtraction formulas, $2 - 5/n^2 + 7/n^3 \to 2$. Likewise, by repeated use of the addition, subtraction and multiplication formulas, we obtain $8 + 9/n - 4/n^3 \to 8$. Next by the division formula we conclude that

$$\frac{2 - \frac{5}{n^2} + \frac{7}{n^3}}{8 + \frac{9}{n} - \frac{4}{n^3}} \to \frac{2}{8} = \frac{1}{4}.$$

So the limit of our sequence is $1/4$.

Note. Once a trick such as the one above is used, it is no longer a trick. You now have a method. You should be able to adapt the method of this example to numerous sequences of this sort. Normally, you do not have to provide an alert every time a limit property from Theorem 9 is being used. It is OK to say

$$\frac{2 - \frac{5}{n^2} + \frac{7}{n^3}}{8 + \frac{9}{n} - \frac{4}{n^3}} \to \frac{1}{4} \text{ by inspection,}$$

but keep in mind that, in the background, you are using Theorem 9 repeatedly.

Now comes a theorem about sequence limits that relates to function continuity.

Theorem 11. *Let f be any function that is continuous at some point p. That is $f(x) \to f(p)$ as $x \to p$ or $\lim_{x \to p} f(x) = f(p)$. We know a multitude of such functions. If x_n is a sequence in the domain of f and $x_n \to p$, then $f(x_n) \to f(p)$.*

Proof. The proof is just a simple matter of keeping track of the limit definitions, including the ϵ-δ definition for continuity of a function.

Take any $\epsilon < 0$. We need a cut-off number K such that

$$|f(x_n) - f(p)| < \epsilon \text{ when } n > K.$$

Since $\lim_{x \to p} f(x) = f(p)$, we do have a $\delta > 0$ such that

$$|f(x) - f(p)| < \epsilon \quad \text{when} \quad |x - p| < \delta.$$

And since $x_n \to p$, we also have a K that gives

$$|x_n - p| < \delta \text{ when } n > K.$$

When $n > K$, our terms x_n are exactly where they need to be in order to guarantee that $|f(x_n) - f(p)| < \epsilon$. □

A mechanical formula that expresses Theorem 11 for a continuous function f is the following:

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right).$$

To put it vaguely, *continuous function pull sequence limits inside them.*

For a simple illustration of how Theorem 11 gets used, take $f(x) = e^x$. We know that $e^x \to 1$ as $x \to 0$. We also know that $\dfrac{1}{n} \to 0$ as $n \to \infty$. By Theorem 11, we see that $e^{1/n} \to 1$ as $n \to \infty$. This type of application of Theorem 11 is done so often and instinctively that we sometimes don't bother to notice it. We often just say: "$e^{1/n} \to 1$ as $n \to \infty$, by inspection".

Now comes a little upgrade of Theorem 11.

Some very nice functions are not continuous at some point p, but still $L = \lim_{x \to p} f(x)$ exists. For example, $f(x) = \dfrac{\sin x}{x}$ is not defined at $p = 0$, and yet $1 = \lim_{x \to 0} \dfrac{\sin x}{x}$. Such functions, you may recall, have a *removable discontinuity* at p. We can make these functions f continuous at p by simply defining, or maybe redefining, $f(p)$ to be L. Let's agree to do that automatically for every function with a removable discontinuity.

Thus we get the following small improvement on Theorem 11.

Theorem 12. *If $f(x) \to L$ as $x \to p$ and a sequence x_n in the domain of f converges to p as $n \to \infty$, then the sequence $f(x_n)$ converges to L as $n \to \infty$.*

For an illustration of Theorem 12, notice that $\dfrac{\sin x}{x} \to 1$ as $x \to 0$. We also know that $1/\sqrt{n} \to 0$ as $n \to \infty$. By Theorem 12

$$\sqrt{n}\sin\left(1/\sqrt{n}\right) = \frac{\sin\left(1/\sqrt{n}\right)}{1/\sqrt{n}} \to 1 \text{ as } n \to \infty.$$

Here is an example of the easy and somewhat dull ways in which Theorems 9, 11 and 12 get used together.

Example 13.

Let us show that $7^{1/2+1/n}\tan((n\pi+1)/4n) \to \sqrt{7}$.

First observe that

$$7^{1/2+1/n} = e^{(\ln 7)(1/2+1/n)}.$$

The addition rule of Theorem 9 tells us that

$$1/2 + 1/n \to 1/2.$$

Then the multiplication rule of Theorem 9 tells us that

$$(\ln 7)(1/2+1/n) \to (\ln 7)/2.$$

By Theorem 11 applied to the function $y = e^x$ we get

$$7^{1/2+1/n} = e^{(\ln 7)(1/2+1/n)} \to e^{(\ln 7)/2} = e^{\ln\sqrt{7}} = \sqrt{7}.$$

By the addition and multiplication formulas in Theorem 9 we get

$$(n\pi+1)/4n = \pi/4 + 1/4n \to \pi/4.$$

Using Theorem 11 for $y = \tan x$ we get

$$\tan((n\pi+1)/4n) \to \tan(\pi/4) = 1.$$

Finally the multiplication rule in Theorem 9 gives

$$7^{1/2+1/n}\cdot\tan((n\pi+1)/4n) \to \sqrt{7}\cdot 1 = \sqrt{7}.$$

Note. The purpose of Example 12 is to demonstrate how frequently Theorems 9, 11 and 12 get used. If you have to find a limit for a given sequence, one strategy might be to put the terms x_n into a form which can accept the rules of Theorems 9, 11 and 12. After that it will not be necessary, as we did in Example 13, to provide a comment every time some aspect of these theorems is used. It will suffice to say that limits, such as the $\sqrt{7}$ obtained above, are calculated " by inspection".

Exercises

25. Find the limit of $\sqrt{n^2+n}-n$. It is not required to use the definition of limits, just do some algebraic manipulations and make an inspection based on Theorems 9 and 11.

26. Find $\lim\limits_{n\to\infty}\sqrt{n^2-2n}-n$, by putting this sequence into a form that accepts the rules of Theorems 9 and 11.

27. Take the sequence of cosine iterates as follows:

$$x_0 = \pi/4,\ x_{n+1} = \cos x_n \text{ for } n = 0, 1, 2, \ldots.$$

One can prove that this sequence converges, say to p. Use Theorem 11 to show that $(p, \cos p)$ is the point where the graph of $y = \cos x$ intersects the line $y = x$.

28. Find the limit of $n\left(\sqrt{1+\frac{1}{n}}-1\right)$ by putting this sequence into a form that Theorems 9 and 11 can digest.

29. Find the limit of $n\left(\sqrt[3]{1+\frac{1}{n}}-1\right)$ by putting this sequence into a form that Theorems 9 and 11 can digest. This one is a bit trickier.

30. For each real number a consider the sequence $x_n = \dfrac{1}{1+a^n}$. Depending on a you may get a different limit or none at all. Use your powers of inspection to decide which a's give which limits.

31. Find $\lim\limits_{n\to\infty}(\ln(2+3^n))/2n$.
Hint: First explain and then use the identity $\ln(2+3^n) = n\ln 3 + \ln\left(1+\frac{2}{3^n}\right)$. After that an inspection based on Theorems 9 and 11 ought to do it.

32. Find $\lim_{n\to\infty} n/\ln(1+2e^n)$, showing how you got it.

Sequence limits and horizontal asymptotes

A function $y = f(t)$ defined on an interval of the type $[a, \infty)$ has a horizontal asymptote L as t tends to infinity provided $f(t) \to L$ as $t \to \infty$. This looks an awful lot like the limit of a sequence concept, but there is a technical difference. In the case of a function $y = f(t)$, our independent variable t runs through a full interval of real numbers $[a, \infty)$. In the case of a sequence $\{x_n\}$, our independent variable is the integer n that runs through the discrete values $1, 2, 3, \ldots$. This technical distinction needs to be kept in mind in order to avoid mixups.

For example, take the function $y = \sin(\pi t)$. As $t \to \infty$ the function $\sin(\pi t)$ has no limit since it oscillates between ± 1. However, the sequence $x_n = \sin(\pi n)$ defined for $n = 1, 2, 3, \ldots$ is the same as the zero sequence $0, 0, 0, \ldots$, and the limit of this is surely 0.

Thus limits of sequences do not help us determine horizontal asymptotes. Happily, horizontal asymptotes do help us determine limits of sequences.

Proposition 14. *If f is a function such that $f(t) \to L$ as $x \to \infty$ and x_n is the sequence given by $x_n = f(n)$ where $n = 1, 2, 3, \ldots$, then $x_n \to L$ too.*

Informally speaking, Proposition 14 is true simply because the n's are just particular t's.

For instance we can use L'Hôpital's rule to get that $t^2/e^t \to 0$ as $t \to \infty$. Thus the sequence $n^2/e^n \to 0$ as $n \to \infty$.

Exercises

33. Find the limit of the following sequences:

    (a) $\sqrt{n^2+n} - n$ (b) $ne^{1/n} - n$ (c) $(1+2/n)^n$ (d) $\dfrac{(\ln n)^2}{n}$.

Monotonic sequences, a gateway to special limits

You may have noticed that, up to this point, our sequence limits have by and large been rather familiar numbers such as 0 or $1/3$. However, for some fairly simple sequences the limits can be rather surprising. For instance take the sequence $\{x_n\}$ given recursively by the rule

$$x_1 = \cos 1 \text{ and } x_{n+1} = \cos(x_n) \text{ for } n \geq 1.$$

try computing the first 50 or so terms of this sequence, using a calculator. Using radian mode punch in $\cos 1$. Then take the $\cos$ of the output, then take the $\cos$ of that, then the $\cos$ of that etc. You quickly see a limit emerging as the outputs begin to stabilize. That limit p is a new number with no name. In fact $p \approx 0.7390851332151607$. Suppose $x_n \to p$. Then $\cos x_n \to \cos p$. On the other hand $\cos x_n = x_{n+1}$ and $x_{n+1} \to p$. We get that $\cos p = p$. Our recursive little sequence has just solved the difficult equation $\cos x = x$.

For such exotic new numbers it is the sequence itself that provides the mechanism for estimating and thereby understanding the limit. If we have no clue what the limit of a sequence could be, the best we could ask for is to decide whether or not a given sequence simply has a limit. From this point of view, a sequence becomes a machine that uncovers previously unknown numbers.

When we do not know the limit in advance, the most basic method for deciding that a sequence has a limit is the so called *monotone convergence principle* or the *monotonic sequence theorem*. We will now discuss that principle.

We say that a sequence $\{x_n\}$ is *increasing* provided

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots$$

If the inequalities are strict (i.e $<$ instead of $\leq$) we say that the sequence is *strictly increasing*. Naturally, if the inequalities go the other way the sequence is called *decreasing*. A sequence that is either increasing or decreasing is known as a *monotonic sequence*.

Recall that a sequence $\{x_n\}$ is *bounded* provided that all terms satisfy

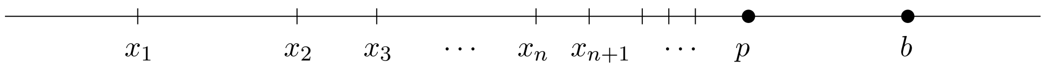
$$a \leq x_n \leq b \text{ for some numbers } a \text{ and } b,$$

called lower and upper bounds respectively. Bounded sequences need not converge, for instance take the bounded sequence $0, 1, 0, 1, 0, \ldots$ which does not converge. Monotonic sequences need not converge, for instance take the increasing sequence $1, 2, 3, 4, \ldots$. However a sequence with both these two properties together (i.e. bounded and monotonic) always has a limit. Let us now state and prove this important result.

Theorem 15 (Monotonic sequence theorem). *If a sequence $\{x_n\}$ is bounded and monotonic, then $\{x_n\}$ has a limit.*

Proof. Let's say the sequence is increasing. Since the sequence is bounded there has to be an upper bound. Of all possible upper bounds for the x_n terms, let p be the smallest one. Thus p is a number with the fundamental property that

$$\text{all } \ x_n \leq p, \ \text{ and numbers less than } p \text{ are not upper bounds.}$$



$p =$ least upper bound of all x_n

$b =$ any other upper bound of the x_n

It turns out that p is our limit. To see that, we go to the limit definition. Take any $\epsilon > 0$. We need to show that

$$|x_n - p| < \epsilon \text{ eventually.}$$

Notice that $p - \epsilon$ is is not an upper bound of the sequence, because $p - \epsilon$ is less than p. This means that $p - \epsilon < x_K$ for some index K. Now remember that x_n increases. Therefore, when $n > K$ we have

$$p - \epsilon < x_K \leq x_n \leq p < p + \epsilon.$$

From this it follows that

$$|x_n - p| < \epsilon \text{ once } n \text{ goes past the cut-off } K\text{, which is to say, eventually.}$$

A similar proof, based on taking greatest lower bounds, works for decreasing sequences. □

Here come some examples of bounded, monotone sequences and their limits.

Example 16.

Suppose that the sequence $\{x_n\}$ is specified recursively by the formula

$$x_1 = 1,\ x_{n+1} = \sqrt{3 + 2x_n} \text{ for } n = 1, 2, 3, \ldots.$$

The first few terms are

$$1, \sqrt{5} \approx 2.24, \sqrt{3 + 2\sqrt{5}} \approx 2.73, \sqrt{3 + 2\sqrt{3 + 2\sqrt{5}}} \approx 2.91, \ldots$$

A preliminary inspection might suggest that this sequence is increasing. Furthemore it seems to not be growing too fast. Maybe it is bounded. After some sampling we suspect that the sequence stays below a bound such as 4. Let use the method of *mathematical induction* to show that

$$x_n < x_{n+1} < 4 \text{ for all } n = 1, 2, 3, \ldots.$$

This will ensure that $\{x_n\}$ is a bounded, increasing sequence. For $n = 1$ we certainly have

$$x_1 = 1 < x_2 = \sqrt{5} < 4.$$

Next suppose that up to a given n we have

$$x_n < x_{n+1} < 4.$$

We must verify the next statement that

$$x_{n+1} < x_{n+2} < 4.$$

We can build up to what we want from our previous inductive assumption. Since

$$x_n < x_{n+1} < 4,$$

we deduce that

$$3 + 2x_n < 3 + 2x_{n+1} < 3 + 2 \cdot 4 = 13.$$

Hence

$$\sqrt{3 + 2x_n} < \sqrt{3 + 2x_{n+1}} < \sqrt{13}.$$

Taking into account how x_n was built recursively we have just discovered that

$$x_{n+1} < x_{n+2} < \sqrt{13} < 4,$$

just what we wanted!

Now the monotonic sequence principle guarantees that the sequence $\{x_n\}$ converges. Could we find the limit p? We know that $x_{n+1} \to p$ also, because the sequence $\{x_{n+1}\}$ is just the original sequence with the first term dropped. On the other hand $x_{n+1} = \sqrt{3 + 2x_n}$. Using the limit properties in Theorems 9 and 11, we see that $x_{n+1} \to \sqrt{3 + 2p}$. Hence

$$p = \sqrt{3 + 2p}.$$

Squaring we obtain $p^2 - 2p - 3 = 0$. Factoring we get $(p + 1)(p - 3) = 0$ and thus $p = -1$ or $p = 3$. However, p cannot be -1 because x_n increases starting with $x_1 = 1$, so $p = 3$. We have recursively built a monotonic, bounded sequence that tends to 3.

Example 17.

Take the sequence given by the rule

$$x_0 = 1,\ x_n = x_{n-1} + \frac{1}{n!} \text{ for } n = 1, 2, 3, \ldots.$$

For instance, the first few terms are

$$1,\ 1+1,\ 1+1+\frac{1}{2},\ 1+1+\frac{1}{2}+\frac{1}{6},\ 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24},$$
$$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120},\ 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720},\ldots$$

which simplify to

$$1,\ 2,\ \frac{5}{2},\ \frac{8}{3},\ \frac{65}{24},\ \frac{163}{60},\ \frac{1957}{720},\ldots$$

You can check with your calculator that these numbers seem to approximate the decimal expansion for the special number e. In fact this sequence does tend to e. This sequence becomes a decent mechanism for using fractions to estimate e to any desired level of accuracy.

We will now satisfy ourselves that this sequence converges by using the monotonic sequence theorem. After starting with $x_0 = 1$, each term of the sequence is obtained by adding a positive amount to the previous term. So there is no doubt that the sequence is increasing. In order to be sure the sequence converges, all we need to do is check that the sequence is bounded. For sure $0 \leq x_n$. To get an upper bound notice that for every positive integer k we have

$$k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k \geq \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{k-1 \text{ 2s here}} = 2^{k-1}.$$

Thus $\dfrac{1}{k!} \leq \dfrac{1}{2^{k-1}}$, and therefore

$$x_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}.$$

After recalling the sum formula for a geometric series we see that

$$x_n \leq 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}\right) = 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \leq 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

Thus we discover that $0 \leq x_n \leq 3$, making our sequence bounded. Being both bounded and increasing our sequence has to converge.

Some mathematicians make the declaration that e, by definition, is the limit of the sequence in Example 17. If e is thus defined, we should notice from the proof of the monotonic sequence theorem that e is the smallest of the upper bounds for x_n. We saw in the proof that 3 is an upper bound for x_n, and we can see that 2 is not an upper bound for x_n, since x_3 is already past 2. Therefore the smallest upper bound has to be bigger than 2 and no bigger than 3. In this way we come to understand why the crude estimate $2 < e \leq 3$ is true.

Example 18.

Most have encountered the symbolism of an infinite decimal expansion:

$$p = 0.a_1a_2a_3\cdots a_n\cdots \quad \text{where the } a_n \text{ are arbitrary digits from } 0 \text{ to } 9.$$

For instance we have noted already in Example 5 that $0.333\cdots$ represents the number $p = 1/3$ in the sense that the sequence of finite decimals tends to $1/3$. The monotone convergence theorem is the result that gives a working meaning to any infinite decimal expansion. Here is how an infinite decimal expansion is to be interpreted. First you select any sequence of integer digits

$$a_1, a_2, a_3, \ldots, a_n, \ldots \quad \text{from } 0 \text{ to } 9.$$

Any sequence of digits, chosen in any way you like, will do. Now form the sequence of finite partial decimals:

$$\begin{aligned}
x_1 &= 0.a_1 \\
x_2 &= 0.a_1a_2 \\
x_3 &= 0.a_1a_2a_3 \\
&\vdots \\
x_n &= 0.a_1a_2a_3\cdots a_n \\
&\vdots
\end{aligned}$$

The sequence x_n is clearly increasing, because we are adding a decimal digit each time. Furthermore all x_n are bounded above by 1, since the digit to the left of each decimal is 0. The x_n are bounded below by 0. By the monotonic sequence theorem, we conclude that $x_n \to p$ for some real number p. This limit p of the bounded monotonic sequence x_n is exactly what is meant by the infinite decimal symbolism

$$p = 0.a_1a_2a_3\cdots a_n\cdots$$

The monotonic sequence theorem is the principle that justifies the common practice of thinking that an infinite decimal expansion produces a real number.

Example 19.

You may have wondered how your calculator implements the square root function: $\sqrt{}$. What it does, is generate a simple numerical sequence with the desired square root as its limit. The example to follow illustrates the mathematics that could well be going on inside your calculator. This process, typically known as an algorithm, deserves our attention at least for the sake of its practicality.

We propose to build a bounded, monotonic sequence of ordinary fractions that converges to $\sqrt{10}$. This will give us a method of computing $\sqrt{10}$ to any desired accuracy. Here is our sequence defined recursively:

$$x_1 = 10,\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right) \ \text{ for } n = 1, 2, 3, \ldots.$$

For instance, the first few terms of the sequence are:

$$10, \frac{11}{2}, \frac{161}{44}, \frac{45281}{14168}, \frac{4057691201}{1283082416}, \ldots.$$

Using the monotone convergence principle we can show that this sequence converges. First notice that $0 < x_n$, just by inspecting how each successive term of the sequence is built. Using some careful algebra let us now show that this sequence decreases. To that end we must prove $x_{n+1} \leq x_n$ for $n = 1, 2, 3, \ldots$. Given how we built x_{n+1}, we are obliged to show that $\frac{1}{2}\left(x_n + \frac{10}{x_n}\right) \leq x_n$. Because $0 < x_n$, you should check that this simplifies down to proving

$$10 \leq x_n^2 \ \text{ for each } n = 1, 2, 3, \ldots.$$

For $n = 1$ we obviouly have $10 \leq x_1^2 = 100$, while for $n > 1$ we know that $x_n = \frac{1}{2}\left(x_{n-1} + \frac{10}{x_{n-1}}\right)$. Now watch closely:

$$\begin{aligned}
x_n^2 &= \frac{1}{4}\left(x_{n-1} + \frac{10}{x_{n-1}}\right)^2 \\
&= \frac{1}{4}\left(x_{n-1}^2 + 20 + \frac{100}{x_{n-1}^2}\right) \\
&= 10 + \frac{1}{4}\left(x_{n-1}^2 - 20 + \frac{100}{x_{n-1}^2}\right), \ \text{ because } \frac{20}{4} = 10 - \frac{20}{4} \\
&= 10 + \frac{1}{4}\left(x_{n-1} - \frac{10}{x_{n-1}}\right)^2 \\
&\geq 10, \ \text{ since the term dropped is a square and thus} \geq 0.
\end{aligned}$$

So $10 \leq x_n^2$, and our explanation above shows that x_n decreases. Since $0 < x_n$ our sequence has a lower bound, and because it decreases it certainly has an upper bound, namely the first term 10. The monotone convergence principle kicks in! The sequence x_n has a limit, call it p.

What could the limit p possibly be? Because $x_n \to p$ we get that $x_{n+1} \to p$ also. After all, x_{n+1} is just the original sequence with the first term dropped. However $x_{n+1} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right)$. Consequently $x_{n+1} \to \frac{1}{2}\left(p + \frac{10}{p}\right)$, using the familiar limit properties of Theorem 9. We conclude that $p = \frac{1}{2}\left(p + \frac{10}{p}\right)$. Solving this equation using simple algebra we get $p = \pm\sqrt{10}$. It cannot be that $p = -\sqrt{10}$ since the sequence x_n is always positive, and therefore gets nowhere near to $-\sqrt{10}$. It must be that $p = \sqrt{10}$. We have therefore built a sequence of ordinary fractions that converges to $\sqrt{10}$.

The method illustrated here for finding $\sqrt{10}$ can readily be adapted to finding $\sqrt{a}$ for any $a > 1$. You simply replace the 10 in the definition of x_n by the a that you have in mind. What is truly remarkable about this sequence-based algorithm for finding square roots is the fantastic speed at which the recursive sequence converges. It turns out that after the recursion gets going a little bit, the number of decimal places of accuracy in estimating $\sqrt{a}$ will roughly double as we pass from x_n to x_{n+1}. To illustrate with $\sqrt{10}$, it turns out that $x_4 \approx \sqrt{10}$ to 1 decimal place, $x_5 \approx \sqrt{10}$ to 3 decimal places, and $x_6 \approx \sqrt{10}$ to at least 7 decimal places. This kind of ultra fast convergence is sometimes known as quadratic convergence. What makes your calculator work so well is not just the high speed electronics, but also the intelligent mathematics that the programs within it exploit.

Exercises

34. Let x_n be defined recursively by the rule

$$x_1 = 0,\ x_{n+1} = 1 - x_n \text{ for } n \geq 1.$$

Let $x_n \to p$. Therefore $x_{n+1} \to p$ as well. Since $x_{n+1} = 1 - x_n$ we conclude $p = 1 - p$ and thus $p = 1/2$. So our sequence tends to $1/2$. However, by calculating a few terms we see that our sequence is $0, 1, 0, 1, 0, \ldots$, which has no limit for sure. What is wrong with the argument that gave $p = 1/2$ as the limit?

35. If a sequence converges, must the sequence be bounded and monotonic? If so prove, if not give an example.

36. Let $\{x_n\}$ be the sequence defined recursively by

$$x_1 = 2 \text{ and } x_{n+1} = \frac{1}{2}(x_n + 6) \text{ for } n = 1, 2, 3, \ldots.$$

Show that this is an increasing and bounded sequence. Find its limit.
Hint: 10 should work as an upper bound.

37. We can see that $\dfrac{n}{n^2+1}$ converges, to 0 in fact. Use the monotone sequence theorem to show this sequence converges by proving that $x_n > x_{n+1}$ and that x_n is bounded below by some number.

38. Suppose that

$$x_1 = \sqrt{2} \text{ and } x_{n+1} = \sqrt{2 + x_n} \text{ for } n = 1, 2, 3, \ldots.$$

Prove that x_n converges. Then find the limit of this sequence.

39. We can see that the sequence $(2n-3)/(3n+4)$ converges to $2/3$. Thus it must be bounded. Decide if the sequence monotonic by sampling a few values, then making a decision, and then proving your decision.

40. Apply the monotone convergence principle to show that the sequence given recursively by

$$x_1 = 2 \text{ and } x_{n+1} = \frac{1}{3 - x_n} \text{ for } n = 1, 2, 3, \ldots.$$

converges. Then find its limit.

41. Take the recursive sequence

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{2 + \sqrt{x_n}} \text{ for } n = 1, 2, 3, \ldots.$$

Prove using induction that

$$x_n < x_{n+1} < 2.$$

Since the sequence is increasing and bounded, it has limit p. Show that p is a solution of the equation $x^4 - 4x^2 - x + 4 = 0$. By plugging in the value 1, we see that 1 is also a solution of this equation. Explain why $p \neq 1$. By factoring $x - 1$ from $x^4 - 4x^2 - x + 4$, find a cubic polynomial that has p as its root.

42. Explain why the sequence $x_n = \arctan n$ converges. Find its limit and give an intuitive reason for your answer.

43. Take the recursive sequence given by

$$x_1 = 2 \text{ and } x_{n+1} = (x_n^2 + 1)/2 \text{ for } n = 1, 2, 3, \ldots.$$

Show that this sequence increases. If this sequence is bounded, what must the limit be? Explain why such a limit is impossible. Is this sequence bounded?

44. Imitate the algorithm of Example 17 to estimate $\sqrt{50}$ as a fraction. Keep going with your algorithm until your estimate agrees with the calculator value of $\sqrt{50}$ to 3 decimals.