

Second Order Linear Differential Equations

§12.1. Homogeneous Equations

A **differential equation** is a relation involving variables x, y, y', y'', \dots . A **solution** is a function $f(x)$ such that the substitution $y = f(x)$, $y' = f'(x)$, $y'' = f''(x)$, ... gives an identity. The differential equation is said to be **linear** if it is linear in the variables y, y', y'', \dots . We have already seen (in section 6.4) how to solve first order linear equations; in this chapter we turn to second order linear equations with constant coefficients. The general form of such an equation is

$$(12.1) \quad y'' + ay' + by = g(x) ,$$

where a and b are constants, and $g(x)$ is a differentiable function of x . In chapter 6.4, we saw that a first order equation has a one-parameter family of solutions, and that the specification of an initial condition $y(x_0) = y_0$ uniquely determines a solution. In the case of second order equations, the basic theorem is this:

Theorem 12.1 *Given x_0 in the domain of the differentiable function g , and numbers y_0, y'_0 , there is a unique function $f(x)$ which solves the differential equation (12.1) and satisfies the initial conditions $f(x_0) = y_0, f'(x_0) = y'_0$.*

In this section we shall see how to completely solve equation (12.1) when the function on the right hand side is zero:

$$(12.2) \quad y'' + ay' + by = 0 .$$

This is called the **homogeneous equation**. An important first step is to notice that if $f(x)$ and $g(x)$ are two solutions, then so is the sum; in fact, so is any linear combination $Af(x) + Bg(x)$. Thus, once we know two solutions (they must be *independent* in the sense that one isn't a constant multiple of the other) we can solve the initial value problem in theorem 12.1 by solving for A and B .

Example 12.1 Solve $y'' + y = 0$, $y(0) = 4$, $y'(0) = -1$.

Now, we know that $\cos x$ and $\sin x$ are solutions of the equation, so we try a solution of the form $y(x) = A \cos x + B \sin x$. Evaluating at $x = 0$, we find that $A = 4$. Differentiate, getting $y'(x) = -A \sin x + B \cos x$, and evaluating at $x = 0$, we find $B = -1$. Thus the solution is $y(x) = 4 \cos x - \sin x$.

The reason the answer worked out so easily is that $y_1 = \cos x$ is the solution with the particular initial values $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2 = \sin x$ is the solution with $y_2(0) = 0$, $y_2'(0) = 1$. Then the solution with initial values $y(0)$ and $y'(0)$ is

$$(12.3) \quad y(x) = y(0)\cos x + y'(0)\sin x$$

Example 12.2 Solve $y'' - y = 0$, with given initial values $y(0), y'(0)$.

Now e^x and e^{-x} are solutions of this differential equation, so the general solution is a linear combination of these. But we won't have as easy a time finding a solution like (12.3), since these functions do not have the initial values 1, 0; 0, 1 respectively. However if we introduce the functions

$$(12.4) \quad \cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \sinh x = \frac{1}{2}(e^x - e^{-x})$$

these do have the right initial values:

$$(12.5) \quad \cosh 0 = 1, \quad \sinh 0 = 0$$

$$(12.6) \quad \frac{d}{dx}(\cosh x) = \sinh x, \quad \frac{d}{dx}(\sinh x) = \cosh x$$

so $(\cosh)'(0) = 0$, $(\sinh)'(0) = 1$. Thus, the solution to our problem is

$$(12.7) \quad y(x) = y(0)\cosh x + y'(0)\sinh x.$$

This particular differential equation comes up so often that it is important to remember these functions, $\cosh x$, $\sinh x$, called the **hyperbolic functions** and their basic properties: equation (12.6) and

$$(12.8) \quad \cosh^2 x - \sinh^2 x = 1.$$

Because of (12.8) these functions parametrize the standard hyperbola (and it is for this reason that they are called hyperbolic functions).

We now return to the general second order equation.

Proposition 12.1 Let r be a root of the equation

$$(12.9) \quad r^2 + ar + b = 0.$$

Then e^{rx} is a solution to the homogeneous equation:

$$(12.10) \quad y'' + ay' + by = 0.$$

Equation (12.9) is called the **auxiliary equation** of the differential equation (12.10). To verify the proposition, let $y = e^{rx}$ so that $y' = re^{rx}$, $y'' = r^2e^{rx}$. Substituting into equation (12.10):

$$(12.11) \quad r^2e^{rx} + are^{rx} + be^{rx} = e^{rx}(r^2 + ar + b) = 0$$

if and only if r is a root of the auxiliary equation.

Now unfortunately a quadratic equation does not necessarily always have two real roots, so we have to examine the cases separately.

Case of two real roots. If the discriminant $a^2 - 4b > 0$, then there are two real roots, and it is straightforward to find the solution of the corresponding initial value problem.

Example 12.3 Solve $y'' + 6y' + 5y = 0$, $y(0) = 4$, $y'(0) = -1$.

The auxiliary equation, $r^2 + 6r + 5 = 0$ has the roots $r = -1, -5$, so e^{-x} and e^{-5x} are solutions. The general solution is

$$(12.12) \quad y = Ae^{-x} + Be^{-5x} \quad \text{with derivative} \quad y' = -Ae^{-x} - 5Be^{-5x}.$$

Evaluating at $x = 0$, we have $4 = A + B$, $-1 = -A - 5B$. Solving this pair of equations, we get $A = 19/4$ and $B = -3/4$, so our solution is

$$(12.13) \quad y = \frac{19}{4}e^{-x} - \frac{3}{4}e^{-5x}$$

Example 12.4 A function $x = x(t)$ satisfies the differential equation

$$(12.14) \quad x'' - 2x' - 15x = 0.$$

Under what conditions on the values of x at $t = 0$ will this function decay to 0 as $t \rightarrow \infty$?

The auxiliary equation $r^2 - 2r - 15$ has the roots $r = -3, 5$. Thus the general solution is $x(t) = Ae^{-3t} + Be^{5t}$. This will decay at infinity only if $B = 0$. Now, evaluating x and x' at 0 gives us the equations

$$(12.15) \quad x(0) = A + B, \quad x'(0) = -3A - 5B.$$

Setting $B = 0$, the condition becomes $x'(0) + 3x(0) = 0$.

Case of complex roots. If the discriminant $a^2 - 4b < 0$, then the roots are two complex conjugate numbers $\alpha + i\beta$, $\alpha - i\beta$.

Let's first look at the case $y'' + y = 0$. Then the roots of $r^2 + 1 = 0$ are $\pm i$, and we'd like to say that the solutions are the functions e^{ix} , e^{-ix} . This does work, and all the algebra in the case of real roots works just as well in this case, once we have given these expressions meaning. First of all, remember equation (12.3): the general solution of $y'' + y = 0$ is

$$(12.16) \quad y(x) = y(0)\cos x + y'(0)\sin x$$

If $y(x) = e^{ix}$ is to represent a solution of this differential equation, we have $y(0) = e^0 = 1$, and $y'(0) = ie^0 = i$, so we must have

$$(12.17) \quad e^{ix} = \cos x + i\sin x$$

Notice that if we differentiate this expression, we get

$$(12.18) \quad -\sin x + i\cos x = i(\cos x + i\sin x),$$

so this expression is consistent with the differentiation rule for the exponential:

$$(12.19) \quad \frac{d}{dx}e^{ix} = ie^{ix}.$$

In fact, defining the complex exponential by (12.17) is consistent with all the rules for exponentials. In particular, if we substitute the Maclaurin series for all the functions in (12.17) we get an identity:

$$(12.20) \quad \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Proposition 12.2 For a complex number $\alpha + i\beta$ if we define the exponential function as

$$(12.21) \quad e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos(\beta x) + i \sin(\beta x)),$$

then all the usual laws of exponents carry through.

Now, of course, we are interested only in real-valued functions. What we have shown is that if $\alpha \pm i\beta$ are the roots of the equation $r^2 + ar + b = 0$, then the functions $e^{(\alpha \pm i\beta)x}$ solve the differential equation $y'' + ay' + b = 0$. But then the real and imaginary parts of this function satisfy the equation as well, which gives us the desired two real-valued solutions.

Proposition 12.3 If the auxiliary equation for the differential equation

$$(12.22) \quad y'' + ay' + b = 0$$

has the complex roots $\alpha \pm i\beta$, then every solution of the differential equation is of the form

$$(12.23) \quad Ae^{\alpha x} \cos(\beta x) + Be^{\alpha x} \sin(\beta x) = e^{\alpha x}(A \cos(\beta x) + B \sin(\beta x))$$

In solving initial value problems, we can work with the complex solutions or solutions of the form (12.23); usually the latter is more convenient.

Example 12.5 Find the general solution $x = x(t)$ of $x'' + \alpha^2 x = 0$.

Since the roots of the auxiliary equation $r^2 = -\alpha^2 = 0$ are $\pm i\alpha$, the general solution is

$$(12.24) \quad x(t) = A \cos \alpha t + B \sin \alpha t$$

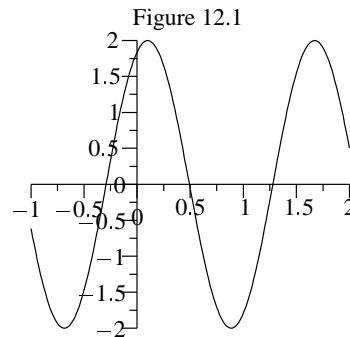
It is easy to see what this function looks like by defining

$$(12.25) \quad C = \sqrt{A^2 + B^2}, \quad \gamma = \arctan(B/A)$$

Then (12.24) becomes

$$(12.26) \quad x(t) = C(\cos \gamma \cos \alpha t + \sin \gamma \sin \alpha t) = C \cos(\alpha x - \gamma) = C \cos(\alpha(x - \gamma/\alpha)).$$

Thus the graph of $x = x(t)$ is a simple cosine curve of amplitude C , and period $2\pi/\alpha$, shifted to the right by the phase γ . (See figure 12.1).



Example 12.6 Find the solution $y = y(x)$ of $y'' + 2y' + 5y = 0$, with the initial values $y(0) = 2$, $y'(0) = -1$.

The auxiliary equation $r^2 + 2r + 5 = 0$ has the solutions $r = -1 \pm 2i$. Thus the general solution is $y = e^{-x}(A \cos(2x) + B \sin(2x))$. To solve for A and B using the initial values we must first differentiate y :

$$(12.27) \quad y' = -e^{-x}(A \cos(2x) + B \sin(2x)) + e^{-x}(-2A \sin(2x) + 2B \cos(2x)) .$$

Substituting the initial values gives the equations $A = 2$, $-A + 2B = -1$, which has the solutions $A = 2$, $B = 1/2$. The answer thus is

$$(12.28) \quad y = e^{-x} \left(2 \cos(2x) - \frac{1}{2} \sin(2x) \right) .$$

Case of a double root. If the discriminant $a^2 - 4b = 0$, then the auxiliary equation has one root r , which gives us only one solution e^{rx} of the differential equation. We find another solution by the technique of variation of parameters. We try $y = ue^{rx}$, where u is a new unknown function. Now, the differential equation is

$$(12.29) \quad y'' - 2ry' + r^2y = 0 .$$

Substituting this y in the equation we get to

$$(12.30) \quad y'' - 2ry' + r^2y = e^{rx} [(u'' + 2u'r + ur^2) - 2r(u' + ur) + r^2u] = e^{rx}u'' = 0$$

Thus $u = Ax + B$.

Proposition 12.4 *If the auxiliary equation for the differential equation*

$$(12.31) \quad y'' + ay' + b = 0$$

has only the root r , then every solution is of the form

$$(12.32) \quad (Ax + B)e^{rx}$$

Example 12.7 Find the solution of $y'' - 4y' + 4y = 0$, with initial values $y(0) = 2$, $y'(0) = -1$.

The auxiliary equation has just the root $r = 2$. The general solution is $y = (Ax + B)e^{2x}$, with derivative $y' = 2(Ax + B)e^{2x} + Ae^{2x}$. Substituting the initial conditions gives the equations

$$(12.33) \quad 2 = B \quad -1 = 2B + A .$$

Thus $A = -5$, $B = 2$ and the answer is

$$(12.34) \quad y = (-5x + 2)e^{2x} .$$

Figure 12.2

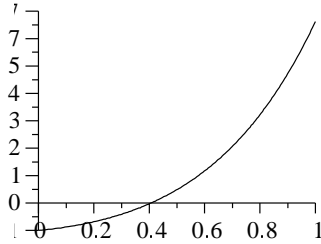


Figure 12.3

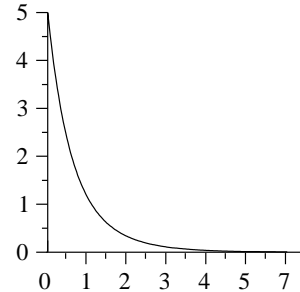


Figure 12.4

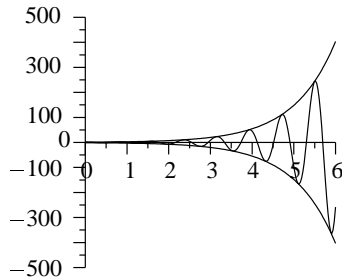
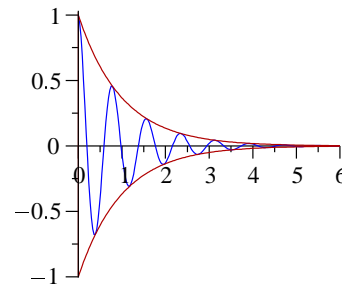


Figure 12.5



§12.2. Behavior of the Solutions

When these differential equations come up in applications, one usually wants to have some idea of the long term behavior of the solution. Since the roots of the auxiliary equation determine the solutions, they also determine their behavior. Here we summarize the results.

Both roots positive. Except for the identically zero solution, all solutions grow exponentially.

Both roots negative. Except for the identically zero solution, all solutions decay exponentially.

A negative and a positive root. All solutions grow exponentially, except for the multiples of the exponential with the negative root.

Both roots imaginary. This is the case of the equation with $a = 0$ and $b > 0$, which we can write as

$$(12.35) \quad y'' + \alpha^2 y = 0,$$

As we saw in example 12.5, the general solution can be written as

$$(12.36) \quad y(x) = A \cos(\alpha x) + B \sin(\alpha x) \quad \text{or} \quad y(x) = C \cos(\alpha x - \gamma),$$

an oscillation of period $2\pi/\alpha$, amplitude C and phase γ (see figure 12.1).

Complex roots. In this case the roots are of the form $\alpha \pm i\beta$ and the general solution can be written in the form (following the previous discussion)

$$(12.37) \quad y(x) = C e^{\alpha x} \cos(\omega x - \gamma).$$

Thus if $\alpha > 0$, this gives an oscillation with exponentially increasing amplitude, and if $\alpha < 0$, this gives an exponentially damped oscillation.

§12.3. Applications

Springs

Suppose that we place a mass m on the end of a vertically hanging spring, and then set it in motion; how can we describe the subsequent motion? As we have seen in section 5.4, the spring is subject to a restoring force proportional to its displacement from equilibrium. By Newton's second law of motion, this force is ma , where a is the acceleration of the mass m . Letting x represent the downward displacement from equilibrium, we have $a = x''$, and if the spring constant is k , this gives us the equation

$$(12.38) \quad mx'' = -kx, \quad \text{or} \quad x'' + \frac{k}{m}x = 0$$

Letting $\omega = \sqrt{k/m}$, this has the solution (see example 12.5)

$$(12.39) \quad x(t) = C \cos(\omega t - \gamma)$$

where C and γ are to be determined by the initial data.

We have to be a little careful about units. In the metric system, when m is measured in kilograms and x in meters, then force is measured in newtons, and the units for the spring constant k are newtons/meter. On a smaller scale, m is in grams, x in centimeters, force in dynes, and k in dynes/meter. However, in the British system it is customary to refer to the *weight* w of the object (in pounds), rather than its mass. Then $m = w/g$, where $g = 32 \text{ ft/sec}^2$ is the acceleration due to gravity. Finally, in the British system, the spring constant is given in lbs/foot.

Example 12.8 Suppose a mass of 10 g hangs from a spring with spring constant $k = 0.4$ dynes/cm. If the spring is extended an additional 8 cm. and then released, give the equation of subsequent motion.

The initial conditions are that when $t = 0$, $x(0) = 8$, $x'(0) = 0$. We also have $\omega = \sqrt{0.4/10} = \sqrt{0.04} = 0.2$. Thus the solution has the form

$$(12.40) \quad x(t) = C \cos(0.2t + \gamma)$$

We get, from the initial conditions

$$(12.41) \quad 8 = C \cos(-\gamma), \quad 0 = -0.2C \sin(-\gamma)$$

so $\gamma = 0$ and $C = 8$, and the equation of motion is

$$(12.42) \quad x(t) = 8 \cos(0.2t)$$

We could have concluded this more quickly, by observing that the initial conditions tell us that $x = 8$ when the velocity is 0, so the maximum extension (the amplitude) has to be 8.

Example 12.9 Suppose that we come upon the above configuration already in motion, and when we make our observation (at time $t = 0$), the mass is displaced 12 cm downward and is traveling downward at a velocity of 1 cm/sec. Find the equation of motion.

Again, the equation has the general form

$$(12.43) \quad x(t) = C \cos(0.2t - \gamma)$$

and the initial conditions give

$$(12.44) \quad 12 = C \cos(-\gamma), \quad 1 = -0.2C \sin(-\gamma).$$

We solve for C and γ as follows. The equations are

$$(12.45) \quad C \cos(-\gamma) = 12, \quad C \sin(-\gamma) = -5.$$

Adding the squares of both equations gives us $C^2 = 12^2 + 5^2 = 169$, so $C = 13$, and dividing one equation by the other gives

$$(12.46) \quad \tan(-\gamma) = -5/12, \quad \text{so that } \gamma = 0.126\pi$$

and the equation of motion is

$$(12.47) \quad x(t) = 13 \cos(0.2t - 0.126\pi).$$

Example 12.10 If a 10 lb. object is hung from a spring with spring constant $k=9$ lbs/foot and then is given an initial velocity of 24 ft/sec, what is the maximum extent of the spring?

Here $m = 16/32$ and $k = 9$, so we have the spring equation

$$(12.48) \quad \frac{1}{2}x'' + 9x = 0$$

so $x = A \cos(3\sqrt{2}t) + B \sin(3\sqrt{2}t)$. The initial conditions $x(0) = 0$, $x'(0) = 24$ lead to $A = 0$, $B = 8\sqrt{2}$. The solution thus is

$$(12.49) \quad x(t) = 8\sqrt{2} \sin(2\sqrt{2}t)$$

whose maximum value is $8\sqrt{2}$ feet.

Now, let us return to our spring with spring constant k and mass m , and suppose that it is inserted in a viscous fluid which imparts a retarding force proportional to the velocity of the mass. Letting $q > 0$ be the constant in this proportion, we see that equation (12.38) is replaced by

$$(12.50) \quad mx'' = -kx - qx', \quad \text{or } x'' + vx' + \omega^2x = 0$$

where $\omega = \sqrt{k/m}$ and $v = q/m$. The roots of this equation are

$$(12.51) \quad r = \frac{-v \pm \sqrt{v^2 - 4\omega^2}}{2}.$$

If $v > 2\omega$, then the roots are both real and negative, and there is exponential decay with no oscillation. If $v < 2\omega$, then we have complex roots, so there are oscillations. But the real part of the roots is $-v/2 < 0$, so the oscillations are exponentially damped. Thus, if we want to have a good damping effect (as for

example in a shock absorber) we should be sure that v is sufficiently large; that is, that the fluid is very viscous.

Example 12.11 A system consisting of a spring in a viscous fluid is installed so as to absorb the shock on a 100 kg mass. The spring constant is $k=2500$ and the constant of viscosity is $q=600$. Suppose a shock is sustained when the system is in equilibrium imparting an instantaneous velocity of 100 cm/sec. How long will it take for the amplitude of the oscillation to be reduced to 1 cm?

The basic differential equation is $100x'' + 600x' + 2500x = 0$. The roots of the auxiliary equation are $r = -3 \pm 4i$, so the general solution is

$$(12.52) \quad x(t) = e^{-3t}(A \cos 4t + B \sin 4t)$$

Now at $t = 0$, $x = 0$, $x' = 100$. Solving for A and B with those initial conditions, we find $x(t) = 25e^{-3t} \sin 4t$. Now the maximum amplitudes of this damped vibration occur at the values $t = k\pi/8$, for k an odd integer. Here is a table of the first few values:

t	0.3927	1.1781	1.9634
A	7.697	0.729	0.069

Thus the first maximum amplitude occurs at 0.3927 seconds and is 7.697 cm, but by 1.1781 the maximum amplitude is less than 1 cm.

§12.4. The Inhomogeneous Equation

We return now to the general second order equation with a nonzero right hand side;

$$(12.53) \quad y'' + ay' + by = g(x),$$

Proposition 12.5 Suppose that $y = y_p(x)$ is a particular solution of the equation (12.53). Then every solution is of the form $y = y_p + y_h$ where y_h is a solution of the homogeneous equation.

Example 12.12 Find the solution of the initial value problem:

$$(12.54) \quad y'' + y = x + 2, \quad y(0) = 4, \quad y'(0) = 2.$$

It is easy to see that $y_p(x) = x + 2$ is a particular solution of this equation. The general solution thus is of the form

$$(12.55) \quad y(x) = x + 2 + A \cos x + B \sin x$$

To find A and B we use the initial conditions (at $x = 0$):

$$(12.56) \quad 4 = 0 + 2 + A \cos(0), \quad 2 = 1 - A \sin(0) + B \cos(0)$$

giving us $4 = 2 + A$, $2 = 1 + B$, so $A = 2$, $B = 1$, and the solution is $y = x + 2 + 2 \cos x + \sin x$.

Example 12.13 Knowing that $y = -(1/3) \cos(2x)$ solves the differential equation $y'' + y = \cos(2x)$ find the solution with initial values $y(\pi/2) = 1$, $y'(\pi/2) = 3$.

We know the solution has the form $y = -(1/3)\cos(2x) + A\cos x + B\sin x$. Putting in the initial values gives us $1 = 1/3 + B$, $3 = -A$, so the solution is $y = -(1/3)\cos(2x) - 3\cos x + (2/3)\sin x$.

In general, we may neither be given a particular solution, nor can we see one by inspection. In general it is very difficult to find that first particular solution. However, here is one technique (basically trial and error) which leads to a particular solution when the inhomogeneous function is an elementary function.

Undetermined coefficients. To solve $y'' + ay' + by = g(x)$, try a function of the same form as $g(x)$. More precisely:

If g is a polynomial of degree n , try the general polynomial of degree n .

If g is an exponential times a polynomial of degree n , try the general exponential times a polynomial of degree n .

If g is a cosine or a sine times a polynomial of degree n , try the general cosine and sine times a polynomial of degree n .

The reason this works is that successive differentiation keeps us in the same form, so we end up equating coefficients and solving for them. There is one caution: if g is a solution of the homogeneous equation, this will fail. In this case we replace the phrase “polynomial of degree n ” by “polynomial of degree n or more”. Let us illustrate

Example 12.14 Find a particular solution of $y'' + 5y' - y = x^2 - 3x + 4$.

We try $y = ax^2 + bx + c$. First we calculate the first and second derivative: $y' = 2ax + b$, $y'' = 2a$. Substituting these in the given equation we obtain

$$(12.57) \quad 2a + 5(2ax + b) - (ax^2 + bx + c) = x^2 - 3x + 4$$

This simplifies to

$$(12.58) \quad -ax^2 + (10a - b)x + 2a + 5b - c = x^2 - 3x + 4$$

We now equate coefficients:

$$(12.59) \quad -a = 1, \quad 10a - b = -3, \quad 2a + 5b - c = 4$$

giving the solutions $a = -1$, $b = -7$, $c = -41$, so the answer is

$$(12.60) \quad y = -x^2 - 7x - 41.$$

Example 12.15 Find a particular solution of $y'' + y' - y = xe^x$.

Try $y = (ax + b)e^x$. Differentiating, $y' = (ax + a + b)e^x$, $y'' = (ax + 2a + b)e^x$. Substituting in the given equation leads to

$$(12.61) \quad (ax + 2a + b)e^x + (ax + a + b)e^x - (ax + b)e^x = xe^x,$$

reducing to the equations $a = 1$, $3a + b = 0$. Thus the answer is

$$(12.62) \quad y = (x - 3)e^x$$

Example 12.16 Find a particular solution of $y'' + y = \cos x$.

Since $\cos x$ satisfies the homogeneous equation, we must try $y = ax\sin x + bx\cos x$. Then $y'' = 2a\cos x - 2b\sin x - ax\sin x - bx\cos x$, and we obtain the equation

$$(12.63) \quad 2a\cos x - 2b\sin x = \cos x$$

so $a = 1/2$, and the solution is $y = (1/2)x\sin x$.