Exponential & Logarithmic Equations

This chapter is about using the inverses of exponentials or logarithms to solve equations involving exponentials or logarithms.

Solving exponential equations

An exponential equation is an equation that has an unknown quantity, usually called x, written somewhere in the exponent of some positive number. Here are three examples of exponential equations: $e^x = 5$, or $2^{3x-5} = 2$, or $3^{5x-1} = 3^x$. In all three of these examples, there is an unknown quantity, x, that appears as an exponent, or as some part of an exponent.

To solve an exponential equation whose unknown quantity is x, the first step is to make the equation look like $a^{f(x)} = c$ where $f(x)$ is some function, and a and c are numbers. Sometimes the equation will already be set up to look like this, as in the first two examples above of $e^x = 5$ or $2^{3x-5} = 2$. Sometimes, you'll have to use the rules of exponentials to make your equation look like $a^{f(x)} = c$. In the third example from the previous paragraph, we could divide the equation $3^{5x-1} = 3^x$ by 3^x to obtain $3^{5x-1-x} = 1$, which is the same thing as $3^{4x-1} = 1$. (In this last sentence we used the rule of exponentials that a^x divided by a^y equals a^{x-y}.) Now all three of our exponential equations have the form $a^{f(x)} = c$.

Once your equation looks like $a^{f(x)} = c$, you can erase the exponential base a on the left side of the equation by applying its inverse function, $\log_a$, to the right side of the equation. That would leave you with the equation $f(x) = \log_a(c)$. Sometimes you can write the number $\log_a(c)$ as a more familiar number. Sometimes you can't. Either way, it's just a number

If $e^x = 5$, then $x = \log_e(5)$. If $2^{3x-5} = 2$, then $3x - 5 = \log_2(2) = 1$. If $3^{4x-1} = 1$, then $4x - 1 = \log_3(1) = 0$.

At this point in the problem, you might already be finished. If not, you should be able to solve for x using techniques that we've learned or reviewed earlier in the semester. In the three examples above, the answers would be $x = \log_e(5)$, $x = 2$, and $x = \frac{1}{4}$ respectively.

$\log_e(5)$ is a perfectly good number. Just as good as say 17, or $-\frac{2}{3}$. There's no way to simplify it. You should be comfortable with it as an answer.

Steps for solving exponential equations

Step 1: Make the equation look like $a^{f(x)} = c$ where $a, c \in \mathbb{R}$ and $f(x)$ is a function.

Step 2: Rewrite the equation as $f(x) = \log_a(c)$.

Step 3: Solve for x.

Example. Let's solve for x if

$$e^{3x-7} = 5e^{x-1}$$

To perform Step 1, we can divide both sides of the equation by e^{x-1}. We'd be left with

$$\frac{e^{3x-7}}{e^{x-1}} = 5$$

But $\frac{e^{3x-7}}{e^{x-1}} = e^{3x-7-(x-1)} = e^{2x-6}$. So we're really left with

$$e^{2x-6} = 5$$

and that completes Step 1.

Step 2 is to erase the exponential function in base e from the left side of the equation $e^{2x-6} = 5$ by applying its inverse, the logarithm base e, to the right side of the equation. To put it more simply, we rewrite $e^{2x-6} = 5$ as

$$2x - 6 = \log_e(5)$$

Step 3 is to solve the equation $2x - 6 = \log_e(5)$ using algebra. We can do this by adding 6 and then dividing by 2. We'll be left with the answer

$$x = \frac{\log_e(5) + 6}{2}$$

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Solving logarithmic equations

A logarithmic equation is an equation that contains an unknown quantity, usually called x, inside of a logarithm. For example, $\log_2(5x) = 3$, and $\log_{10}(\sqrt{x}) = 1$, and $\log_e(x^2) = 7 - \log_e(2x)$ are all logarithmic equations.

To solve a logarithmic equation for an unknown quantity x, you'll want to put your equation into the form $\log_a(\,f(x)\,) = c$ where $f(x)$ is a function of x and c is a number. The logarithmic equations $\log_2(5x) = 3$ and $\log_{10}(\sqrt{x}) = 1$ are already written in the form $\log_a(\,f(x)\,) = c$, but $\log_e(x^2) = 7 - \log_e(2x)$ isn't. To arrange the latter equality into our desired form, we can use rules of logarithms. More precisely, add $\log_e(2x)$ to the equation and use the logarithm rule that $\log_e(x^2) + \log_e(2x) = \log_e(x^2 2x)$. Then the equation becomes $\log_e(2x^3) = 7$, and that's the form we want our logarithmic equations to be in.

Once your equation looks like $\log_a(\,f(x)\,) = c$, use that the base a exponential is the inverse of $\log_a$ to rewrite your equation as $f(x) = a^c$. You might want to simplify the number that appears as a^c in your new equation, but other than that, you're done with exponentials and logarithms at this point in the problem. It's time to solve the equation using techniques we used earlier in the semester.

Let's look at the three examples above. We would rewrite $\log_2(5x) = 3$ as $5x = 2^3 = 8$. Then we solve our new equation to find that $x = \frac{8}{5}$.

We would rewrite $\log_{10}(\sqrt{x}) = 1$ as $\sqrt{x} = 10^1 = 10$. Since squaring is the inverse of the square root, we are left with $x = 10^2 = 100$.

For the third equation, we had $\log_e(2x^3) = 7$. Rewrite it as $2x^3 = e^7$, and then solve for x to find that $x = \sqrt[3]{\frac{e^7}{2}}$.

Steps for solving logarithmic equations

Step 1: Make the equation look like $\log_a(f(x)) = c$ where $a, c \in \mathbb{R}$ and $f(x)$ is a function.

Step 2: Rewrite the equation as $f(x) = a^c$.

Step 3: Solve for x.

Example. Let's solve for x if
$$\log_e(-x^2 + 2x) = \log_e(x) + 4$$
To perform Step 1, we can subtract $\log_e(x)$ from both sides of the equation to get
$$\log_e(-x^2 + 2x) - \log_e(x) = 4$$

Recall that $\log_e(-x^2 + 2x) - \log_e(x) = \log_e(\frac{-x^2 + 2x}{x}) = \log_e(-x + 2)$. That means that
$$\log_e(-x + 2) = 4$$
That's the end of Step 1.

Step 2 is to erase the logarithm base e from the left side of the equation $\log_e(-x + 2) = 4$ by applying the exponential function of base e to the right side of the equation. That is, we rewrite $\log_e(-x + 2) = 4$ as
$$-x + 2 = e^4$$

Step 3 is to solve the equation $-x + 2 = e^4$ using algebra. Subtracting 2 and multiplying by -1 leaves us with the answer

$$x = 2 - e^4$$

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Exercises

Solve the following exponential equations for x.

1.) $10^{3x} = 1000$

2.) $6(14^x) = 30$

3.) $2e^x = 8$

4.) $e^x + 10 = 17$

5.) $(3^x)^5 = 27$

6.) $5^{-\frac{x}{2}} = \frac{1}{5}$

7.) $5^{3x-4} = 125$

8.) $e^{2x} = \frac{e^{x^2}}{e^2}$

9.) $e^{-x^2} = e^{x+5}e^{-11}$

Solve the following logarithmic equations for x.

10.) $\log_3(x - 5) = 2$

11.) $\log_e(x) = -6$

12.) $\log_e(2x) = 24$

13.) $\log_e(\sqrt{x - 4}) = 5$

14.) $\log_2(x^7) = 28$

15.) $\log_{10}((x + 1)^{-5}) = -15$

16.) $5 + \log_e(x^3) = 11$

17.) $\log_2(-x) - \log_2(-x - 4) = 3$

18.) $\log_2(x - 2) = -3$