Fuzzy Sets (Type-1 and Type-2) and their Applications

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(for self use only)

Why Fuzzy Sets

- It enables one to work in uncertain and ambiguous situations and solve ill-posed problems or problems with incomplete information

Example : Fuzzy Image Processing (Humanlike)

- Human visual system is perfectly adapted to handle uncertain information in both data and knowledge
- It will be hard to define quantitatively how an object , such as a car, has to look in terms of geometrical primitives with exact shapes, dimensions and colors.
- We use descriptive language to define features that eventually are subject to a wide range of variations.

Fuzzy Reasoning and Probability

- They are related , but complimentary to each other.
- Say, for example , if we have to define the probability of appearance of an edge in few frames of images, we have to define, what is an edge. Certain threshold for rate of variation has to be taken, which may not be true for other images or noisy images.
- Fuzzy logic, unlike probability, handles imperfection in the informational content of the event.

Two frameworks for Fuzzy Systems

1) Development based on Crisp mathematical model and fuzzifying some quantities :

Model 1 : Fuzzy Mathematical Model

Example : Fuzzy – K means clustering

2) Development based on Fuzzy Inference rules:

Model 2 : Fuzzy Logical Model

Example : Fuzzy decision Support System

1. Definition of fuzzy set

- 1.1 Concept for fuzzy set
  - **Definition (Membership function of fuzzy set)**

In fuzzy sets, each elements is mapped to [0,1]

by membership function.

$$\mu_A : X \to [0, 1]$$

Where [0,1] means real numbers between 0 and 1 (including 0 and 1).

1 Definition of fuzzy set

- Example

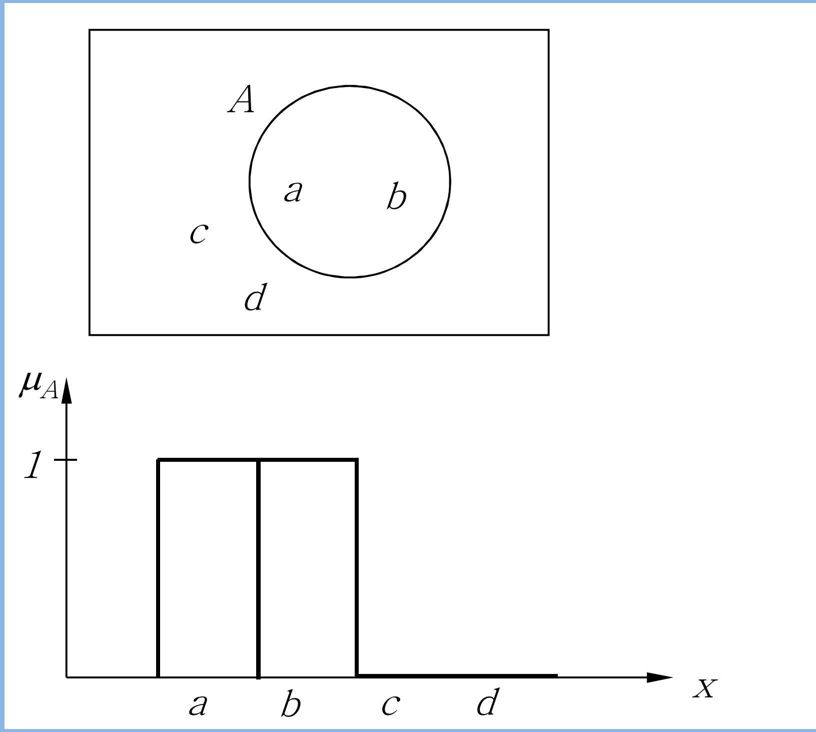


Fig : Graphical representation of crisp set

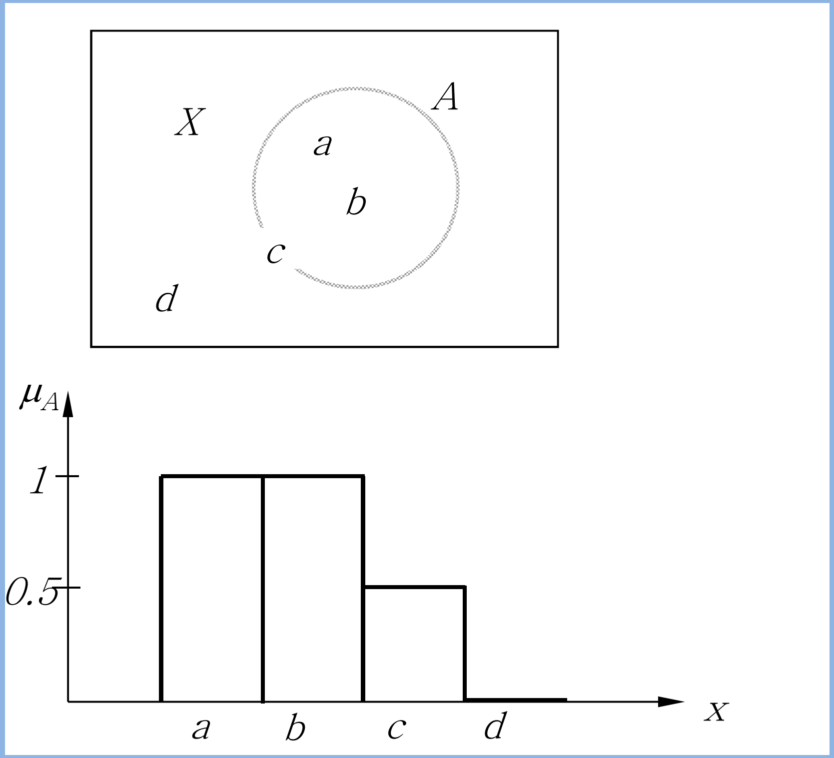


Fig : Graphical representation of fuzzy set

1 Definition of fuzzy set

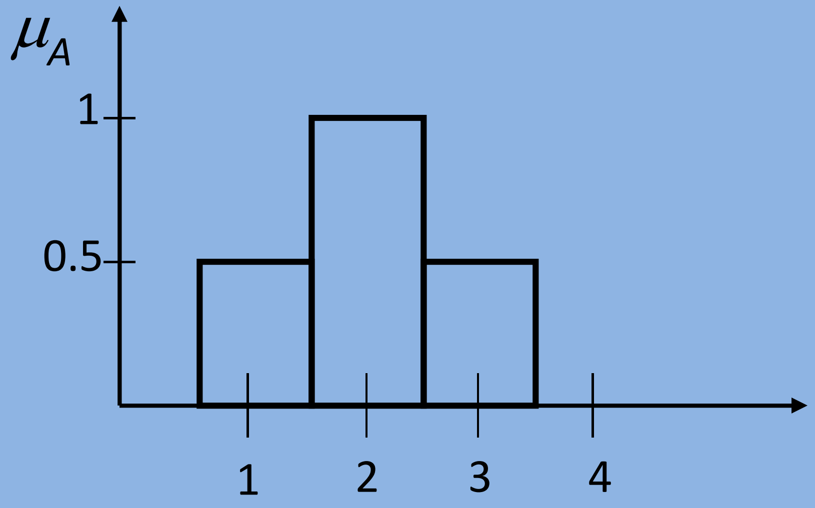
- Example

Consider fuzzy set 'two or so'. In this instance, universal set X are the positive real numbers.

$X = \{1, 2, 3, 4, 5, 6, \dots\}$

- Membership function for A ='two or so' in this universal set *X* is given as follows:

$\mu_A(1) = 0.5,\ \mu_A(2) = 1,\ \mu_A(3) = 0.5,\ \mu_A(4) = 0\dots$



1. Examples of fuzzy set and linguistic terms

- A= "young" , B="very young"

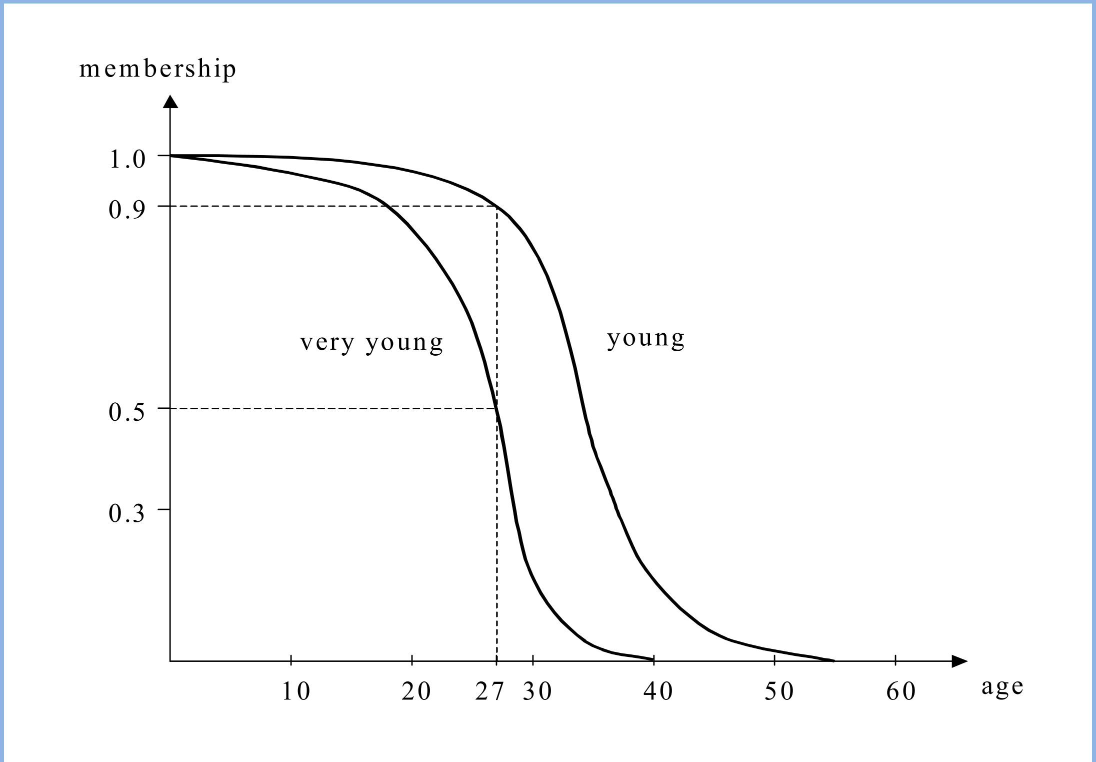


Fig : Fuzzy sets representing “young” and “very young”

1. Examples of fuzzy set

- *A* ={real number near 0}

$$A = \int \mu_A(x)/x \quad \text{where } \mu_A(x) = \frac{1}{1+x^2}$$

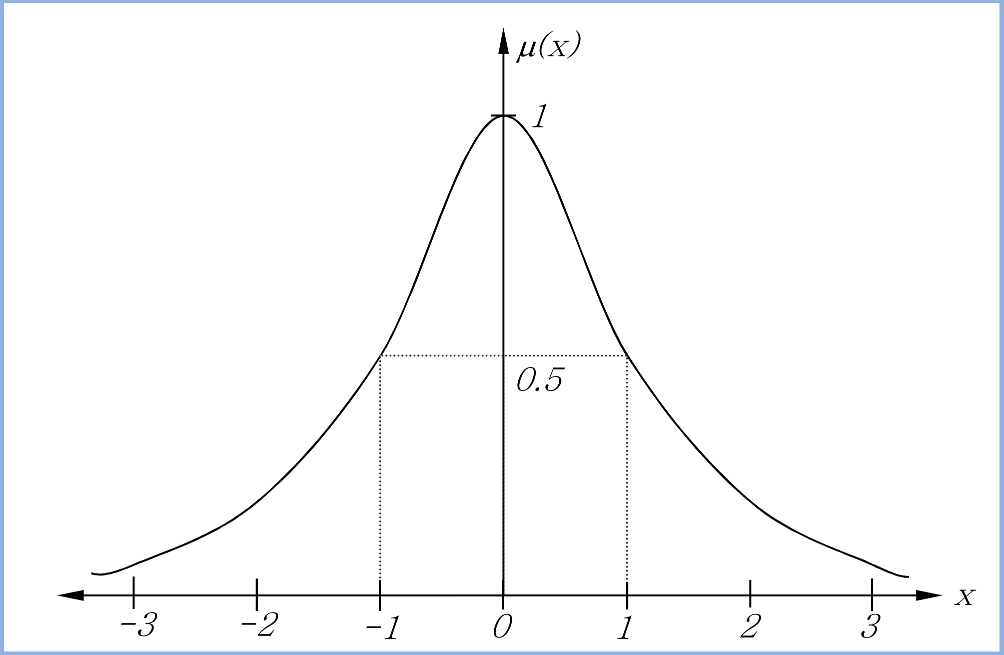


Fig : membership function of fuzzy set "real number near 0"

2. Expansion of fuzzy set

- *Type-n Fuzzy Set*
  - The value of membership degree might include uncertainty. If the value of membership function is given by a fuzzy set, it is a **type-2** fuzzy set.
  - This concept can be extended up to Type-*n* fuzzy set.

Example (Type-n Fuzzy Set)

- Fuzzy sets of type 2: $A : X \to \mathcal{F}([0,1]),$
- $\mathcal{F}([0,1])$: the set of all ordinary fuzzy sets that can be defined with the universal set [0,1].
- $\mathcal{F}([0,1])$ is also called a fuzzy power set of [0,1].

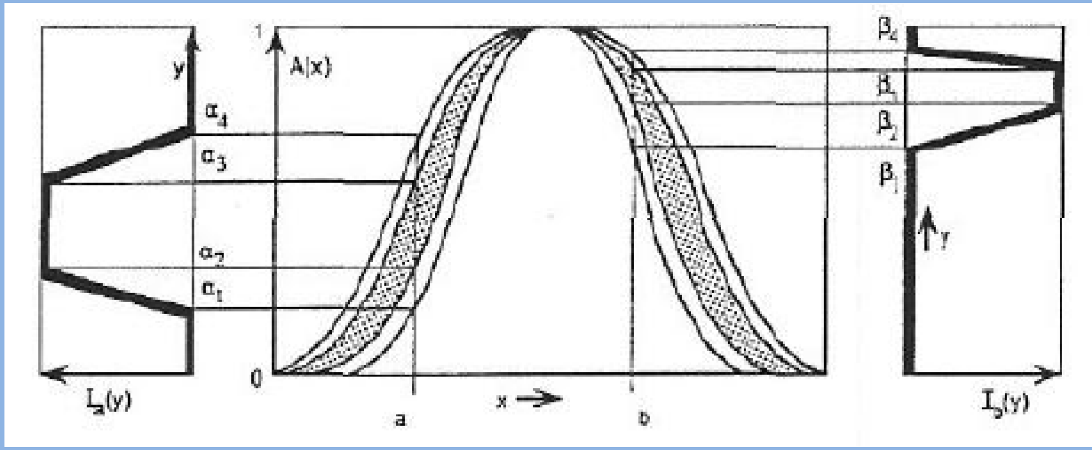


Fig : Fuzzy Set of Type-2

2. Operators: Fuzzy complement

- 2.1 Requirements for complement function
  - Complement function

    $C: [0,1] \rightarrow [0,1]$

    $$\mu_{\overline{A}}(x) = C(\mu_A(x))$$

(Axiom C1) $C(0) = 1,\ C(1) = 0$ (boundary condition)

(Axiom C2) $a, b \in [0,1]$

if $a < b$, then $C(a) \geq C(b)$ (monotonic non-increasing)

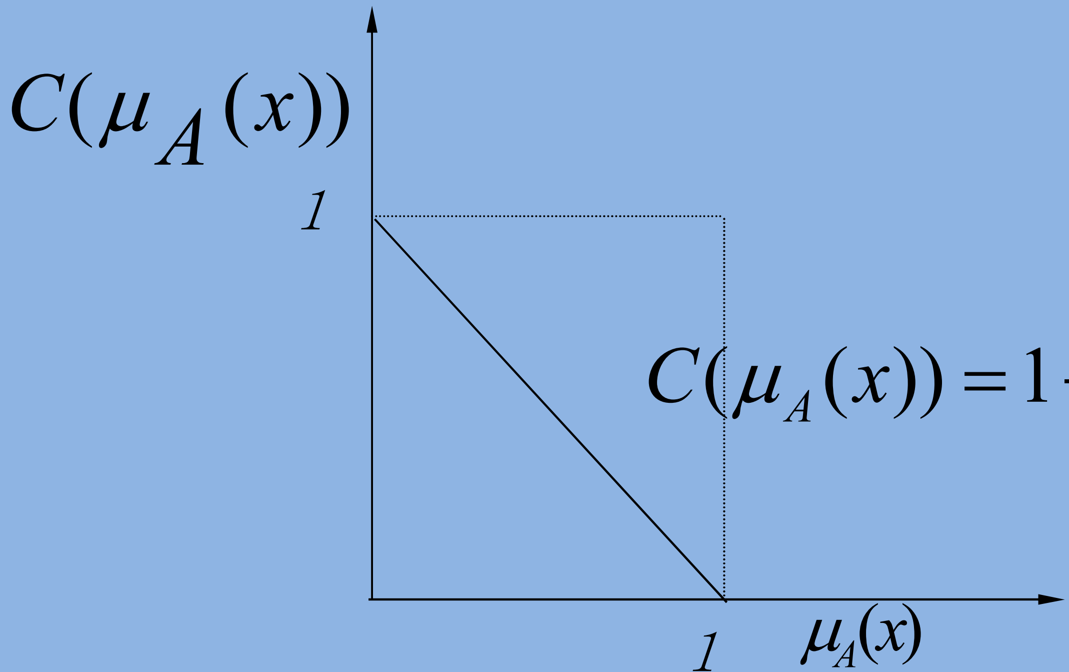
(Axiom C3) C is a continuous function.

(Axiom C4) C is involutive.

$C(C(a)) = a$ for all $a \in [0,1]$

2.1 Fuzzy complement

- **Example of complement function**



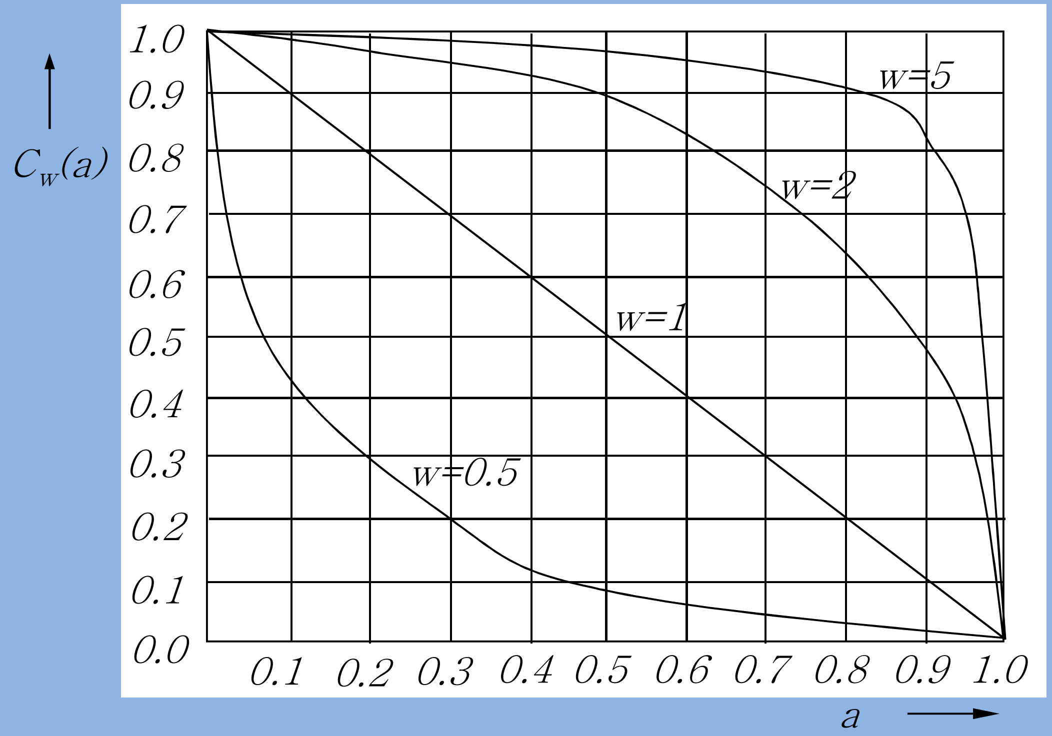
$$C(\mu_A(x)) = 1 - \mu_A(x)$$

Fig : Standard complement set function

2.1 Fuzzy complement

- **Example of complement function**
  - Yager complement function $C_w(a) = (1 - a^w)^{1/w}$

$w \in (-1, \infty)$



2.2 Fuzzy union

- 2.2.1 Axioms for union function

$$U : [0,1] \times [0,1] \rightarrow [0,1]$$

$$\mu_{A \cup B}(x) = U[\mu_A(x), \mu_B(x)]$$

(Axiom U1) $U(0,0) = 0, U(0,1) = 1, U(1,0) = 1, U(1,1) = 1$

(Axiom U2) $U(a,b) = U(b,a)$ (Commutativity)

(Axiom U3) If $a \leq a'$ and $b \leq b'$, $U(a, b) \leq U(a', b')$

Function U is a monotonic function.

(Axiom U4) $U(U(a, b), c) = U(a, U(b, c))$ (Associativity)

(Axiom U5) Function U is continuous.

(Axiom U6) $U(a, a) = a$ (idempotency)

2.2 Fuzzy union

- 2.2.2 Examples of union function

$U[\mu_A(x), \mu_B(x)] = Max[\mu_A(x), \mu_B(x)]$, or $\mu_{A \cup B}(x) = Max[\mu_A(x), \mu_B(x)]$

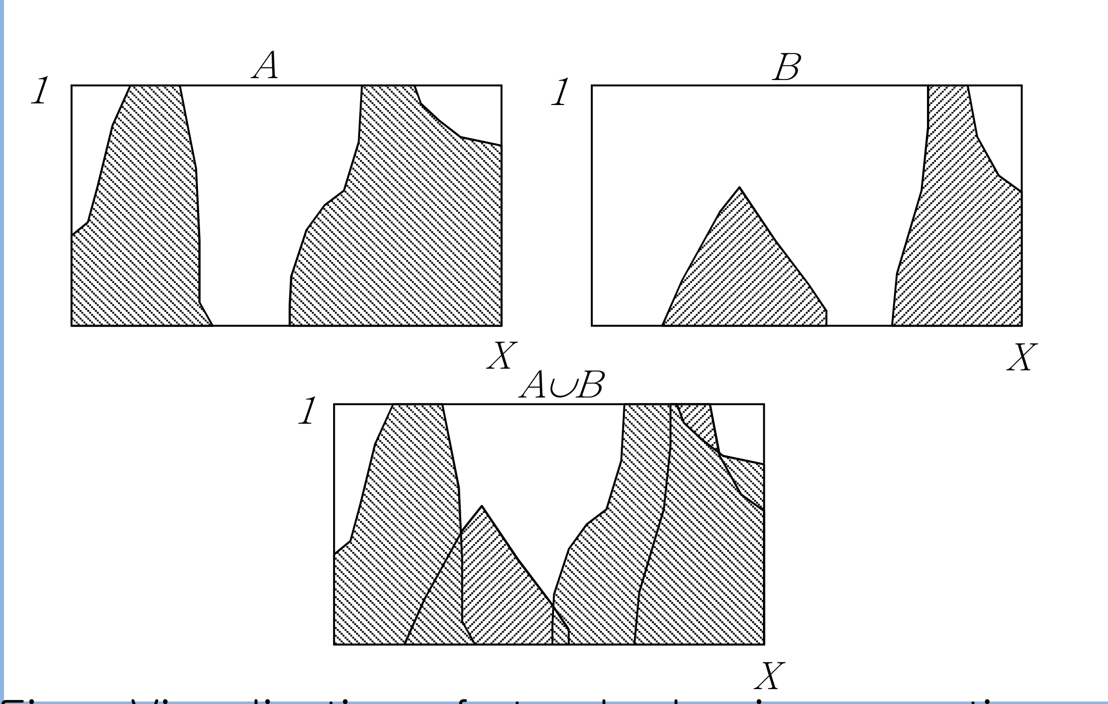


Fig : Visualization of standard union operation

2.2 Fuzzy union

- Yager's union function :

$$U_w(a,b) = \mathrm{Min}[1, (a^w + b^w)^{1/w}] \quad \text{where} \quad w \in (0, \infty)$$

b / a	0	0.25	0.5
1	1	1	1
0.75	0.75	1	1
0.25	0.25	0.5	0.75

$U_1(a,b) = Min[1, a+b]$

$w = 1$

b / a	0	0.25	0.5
1	1	1	1
0.75	0.75	0.79	0.9
0.25	0.25	0.35	0.55

$U_2(a,b) = Min[1, \sqrt{a^2 + b^2}\ ]$

$w = 2$

b / a	0	0.25	0.5
1	1	1	1
0.75	0.75	0.75	0.75
0.25	0.25	0.25	0.5

$U_\infty(a,b) = Max[\ a,\ b]$: standard union function

$w \to \infty$

2.3 Fuzzy intersection

- 2.3.1 Axioms for intersection function

$$I:[0,1] \times [0,1] \rightarrow [0,1]$$

$$\mu_{A \cap B}(x) = I[\mu_A(x), \mu_B(x)]$$

(Axiom I1) $I(1, 1) = 1, I(1, 0) = 0, I(0, 1) = 0, I(0, 0) = 0$

(Axiom I2) $I(a, b) = I(b, a)$, Commutativity holds.

(Axiom I3) If $a \leq a'$ and $b \leq b'$, $I(a, b) \leq I(a', b')$,

Function I is a monotonic function.

(Axiom I4) $I(I(a, b), c) = I(a, I(b, c))$, Associativity holds.

(Axiom I5) I is a continuous function

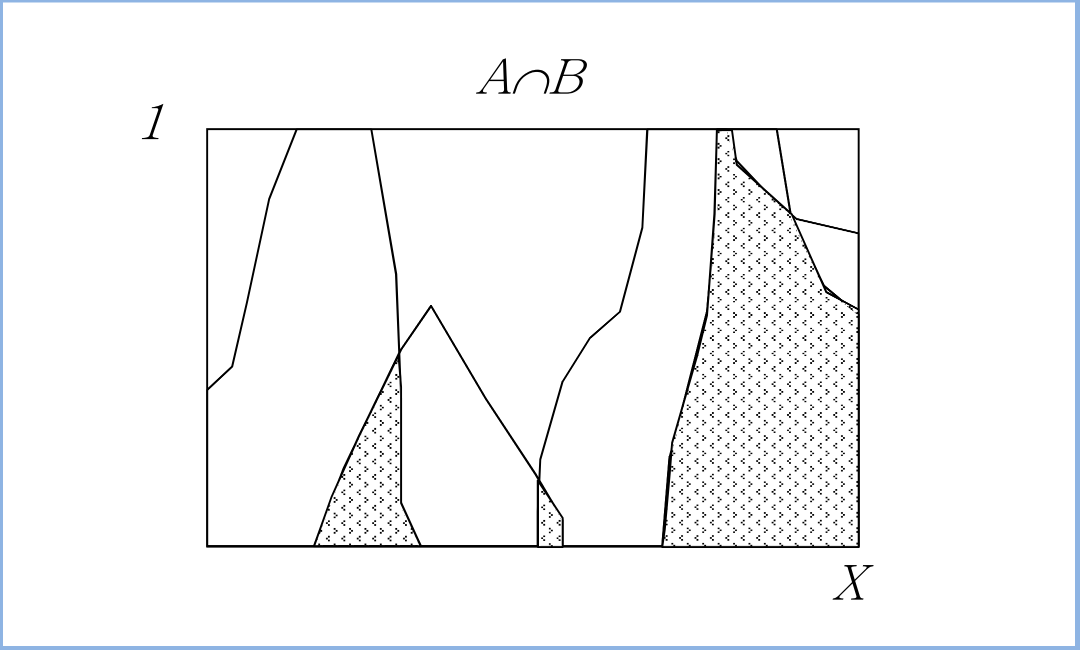
(Axiom I6) $I(a, a) = a$, I is idempotency.

2.3 Fuzzy intersection

- 2.3.2 Examples of intersection
  - standard fuzzy intersection

$I[\mu_A(x), \mu_B(x)] = Min[\mu_A(x), \mu_B(x)]$, or

$\mu_{A\cap B}(x) = Min[\mu_A(x), \mu_B(x)]$



2.3 Fuzzy intersection

- Yager intersection function

$$I_w(a,b) = 1 - \text{Min}[1, ((1-a)^w + (1-b)^w)^{1/w}], w \in (0,\infty)$$

a \ B	0	0.25	0.5
1	0	0.25	0.5
0.75	0	0	0.25
0.25	0	0	0

$I_1(a,b) = 1\text{-Min}[1,\ 2\text{-}a\text{-}b]$

$w = 1$

a \ B	0	0.25	0.5
1	0	0.25	0.5
0.75	0	0.21	0.44
0.25	0	0	0.1

$I_2(a,b) = 1\text{-Min}[1, \sqrt{(1-a)^2 + (1-b)^2}\ ]$

$w = 2$

a	0	0.25	0.5
1	0	0.25	0.5
0.75	0	0.25	0.5
0.25	0	0.25	0.25

$I_\infty(a,b) = \text{Min}[\ a,\ b]$

$w \to \infty$

3. Extension Principle

- **The extension principle is a basic concept of fuzzy set theory that provides a general procedure for extending crisp domains of mathematical expressions to fuzzy domains.**
- **This procedure generalizes an ordinary mapping of a function f to a mapping between fuzzy sets.**

5. Fuzzy Relation and Composition

- Suppose that g is a function from X to Y, and A is a fuzzy set on X defined as

  $A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), . . , (x_n, \mu_A(x_n))\}$

- *Then the extension principle states that the image of fuzzy set A under the mapping f can be* expressed as a fuzzy set B ⊆ Y.
- $B = f(A) = \{(y, \mu_B(y))\}$,

  where $\mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x)$

5.1 Composition of Fuzzy Relations

- We can summarize various kinds of compositions as :

1) Composition of crisp sets A and B. It can represent a relation R between the sets A and B.

$$R = \{(x, y) \mid x \in A, y \in B\}, R \subseteq A \times B$$

2) Composition of fuzzy sets A and B. It is a relation R between fuzzy sets A and B.

$$R = \{((x, y), \mu_R(x, y)) \mid \mu_R(x, y) = \min[\mu_A(x), \mu_B(y)] \text{ or }$$
$$\mu_R(x, y) = \mu_A(x) \bullet \mu_B(y)\}$$

3) Composition of crisp relations R and S

$$S \circ R = \{(x, z) \mid (x, y) \in R, (y, z) \in S\} \text{ where } R \subseteq A \times B, S \subseteq B \times C, \text{ and } S \circ R \subseteq A \times C$$

4) Composition of fuzzy relations R and *S*

$$SR = S \circ R = \{((x, y), \mu_{SR}(x, z))\}$$

where $\mu_{SR}(x, z) = \max_y \min[\mu_R(x, y), \mu_S(y, z)]$

6. Fuzzy Rules and Fuzzy Inference

- R: If *x is A then y is B.,*
- which is sometimes abbreviated as
- R: A → B
- The expression describes a relation between two variables *x and y. This suggests that a fuzzy rule can be defined as a binary relation R on the product space X × Y.*

6.1 Fuzzy Rule

A fuzzy rule can be represented by a fuzzy relation $R = A \rightarrow B$

- R can be viewed as a fuzzy set with a two-dimensional membership function
- $\mu_R(x, y) = f(\mu_A(x), \mu_B(y))$
  where the function f, called the fuzzy implication function, performs the task of transforming the membership degrees of x in A and y in B into those of (x, y) in A × B.
- f is a min operator [Mamdani] and product operator [Larsen]

6.1 Example[2]

- If temperature is high, then humidity is fairly high.

  It is a fuzzy rule and a fuzzy relation.

  To determine the membership function of the rule, let T and H be universe of discourse of temperature and humidity, respectively, and let us define variables $t \in T$ and $h \in H$.

  We represent the fuzzy terms : high, and fairly high. by A and B respectively:

  A = high, $A \subseteq T$ B = fairly high, $B \subseteq H$

  Then the above rule can be rewritten as

  R(t, h): If t is A, then h is B.

  In the rule (relation), we can find two predicate propositions:

  R(t): t is A R(h): h is B

  The rule becomes $R(t, h): R(t) \rightarrow R(h)$

6.1.1 Composition of relations

- The max-min composition

  $R_1 \circ R_2 = \{((x, z), \mu_{R1 \circ R2}(x, z))\}$

  where $\mu_{R1 \circ R2}(x, z) = \max_y \min [\mu_{R1}(x, y), \mu_{R2}(y, z)]$

  $= \bigvee_y [\mu_{R1}(x, y) \wedge \mu_{R2}(y, z)]$

  $x \in X, y \in Y, z \in Z, \quad R_1 \subseteq X \times Y, R_2 \subseteq Y \times Z$

- The max-product composition

  $R_1 \circ R_2 = \{((x, z), \mu_{R1 \circ R2}(x, z))\}$

  where $\mu_{R1 \circ R2}(x, z) = \max_y [\mu_{R1}(x, y) \bullet \mu_{R2}(y, z)]$

  $x \in X, y \in Y, z \in Z, \quad R_1 \subseteq X \times Y, R_2 \subseteq Y \times Z$

Example

t	20	30	40
$\mu_A(t)$	0.1	0.5	0.9

h	20	50	70	90
$\mu_B(h)$	0.2	0.6	0.7	1

	20	50	70	90
20	0.1	0.1	0.1	0.1
30	0.2	0.5	0.5	0.5
40	0.2	0.6	0.7	0.9

Calculated R(t,h) through min operator

Example contd...

- Now suppose, we want to get information about the humidity when there is the following premise about the temperature:

  Temperature is fairly high.

- This fact is rewritten as

  R(t): t is A′ where A′ = fairly high.

  where the fuzzy term $A' \subseteq T$ is defined as below

  Membership function of A′ in T (temperature)

t	20	30	40
$\mu_{A'}(t)$	0.01	0.25	0.81

Example contd

- $R(h) = R(t) \circ R(t, h)$

	20	50	70	90
20	0.1	0.1	0.1	0.1
30	0.2	0.5	0.5	0.5
40	0.2	0.6	0.7	0.9

t	20	30	40
$\mu_{A'}(t)$	0.01	0.25	0.81

h	20	50	70	90
$\mu_{B'}(h)$	0.2	0.6	0.7	0.81

Result of fuzzy inference

Development of Type -1 Fuzzy Systems

1. Fuzzy Mathematical Model

1) Fuzzification of quantities

2) Composition of fuzzy sets

3) Composition of fuzzy relations

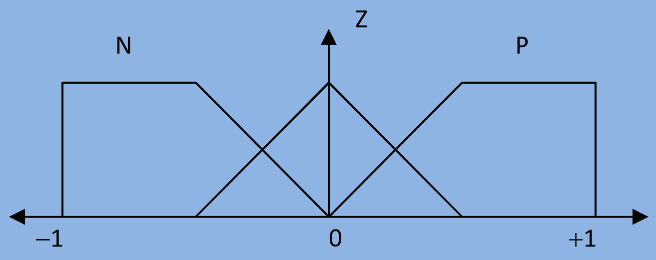
4) Defuzzification of quantities

1.1 Fuzzification of input quantities

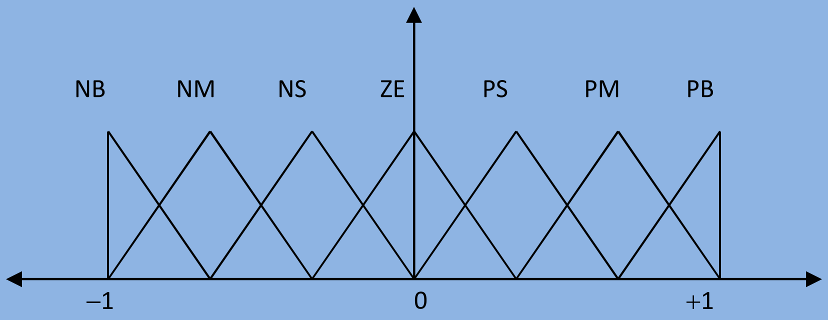
1. discretization and normalization
2. fuzzy partition of spaces
3. membership function of primary fuzzy set

1.2 Fuzzy Partition : Example

- Example of fuzzy partition with linguistic terms



N: negative, Z: zero, P: positive



NB, NM, NS, ZE, PS, PM, PB

7 linguistic terms are often used

- NB: negative big
- NM: negative medium
- NS: negative small
- ZE: zero
- PS: positive small
- PM: positive medium
- PB: positive big

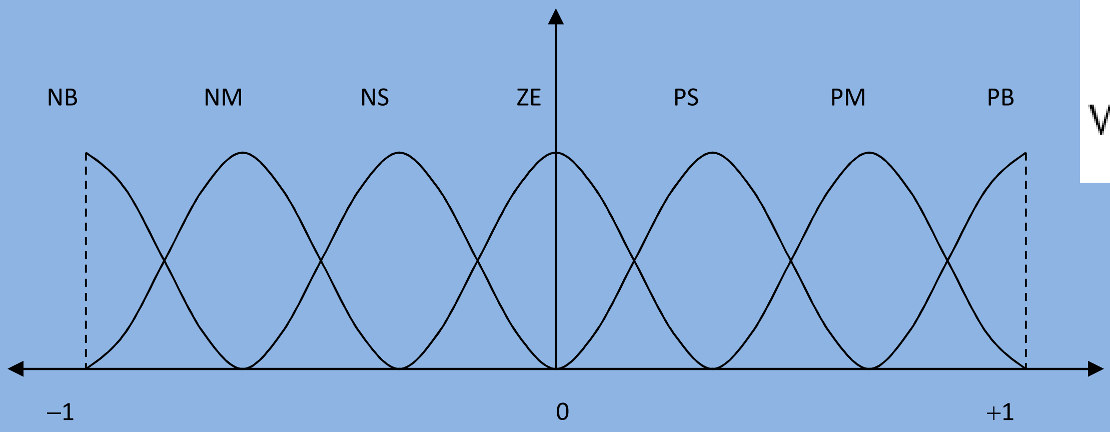
1.2 Fuzzy Membership Function[2]

- Example of bell-shaped membership function

$$\pi(x;\beta,\gamma)=\begin{cases} S(x;\gamma-\beta,\gamma-\beta/2,\gamma) & if \quad x\le\gamma \\ 1-S(x;\gamma,\gamma+\beta/2,\gamma+\beta) & if \quad x>\gamma \end{cases}$$

$$S(x;\alpha,\beta,\gamma)=\begin{cases} 0 \quad if \quad x<\alpha \\ 2\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 \quad if \quad \alpha\le x\le\beta \\ 1-2\left(\frac{x-\gamma}{\gamma-\alpha}\right)^2 if \quad \beta\le x\le\gamma \\ 1 \quad if \quad x>\gamma \end{cases}$$

where $\beta=\frac{\alpha+\gamma}{2}$



1.2 Fuzzy Partition Example

- Fuzzy partition of input and output spaces

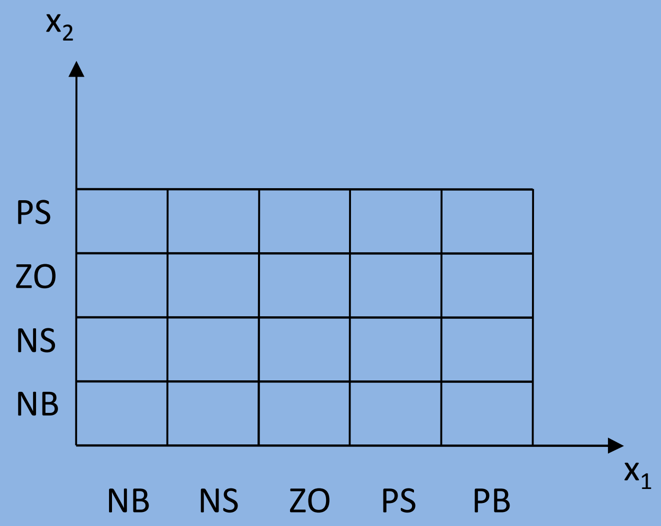


Fig : A fuzzy partition
in 2-dimension input space
the maximum number of control rules = 20 (5x4)

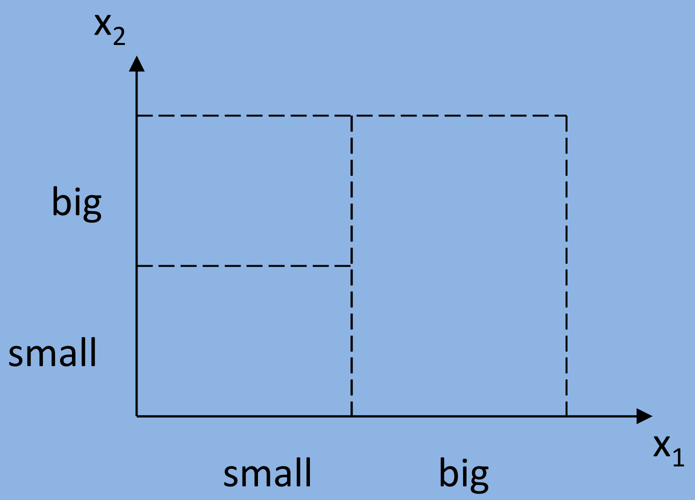


Fig : A fuzzy partition
having three rules

1.3 Defuzzification

- Defuzzification
  - In many practical applications, a control command is given as a crisp value.
  - a process to get a non-fuzzy control action that best represents the possibility distribution of an inferred fuzzy control action.
  - no systematic procedure for choosing a good defuzzification strategy,
  - select one in considering the properties of application case

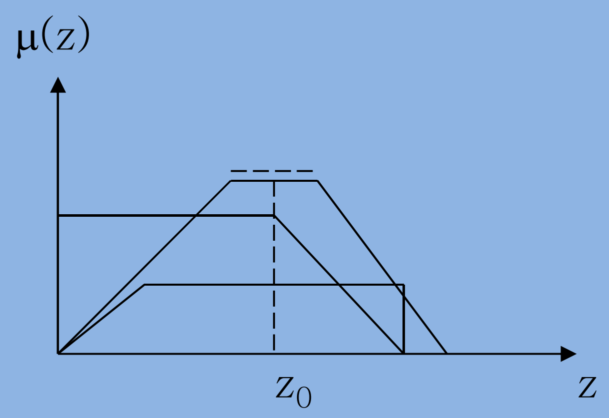
1.3.1 Mean of maximum method (MOM)

- The MOM strategy generates a quantity which represents the mean value of all outputs, whose membership functions reach the maximum.

$$z_0 = \sum_{j=1}^{k} \frac{z_j}{k}$$

z_j: action whose membership functions reach the maximum.

k: number of such actions.



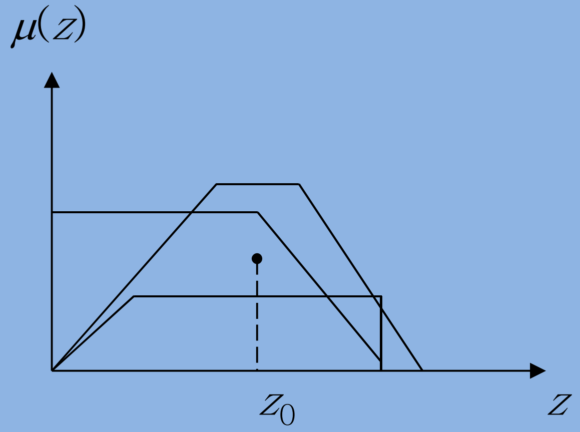
1.3.2 Center of area method (COA)

- The widely used COA strategy generates the center of gravity of the possibility distribution of a fuzzy set *C* .

$$z_0 = \frac{\sum_{j=1}^{n} \mu_C(z_j) \cdot z_j}{\sum_{j=1}^{n} \mu_C(z_j)}$$

n : the number of quantization levels of the output

C : a fuzzy set defined on the output dimension (*z*)

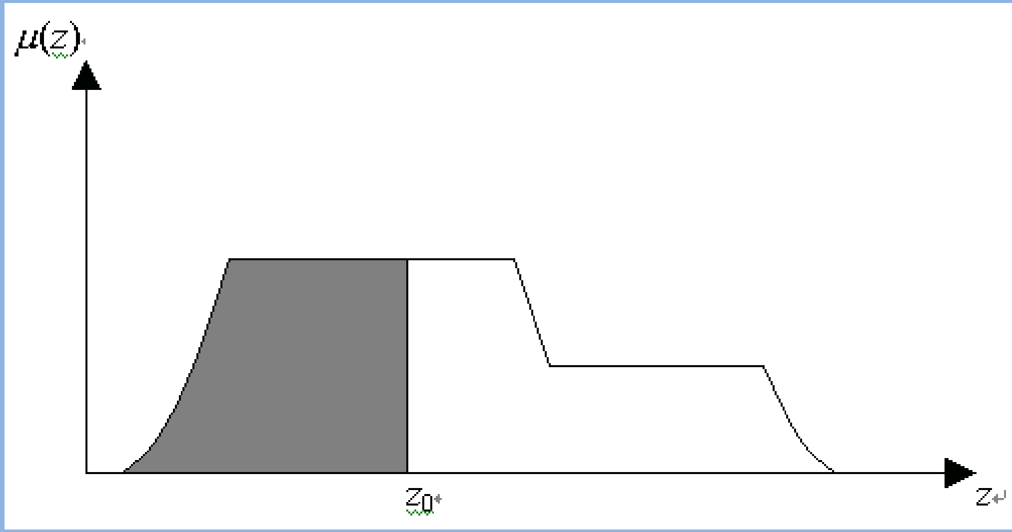


1.3.3 Bisector of area (BOA)

- The BOA generates the action (z0) which partitions the area into two regions with the same area .

$$\int_{\alpha}^{z_0} \mu_C(z)\, dz = \int_{z_0}^{\beta} \mu_C(z)\, dz$$

$\alpha = \min\{z \mid z \in W\}$
$\beta = \max\{z \mid z \in W\}$



Example -1 Fuzzy Image Processing System : Source[5]

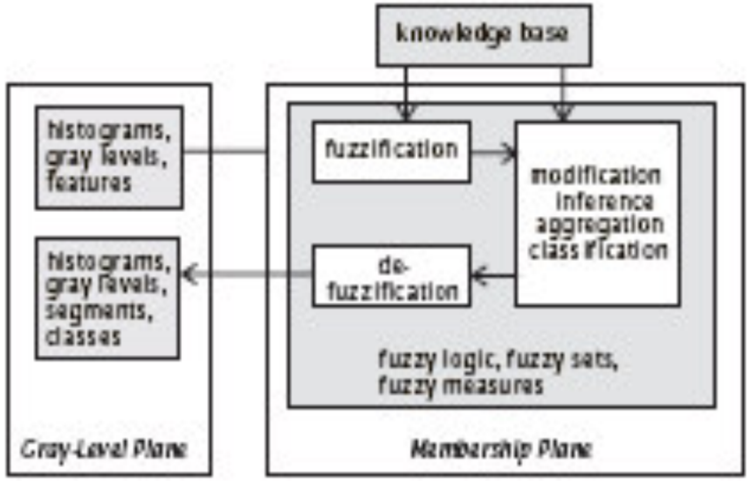


Figure 22.11: General structure of fuzzy image processing systems [29].

Fuzzy Image Processing System

Table 22.1: On relationships between imperfect knowledge and the type of image fuzzification [29].

Problem	Fuzzification	Level	Examples
Brightness ambiguity/ vagueness	histogram	low	thresholding
Geometrical fuzziness	local	intermediate	edge detection, filtering
Complex/ill-defined data	feature	high	recognition, analysis

FIP Systems:

A. Fuzzification: Coding of the Image

B. Processing : Modification, Aggregation, Classification, Modification by IF-Then Rules

C. Defuzzification : Decoding

Example: Fuzzy Image Processing System [5]

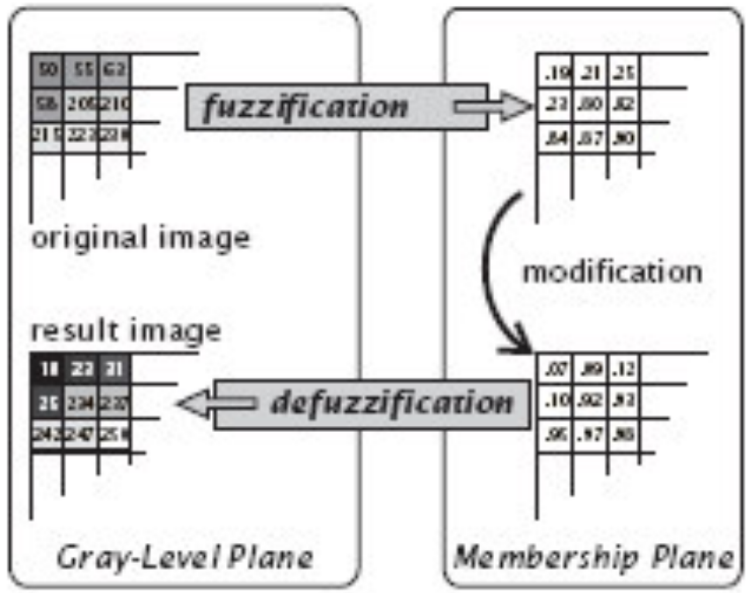


Figure 22.12: Example for modification-based fuzzy image processing [29].

Image Fuzzification[5]

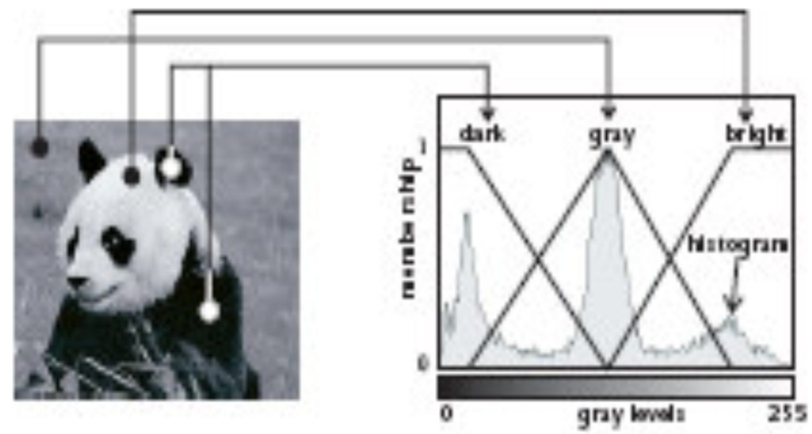


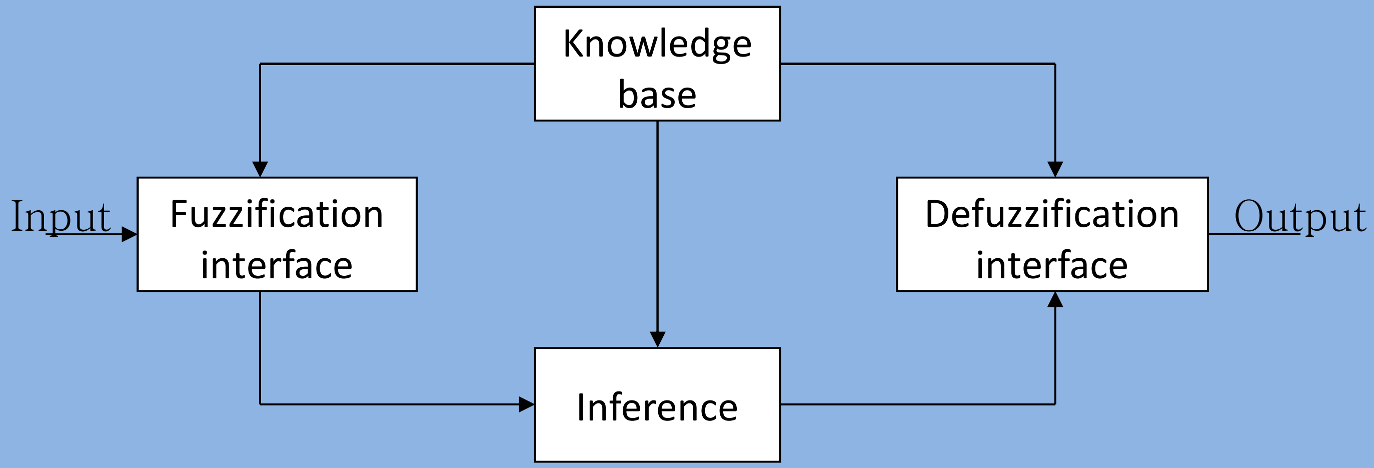
Figure 22.5: Histogram-based gray-level fuzzification. The location of membership functions is determined depending on specific points of image histogram (adapted from [31]).

Image Fuzzification is a nonlinear transformation.

The number, form and location of each membership function should be optimized.

Example 2 (Type 1): Fuzzy logic system

- Configuration of FLS

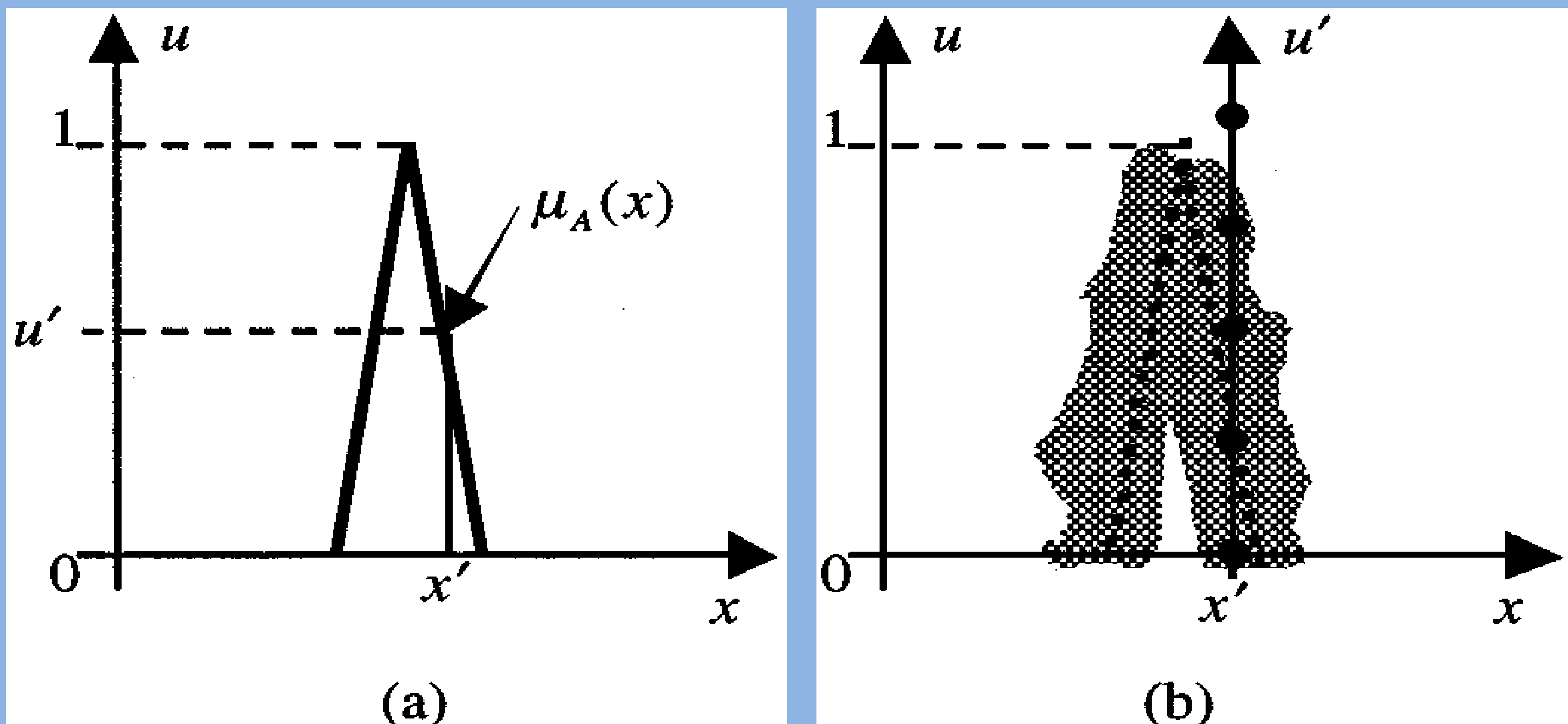


Design of fuzzy logic system

- Four main components
  - (1) The fuzzification interface : transforms input crisp values into fuzzy values
  - (2) The knowledge base : contains a knowledge of the application domain and the goals.
  - (3) The decision-making logic :performs inference for fuzzy control actions
  - (4) The defuzzification interface

Type 2 Fuzzy System Design

Definitions[1]



Why Type-2 Fuzzy sets ?

- Type-2 fuzzy sets allow us to handle linguistic uncertainties, which can be expressed as : “words can mean different things to different people.”
- A fuzzy relation of higher type (e.g., type-2) has been regarded as one way to increase the fuzziness of a relation.
- According to Hisdal, “increased fuzziness in a description means increased ability to handle inexact information in a logically correct manner

Type-2 Fuzzy Sets – definition[1]

- The concept of type-2 fuzzy sets was introduced by Zadeh as an extension of the concept of an ordinary fuzzy set, i.e., a type-1 fuzzy set.
- Type-2 fuzzy sets have grades of membership that are themselves fuzzy . A type-2 membership grade can be any subset in [0,1]—the *primary membership;* and, corresponding to each primary membership, there is a *secondary membership (which can also be in[0,1]) that defines* the possibilities for the primary membership.
- A type-1 fuzzy set is a special case of a type-2 fuzzy set; its secondary membership function is a subset with only one element—unity.

Interval Type-2 Fuzzy Sets [1]

- General type-2 FLSs are computationally intensive because type-reduction is very intensive.

- Things simplify a lot when secondary membership functions (MFs) are interval sets (in this case, the secondary memberships are either zero or one and we call them interval type-2 sets)

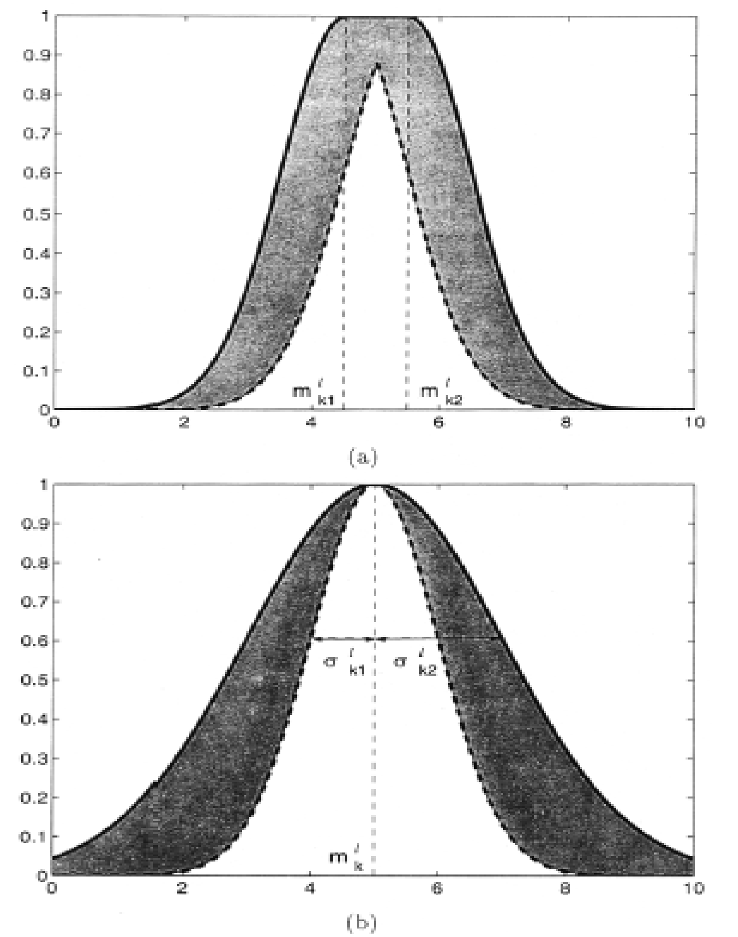


Fig. 1. The type-2 MFs for (a) Example 1 and (b) Example 2. The thick solid lines denote upper MFs and the thick dashed lines denote lower MFs. The shaded regions are the footprints of uncertainty for interval secondaries. In (a), the centers of Gaussian MFs vary from 4.5–5.5; in (b), the center of the Gaussian MFs is 5 and the variance varies from 1.0–2.0.

Type-2 Fuzzy Logic System

- A type-2 FLS includes fuzzifier, rule base, fuzzy inference engine, and output processor.
- The output processor includes type-reducer and defuzzifier; it generates a type-1 fuzzy set output (from the type-reducer) or a crisp number (from the defuzzifier).
- A type-2 FLS is again characterized by IF–THEN rules, but its antecedent or consequent sets are now type-2.

Meet and Join of Interval Sets

- Theorems prove that meet and join operations of interval sets are determined just by the two end-points of each interval set. In a type-2 FLS, the two end-points are associated with two type-1 MFs that we refer to as upper and lower MFs

Upper and Lower Membership functions

- Definition 1 (Footprint of Uncertainty of a Type-2 MF): Uncertainty in the primary membership grades of a type-2 MF consists of a bounded region that we call the footprint of uncertainty of a type-2 MF.
- It is the union of all primary membership grades.

- Definition 2 (Upper and Lower MFs): An upper MF and a lower MF are two type-1 MFs that are bounds for the footprint of uncertainty of an interval type-2 MF. The upper MF is a subset that has the maximum membership grade of the footprint of uncertainty; and the lower MF is a subset that has the minimum membership grade of the footprint of uncertainty

Upper and Lower MF

- The upper MF is a subset that has the maximum membership grade of the footprint of uncertainty; and
- the lower MF is a subset that has the minimum membership grade of the footprint of uncertainty
- We use an overbar (underbar) to denote the upper (lower) MF Liang&Mendel TFS 10-00 #2.pdf

Type-2 FLS : Input and Antecedent Operations

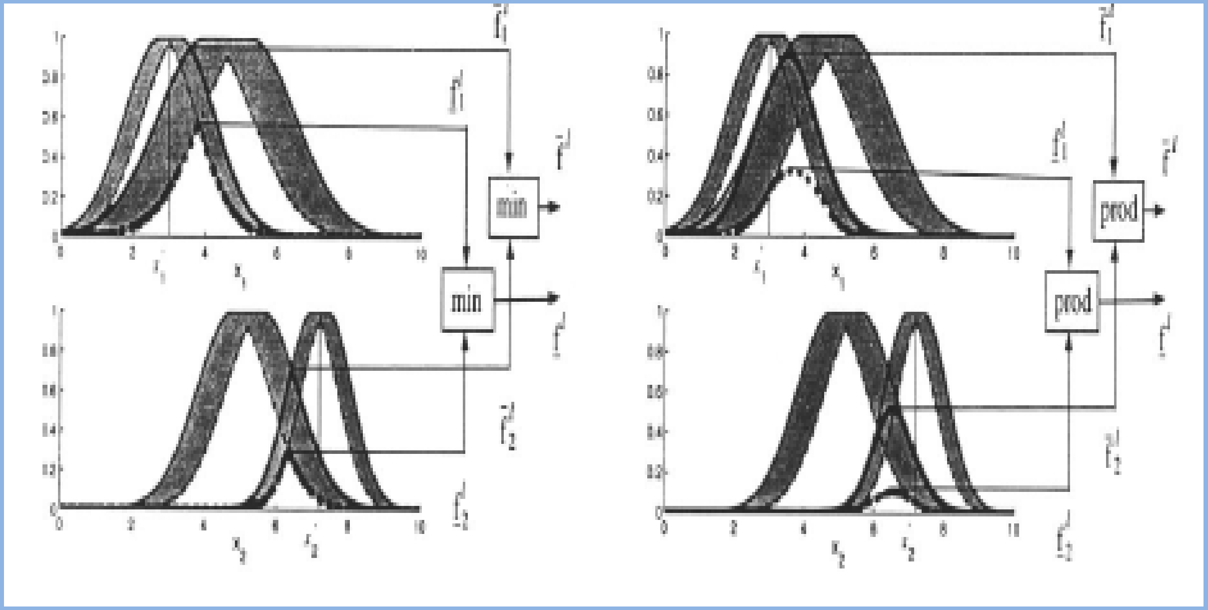


Figure : NS (non-singleton) type-2 fuzzification with minimum t-norm; (left) ; and NS type-2 fuzzification with product t-norm(right). The dark shaded regions depict the meet between input and antecedent

Type Reduction

- Type-reduction was proposed by Karnik and Mendel and by others.
- It is an “extended version” [using the extension principle , of type-1 defuzzification methods and is called type reduction because this operation takes us from the type-2 output sets of the FLS to a type-1 set that is called the “type-reduced set.” This set may then be defuzzified to obtain a single crisp number;
- The Type reduced set may be more important than a single crisp number since it conveys a measure of uncertainties that have flown through the type-2 FLS.
- There exist many kinds of type-reduction, such as centroid, center-of-sets, height, and modified height.

Application of Type-2 Fuzzy Sets

- Type-2 sets and FLSs have been used in decision making , solving fuzzy relation equations, survey processing, time-series forecasting , function approximation, time-varying channel equalization , control of mobile robots , etc.

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