Analysis of Statically Indeterminate Structures by the Displacement Method

Methods of structural analysis

- The methods are classified into two groups:
  1. Force method of analysis (Statics of building structures I)
     - Primary unknowns are forces and moments
     - Deformation conditions are written depending on pre-selected statically indeterminate reactions.
     - The unknown statically indeterminate reactions are evaluated solving these equations.
     - The remaining reactions are obtained from the equilibrium equations.

Methods of structural analysis

- The methods are classified into two groups:
  2. <u>Displacement method of analysis</u>
     - Primary unknowns are displacements.
     - Equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load-displacement relations.
     - Unknown joint displacements are calculated by solving equilibrium equations.
     - In the next step, the unknown reactions are computed from compatibility equations using force displacement relations.

Displacement method

- This method follows essentially the same steps for both statically determinate and indeterminate structures.
- Once the structural model is defined, the unknowns (joint rotations and translations) are automatically chosen unlike the force method of analysis (hence, this method is preferred to computer implementation).

Displacement method

1. Slope-Deflection Method
    - In this method it is assumed that all deformations are due to bending only. Deformations due to axial forces are neglected.
2. Direct Stiffness Method
    - Deformations due to axial forces are not neglected.

The Slope-deflection method was used for many years before the computer era. After the revolution occurred in the field of computing direct stiffness method is preferred.

Slope-Deflection Method: Beams

Slope-Deflection Method: Beams

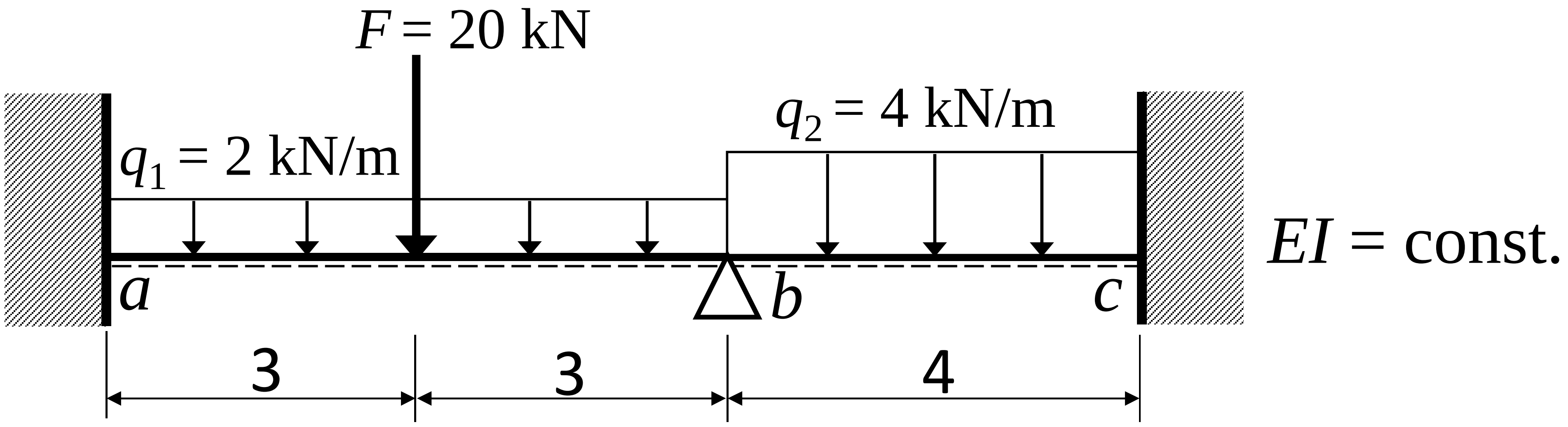
- Application of Slope-Deflection Equations to Statically Indeterminate Beams:
  - The procedure is the same whether it is applied to beams or frames.
  - It may be summarized as follows:

1. Identify all kinematic degrees of freedom for the given problem. Degrees of freedom are treated as unknowns in slope-deflection method.
2. Determine the fixed end moments at each end of the span to applied load (using table).
3. Express all internal end moments by slope-deflection equations in terms of:
   - fixed end moments
   - near end and far end joint rotations

Slope-Deflection Method: Beams

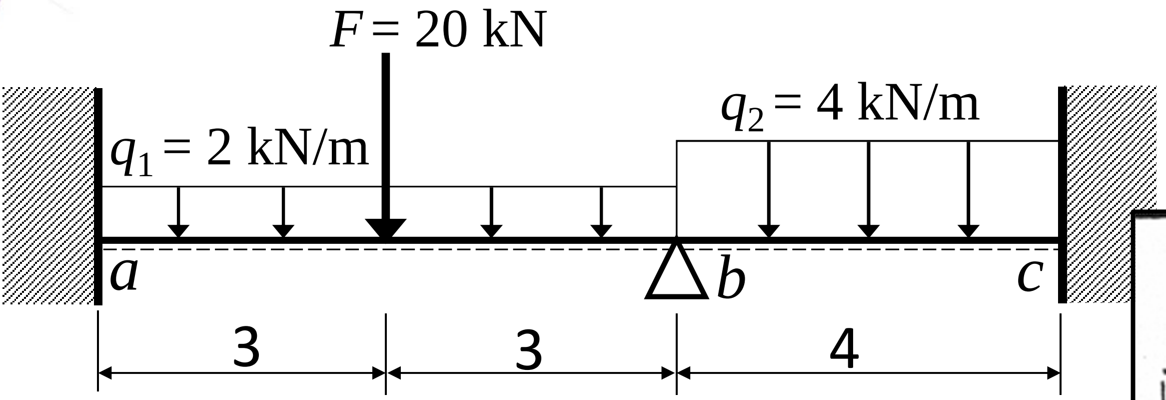
4. Write down one equilibrium equation for each unknown joint rotation. Write down as many equilibrium equations as there are unknown joint rotations. Solve the set of equilibrium equations for joint rotations.
5. Now substituting these joint rotations in the slope-deflection equations evaluate the end moments.
6. Evaluate shear forces and reactions.
7. Draw bending moment and shear force diagrams.

Slope-Deflection Method: Beams Example 1



1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to first degree. Only one joint rotation φ_b is unknown.

Slope-Deflection Method: Beams
Example 1



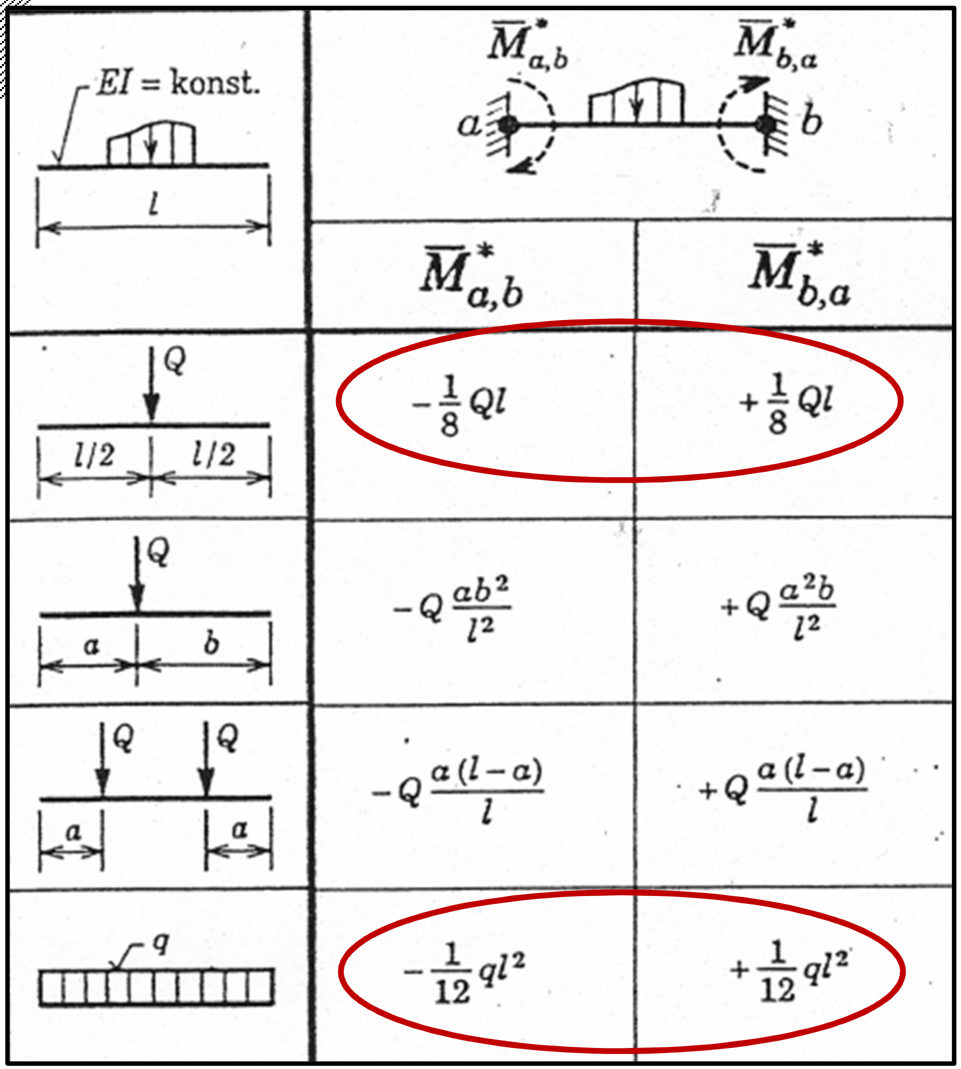
2. Fixed end moments are calculated referring to the table.

$$\overline{M}_{ab} = -\frac{1}{8}\cdot F\cdot l_{ab} - \frac{1}{12}\cdot q_1\cdot l_{ab}^2$$

$$\overline{M}_{ba} = +\frac{1}{8}\cdot F\cdot l_{ab} + \frac{1}{12}\cdot q_1\cdot l_{ab}^2$$

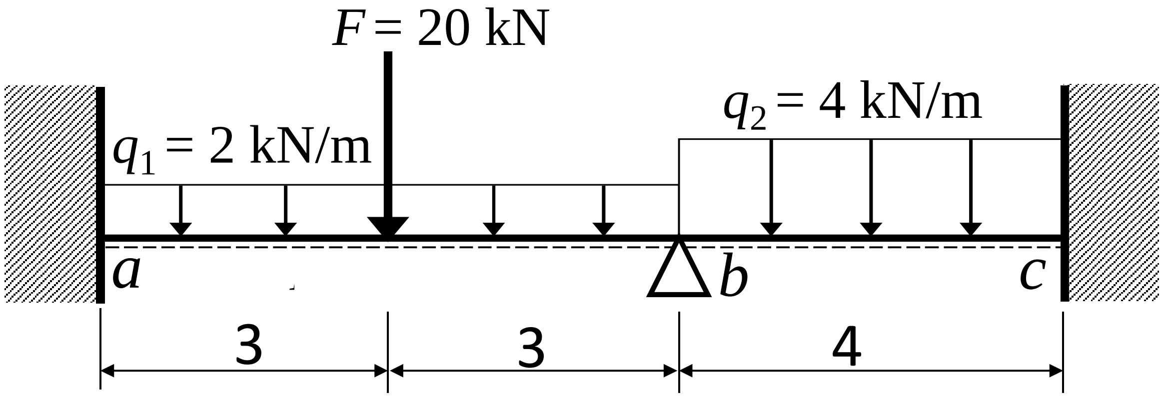
$$\overline{M}_{bc} = -\frac{1}{12}\cdot q_2\cdot l_{bc}^2$$

$$\overline{M}_{cb} = +\frac{1}{12}\cdot q_2\cdot l_{bc}^2$$



EI = konst. l	$\overline{M}^*_{a,b}$ a	$\overline{M}^*_{b,a}$ b
	$\overline{M}^*_{a,b}$	$\overline{M}^*_{b,a}$
Q $l/2$ $l/2$	$-\frac{1}{8}Ql$	$+\frac{1}{8}Ql$
Q a b	$-Q\frac{ab^2}{l^2}$	$+Q\frac{a^2b}{l^2}$
Q Q a a	$-Q\frac{a(l-a)}{l}$	$+Q\frac{a(l-a)}{l}$
q	$-\frac{1}{12}ql^2$	$+\frac{1}{12}ql^2$

Slope-Deflection Method: Beams
Example 1



3. Express internal end moments by slope-deflection equations.

$$M_{ab} = \boxed{\overline{M}_{ab}} + \frac{2\boxed{EI}}{l}(2\cdot\boxed{\varphi_a}+\boxed{\varphi_b})$$

far end joint rotation

fixed end moment

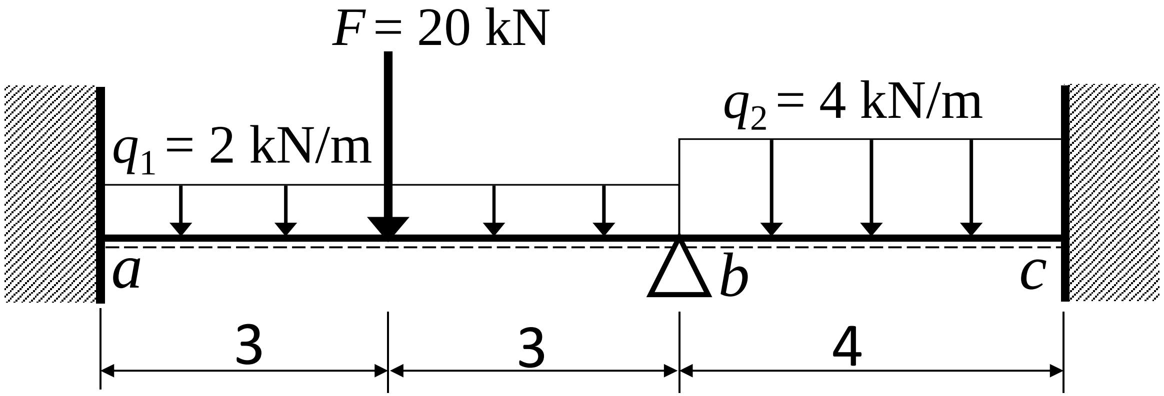
flexural rigidity

near end joint rotation

$$M_{ab} = \overline{M}_{ab} + \frac{2EI}{l_{ab}}(2\cdot\varphi_a+\varphi_b) \qquad M_{bc} = \overline{M}_{bc} + \frac{2EI}{l_{bc}}(2\cdot\varphi_b+\varphi_c)$$

$$M_{ba} = \overline{M}_{ba} + \frac{2EI}{l_{ab}}(2\cdot\varphi_b+\varphi_a) \qquad M_{cb} = \overline{M}_{cb} + \frac{2EI}{l_{bc}}(2\cdot\varphi_c+\varphi_b)$$

Slope-Deflection Method: Beams Example 1



4. Equilibrium equations (write one equilibrium equation for each unknown joint rotation)
    - End moments are expressed in terms of unknown rotation φ_b. Now, the required equation to solve for the rotation φ_b is the moment equilibrium equation at support b.

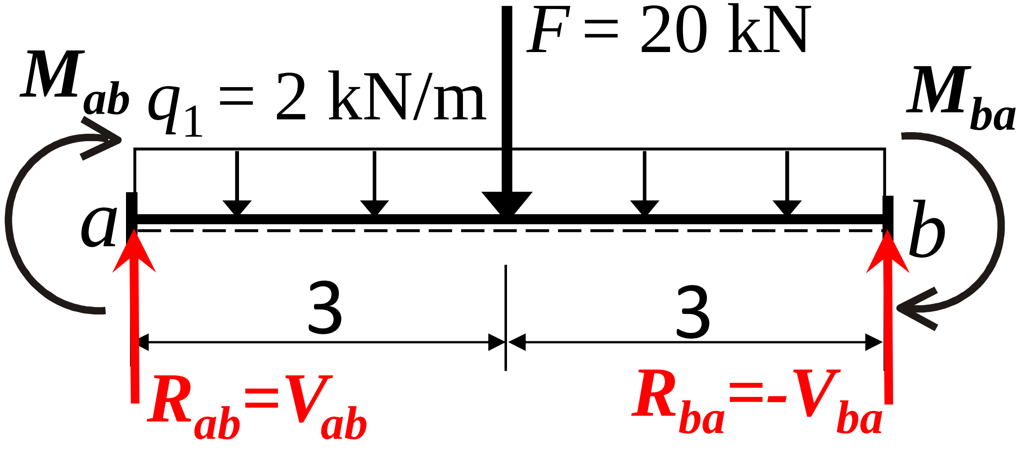
$$\sum M_b = 0 : M_{ba} + M_{bc} = 0 \Rightarrow \varphi_b$$

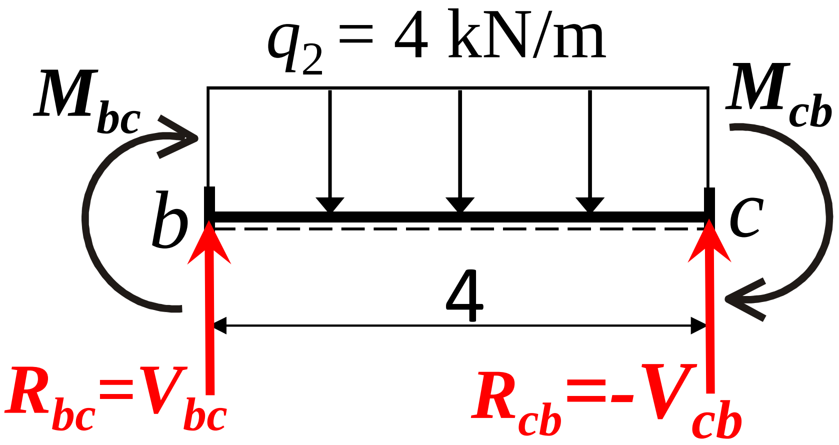
5. End moments
    - After evaluating φ_b, substitute it to evaluate beam end moments.

Slope-Deflection Method: Beams Example 1

6. Shear forces and reactions

- Now, reactions at supports are evaluated using equilibrium equations. Shear forces are equal to plus/minus this reactions.





$$V_{ab} = R_{ab} = \frac{1}{l_{ab}}\left(F \cdot \frac{l_{ab}}{2} + q_1 \cdot \frac{l_{ab}^2}{2} - M_{ab} - M_{ba}\right)$$

$$V_{bc} = R_{bc} = \frac{1}{l_{bc}}\left(q_2 \cdot \frac{l_{cb}^2}{2} - M_{bc} - M_{cb}\right)$$

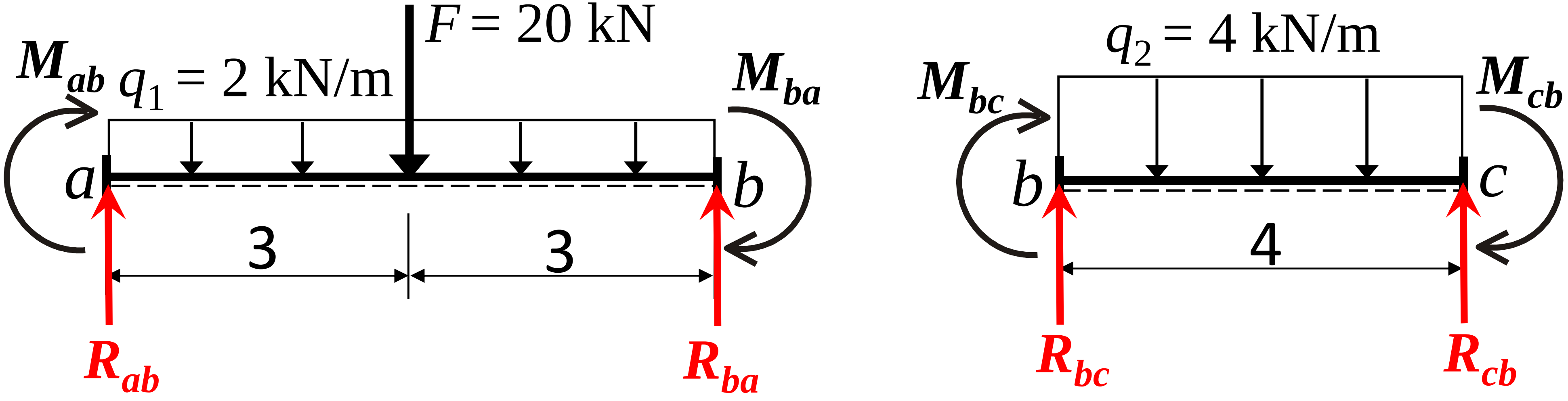
$$V_{ba} = -R_{ba} = -\frac{1}{l_{ab}}\left(F \cdot \frac{l_{ab}}{2} + q_1 \cdot \frac{l_{ab}^2}{2} + M_{ab} + M_{ba}\right)$$

$$V_{cb} = -R_{cb} = -\frac{1}{l_{bc}}\left(q_2 \cdot \frac{l_{cb}^2}{2} + M_{bc} + M_{cb}\right)$$

Slope-Deflection Method: Beams
Example 1

6. Shear forces and reactions

- Now, reactions at supports are evaluated using equilibrium equations. Shear forces are equal to plus/minus this reactions.

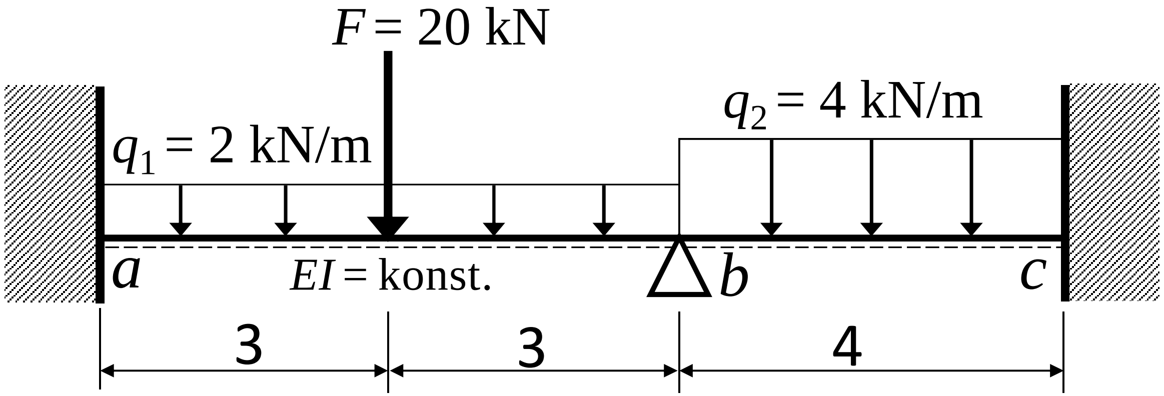


$$R_{az} = R_{ab}$$

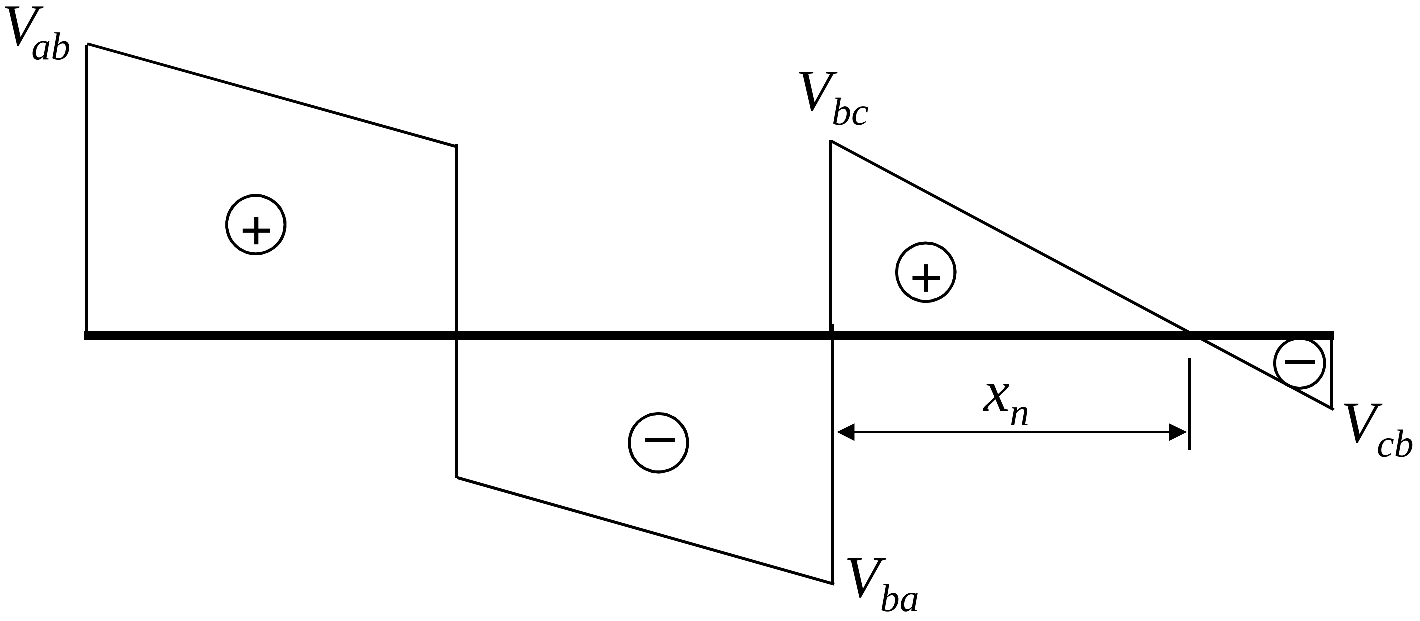
$$R_{bz} = R_{ba} + R_{bc}$$

$$R_{cz} = R_{cb}$$

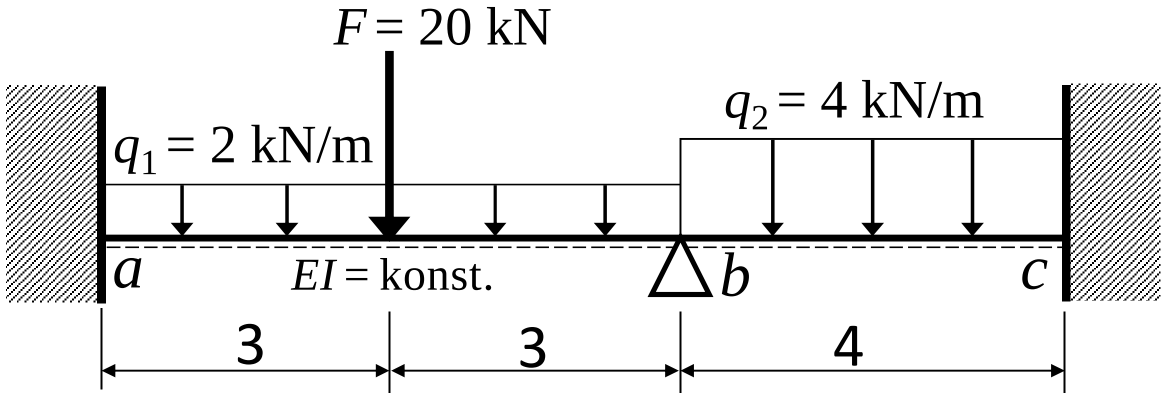
Slope-Deflection Method: Beams
Example 1

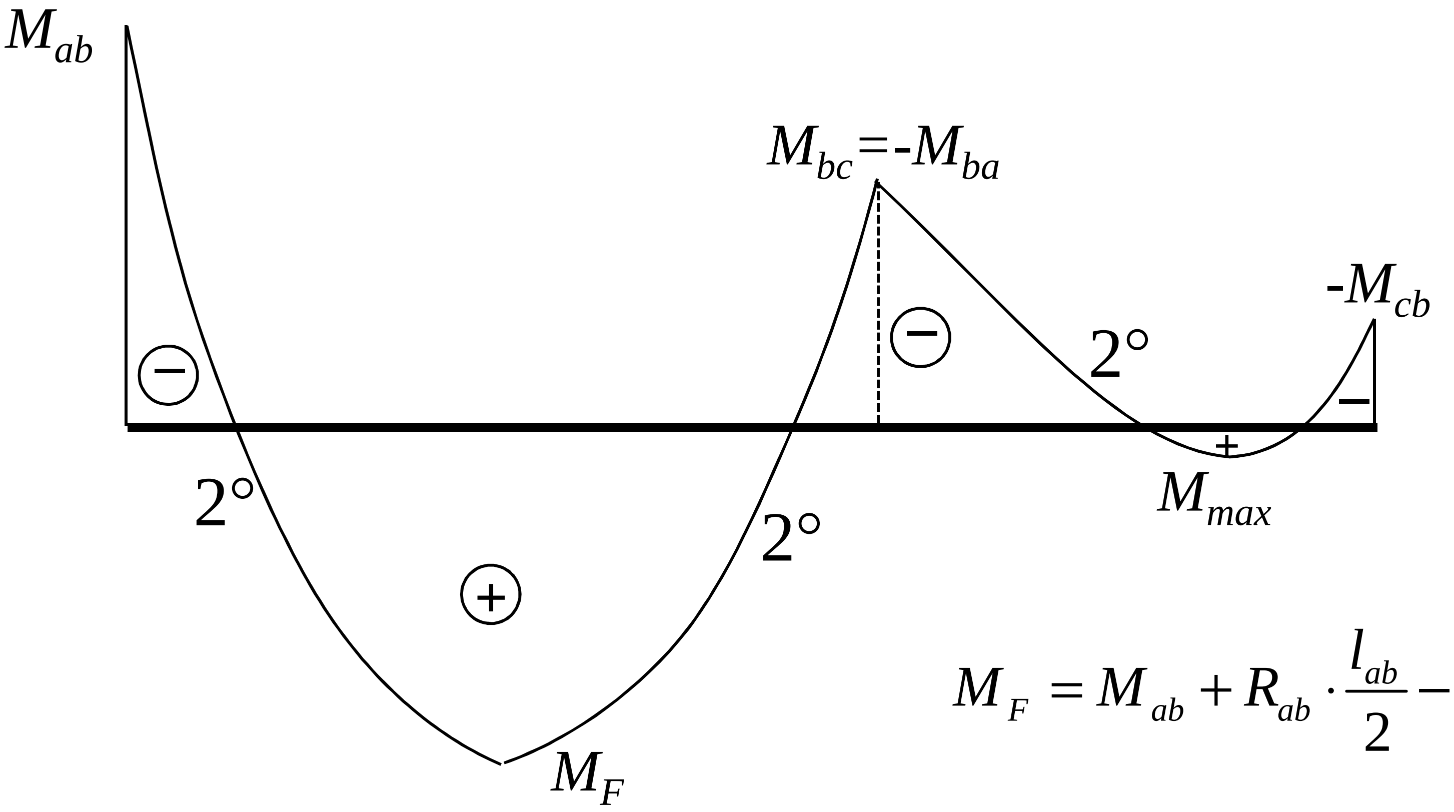


7. Draw shear force and bending moment diagrams.



Slope-Deflection Method: Beams
Example 1

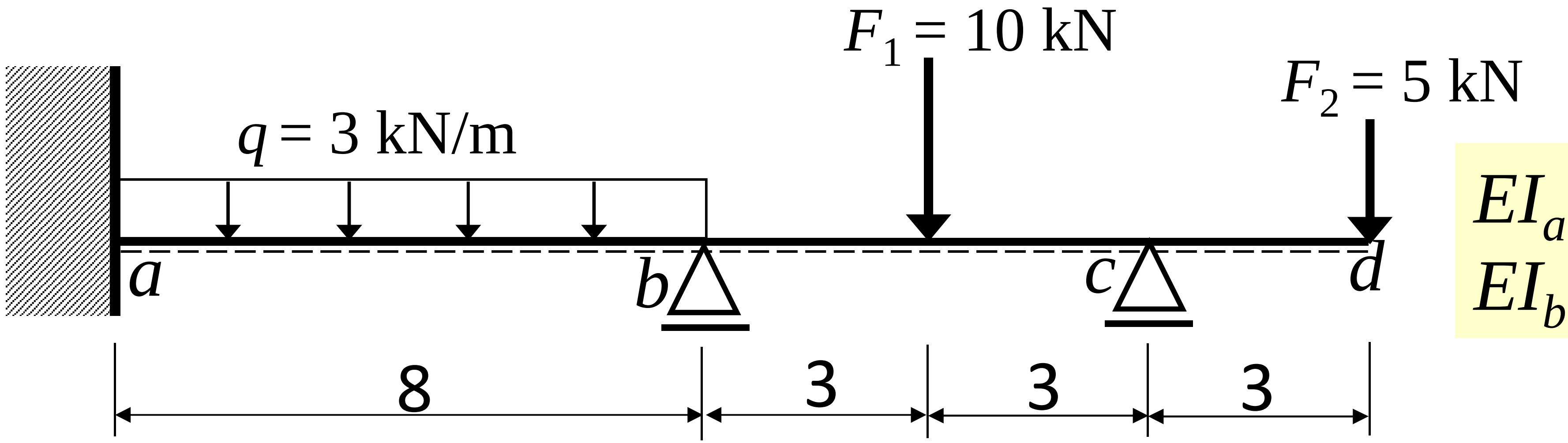




$$M_F = M_{ab} + R_{ab} \cdot \frac{l_{ab}}{2} - q_1 \cdot \left(\frac{l_{ab}}{2}\right)^2 \cdot \frac{1}{2}$$

$$M_{max} = M_{bc} + R_{bc} \cdot x_n - q_2 \cdot \frac{x_n^2}{2}$$

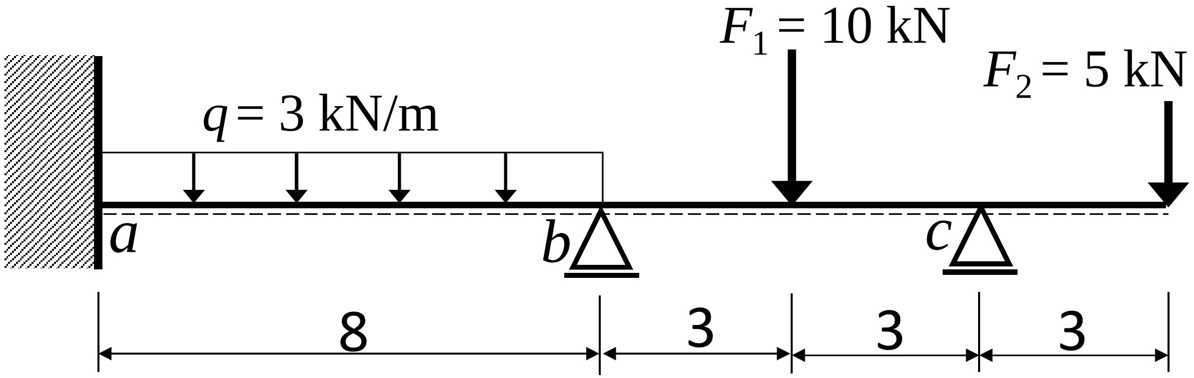
Slope-Deflection Method: Beams Example 2



$EI_{ab} = EI$
$EI_{bc} = 2EI$

1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to second degree.
     - The 1st **possibility of solution** – two unknown joint rotation φ_b, φ_c ($\varphi_a = 0$) - two required equations to solve for the rotation φ_b, φ_c are the moment equilibrium equations at support b and c.

Slope-Deflection Method: Beams Example 2



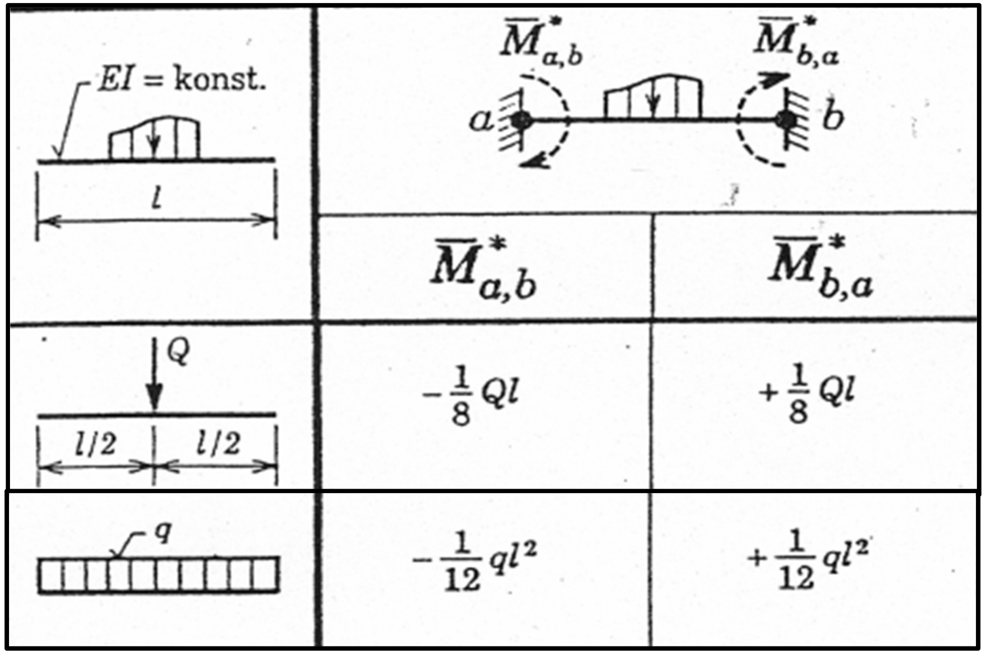
2. Fixed end moments are calculated referring to the table.

$$\overline{M}_{ab} = -\frac{1}{12} \cdot q \cdot l_{ab}^2$$

$$\overline{M}_{ba} = +\frac{1}{12} \cdot q \cdot l_{ab}^2$$

$$\overline{M}_{bc} = -\frac{1}{8} \cdot F \cdot l_{bc}$$

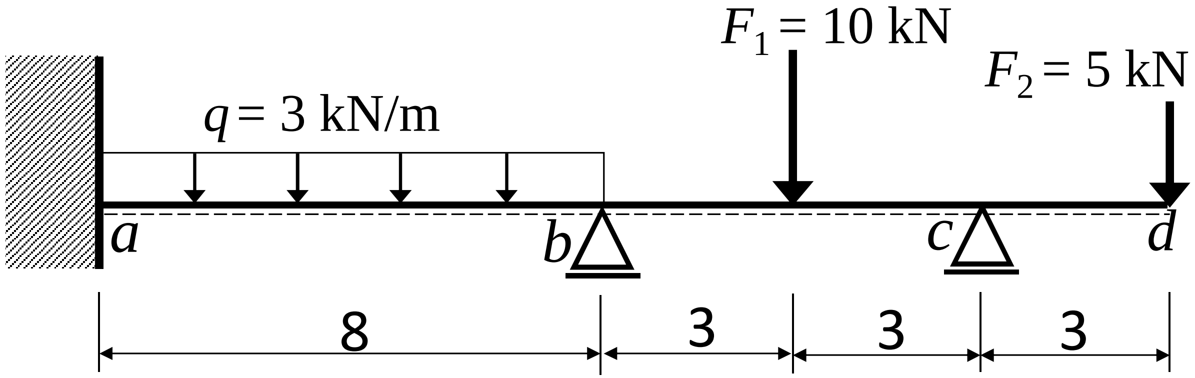
$$\overline{M}_{cb} = +\frac{1}{8} \cdot F \cdot l_{bc}$$



$EI =$ konst.	$\overline{M}^*_{a,b}$	$\overline{M}^*_{b,a}$
Q, $l/2$, $l/2$	$-\frac{1}{8}Ql$	$+\frac{1}{8}Ql$
q	$-\frac{1}{12}ql^2$	$+\frac{1}{12}ql^2$

Slope-Deflection Method: Beams
Example 2

$EI_{ab} = EI$
$EI_{bc} = 2EI$



3. Express internal end moments by slope-deflection equations.

$$M_{ab} = \overline{M}_{ab} + \frac{2EI_{ab}}{l_{ab}}(2\cdot\varphi_a + \varphi_b)$$

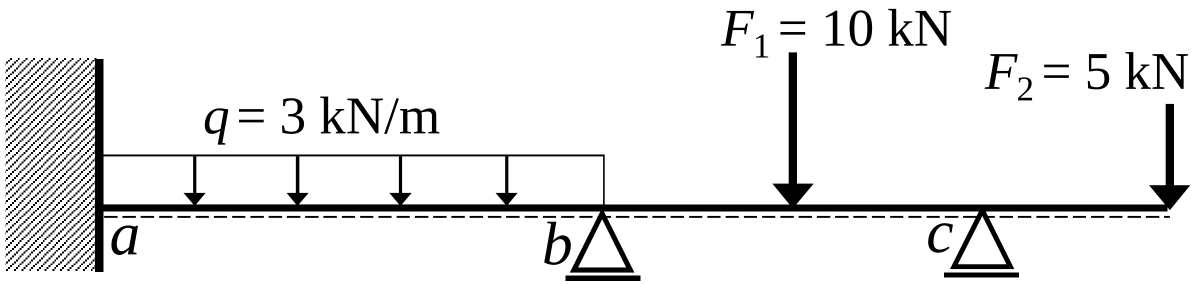
$$M_{ba} = \overline{M}_{ba} + \frac{2EI_{ab}}{l_{ab}}(2\cdot\varphi_b + \varphi_a)$$

$$M_{bc} = \overline{M}_{bc} + \frac{2EI_{bc}}{l_{bc}}(2\cdot\varphi_b + \varphi_c)$$

$$M_{cb} = \overline{M}_{cb} + \frac{2EI_{bc}}{l_{bc}}(2\cdot\varphi_c + \varphi_b)$$

$$M_{cd} = -F_2\cdot 3$$

Slope-Deflection Method: Beams
Example 2



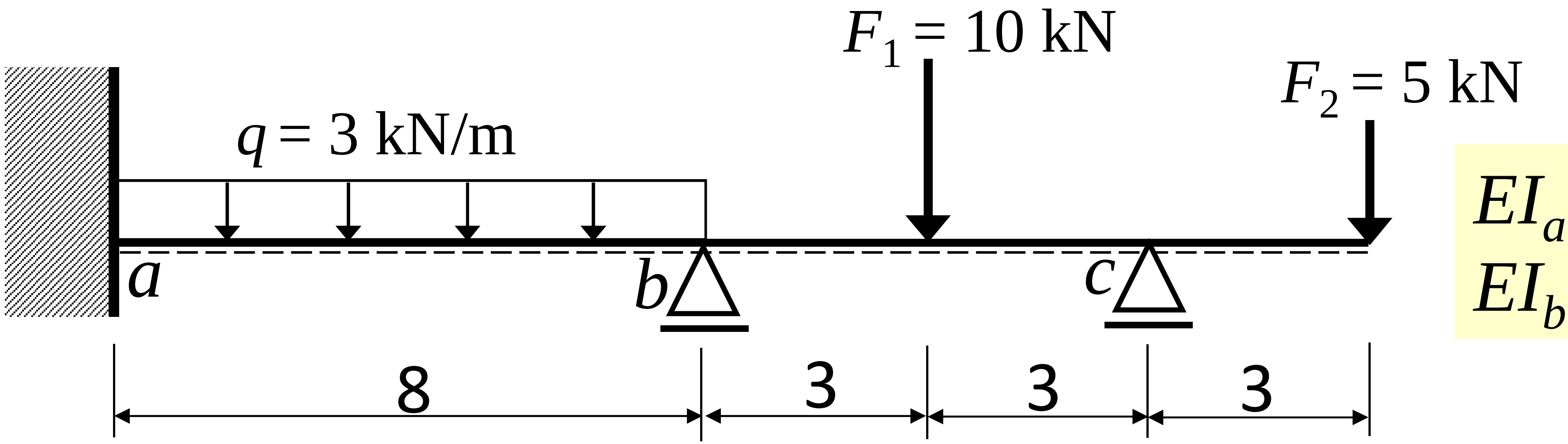
4. Equilibrium equations (write two equilibrium equations for two unknown joint rotations)
   - End moments are expressed in terms of unknown rotations. Now, the required equations to solve for the rotations are the moment equilibrium equations at supports b and c.

$$\left.\begin{aligned} \sum M_b = 0: \quad M_{ba} + M_{bc} = 0 \\ \sum M_c = 0: \quad M_{cb} + M_{cd} = 0 \end{aligned}\right\} \Rightarrow \varphi_b, \varphi_c$$

5. End moments
   - After evaluating φ_b, φ_c, substitute them to evaluate end moments.

Then the procedure is the same as for the 2nd possibility of solution.

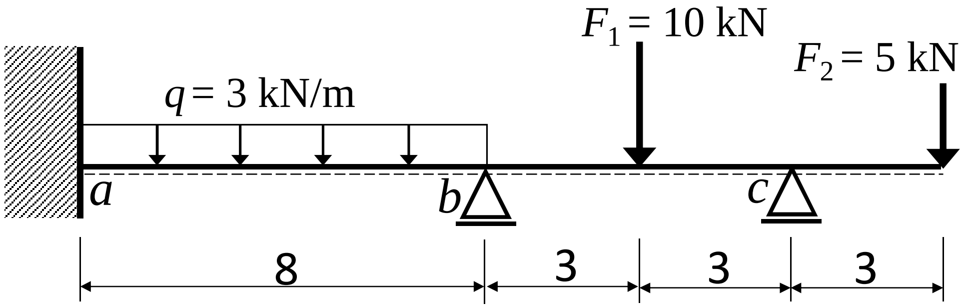
Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution

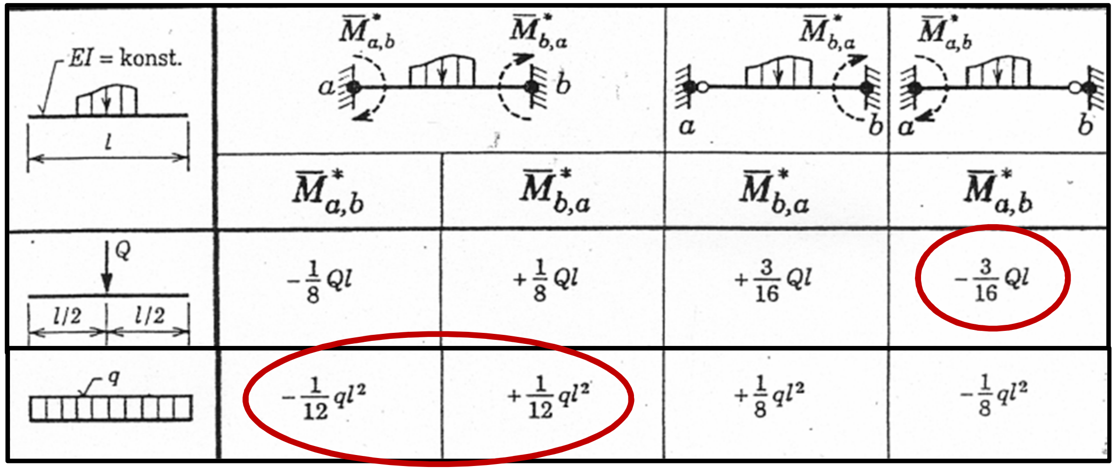


$EI_{ab} = EI$
$EI_{bc} = 2EI$

1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to second degree.
     - The 2nd possibility of solution – solve only one unknown joint rotation φ_b ($\varphi_a = 0$, joint rotation φ_c is not necessary to solution because the moment in the cantilever portion M_c is known $\Rightarrow$ beam portion $\boldsymbol{bc}$ is taken as fixed - hinged).

Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution





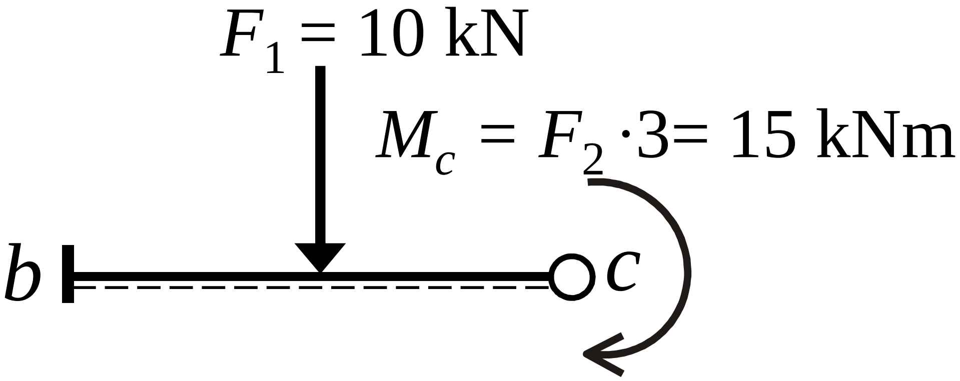
EI = konst. l	$\overline{M}^*_{a,b}$	$\overline{M}^*_{b,a}$	$\overline{M}^*_{b,a}$	$\overline{M}^*_{a,b}$
Q l/2 l/2	$-\frac{1}{8}Ql$	$+\frac{1}{8}Ql$	$+\frac{3}{16}Ql$	$-\frac{3}{16}Ql$
q	$-\frac{1}{12}ql^2$	$+\frac{1}{12}ql^2$	$+\frac{1}{8}ql^2$	$-\frac{1}{8}ql^2$

2. Fixed end moments are calculated referring to the table.

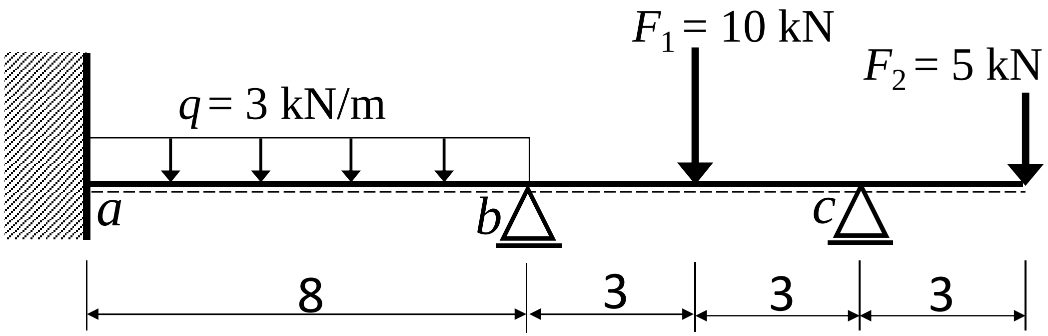
$$\overline{M}_{ab} = -\frac{1}{12}\cdot q\cdot l_{ab}^2$$

$$\overline{M}_{ba} = +\frac{1}{12}\cdot q\cdot l_{ab}^2$$

$$\overline{M}_{bc} = -\frac{3}{16}\cdot F\cdot l_{bc} + \frac{M_c}{2}$$



Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution



$EI_{ab} = EI$
$EI_{bc} = 2EI$

3. Express internal end moments by slope-deflection equations.

$$M_{ab} = \overline{M}_{ab} + \frac{2\,EI}{l}\left(2\cdot\varphi_a + \varphi_b\right)$$

$$\boxed{M_{ab} = \overline{M}_{ab} + \frac{3\,EI}{l}\cdot\varphi_a}$$

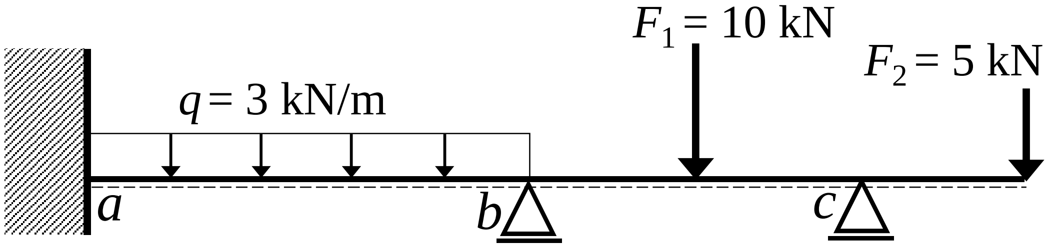
fixed - hinged

$$M_{ab} = \overline{M}_{ab} + \frac{2EI_{ab}}{l_{ab}}\left(2\cdot\varphi_a + \varphi_b\right)$$

$$M_{ba} = \overline{M}_{ba} + \frac{2EI_{ab}}{l_{ab}}\left(2\cdot\varphi_b + \varphi_a\right)$$

$$M_{bc} = \overline{M}_{bc} + \frac{3EI_{bc}}{l_{bc}}\cdot\varphi_b$$

Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution



4. Equilibrium equations (write one equilibrium equation for each unknown joint rotation)
   - End moments are expressed in terms of unknown rotation φ_b. Now, the required equation to solve for the rotation φ_b is the moment equilibrium equation at support b.

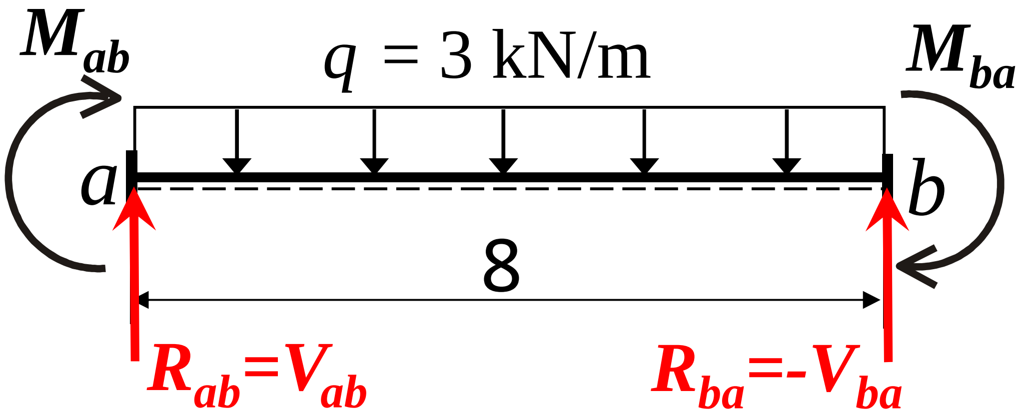
$$\sum M_b = 0 : M_{ba} + M_{bc} = 0 \Rightarrow \varphi_b$$

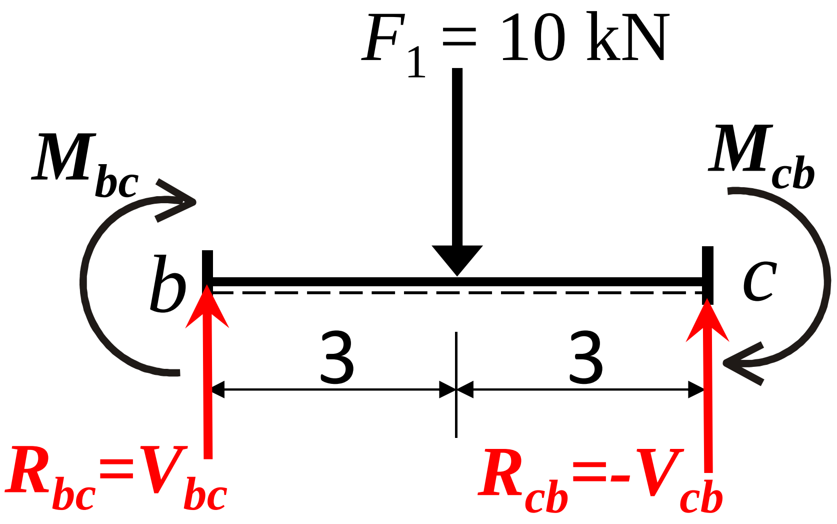
5. End moments
   - After evaluating φ_b, substitute it to evaluate beam end moments.

Then the procedure is the same as
for the 1st possibility of solution.

Slope-Deflection Method: Beams Example 2

6. Shear forces and reactions.





$$V_{ab} = R_{ab} = \frac{1}{l_{ab}}\left(q_1 \cdot \frac{l_{ab}^2}{2} - M_{ab} - M_{ba}\right)$$

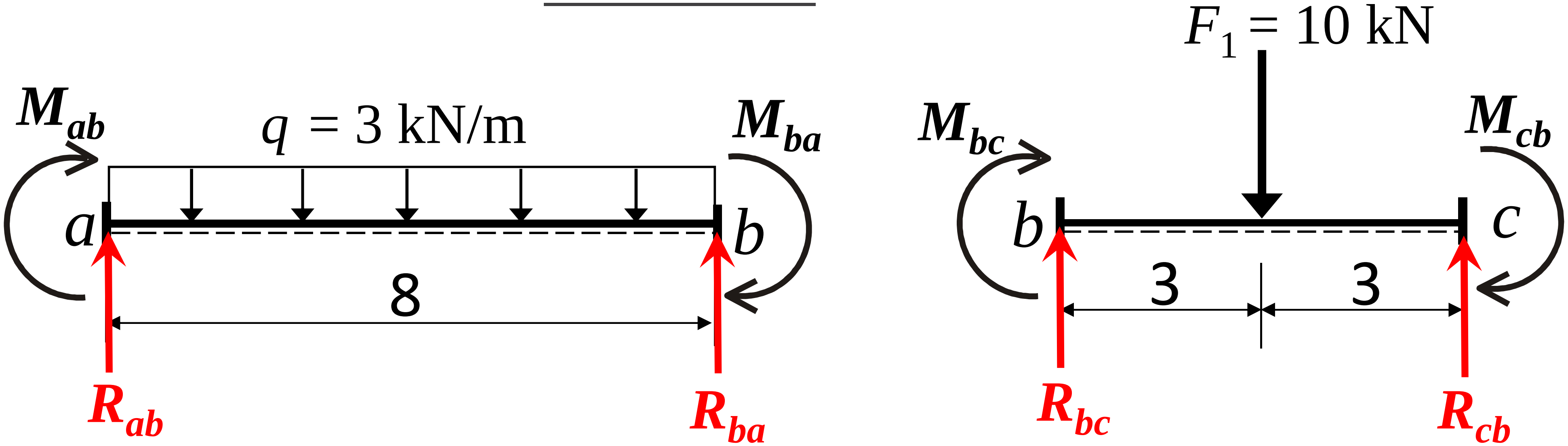
$$V_{ba} = -R_{ba} = -\frac{1}{l_{ab}}\left(q_1 \cdot \frac{l_{ab}^2}{2} + M_{ab} + M_{ba}\right)$$

$$V_{bc} = R_{bc} = \frac{1}{l_{bc}}\left(F_1 \cdot \frac{l_{bc}}{2} - M_{bc} - M_{cb}\right)$$

$$V_{cb} = -R_{cb} = -\frac{1}{l_{bc}}\left(F_1 \cdot \frac{l_{bc}}{2} + M_{bc} + M_{cb}\right)$$

Slope-Deflection Method: Beams Example 2

6. Shear forces and reactions.

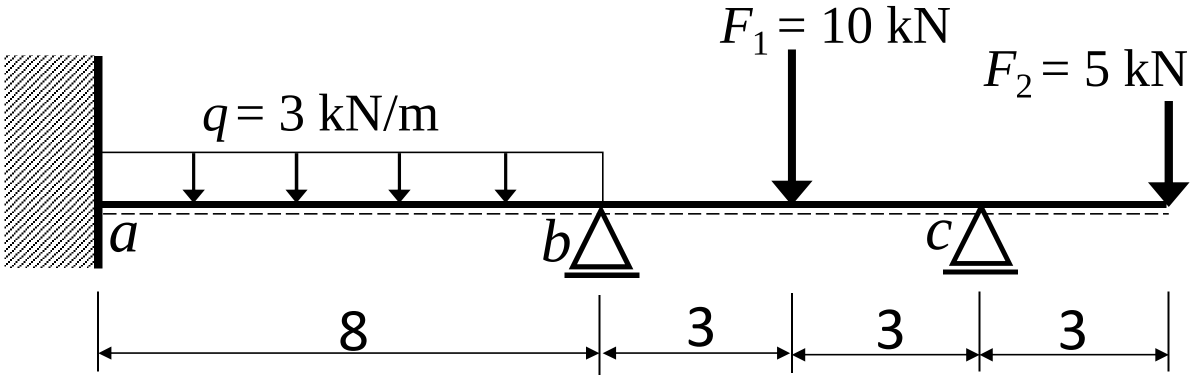


$$R_{az} = R_{ab}$$

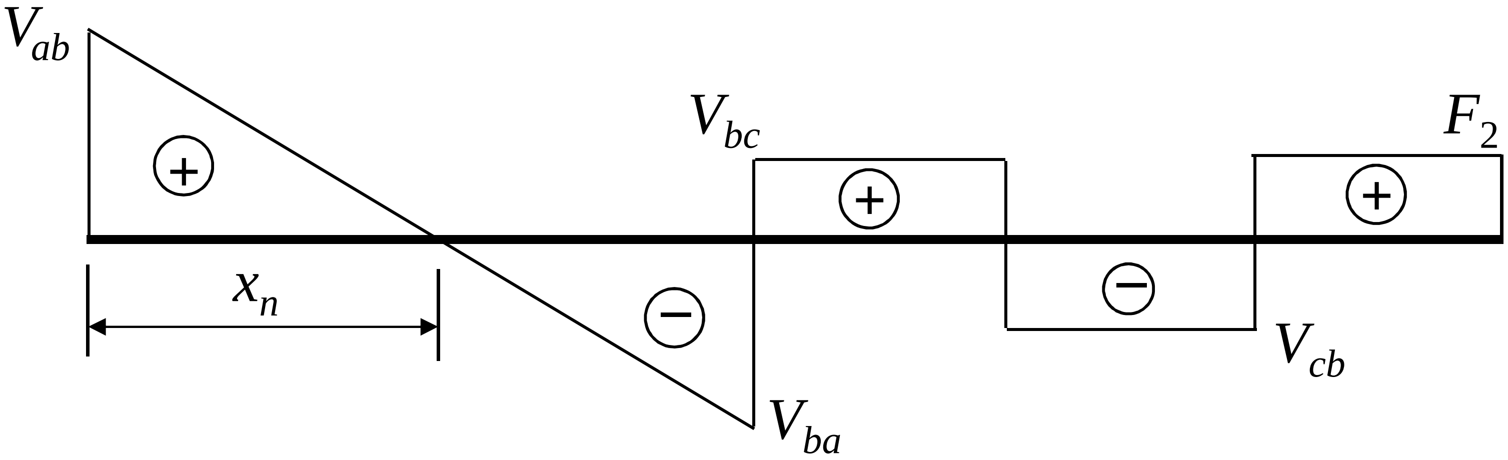
$$R_{bz} = R_{ba} + R_{bc}$$

$$R_{cz} = R_{cb} + F_2$$

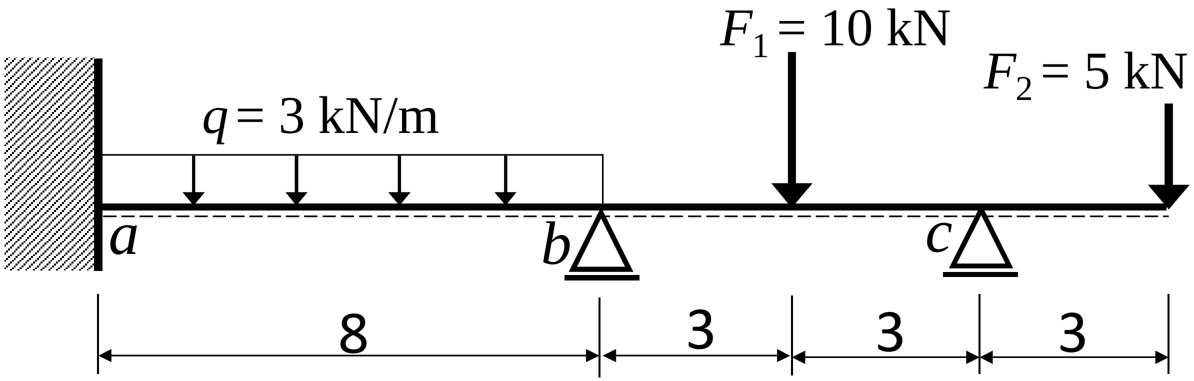
Slope-Deflection Method: Beams
Example 2

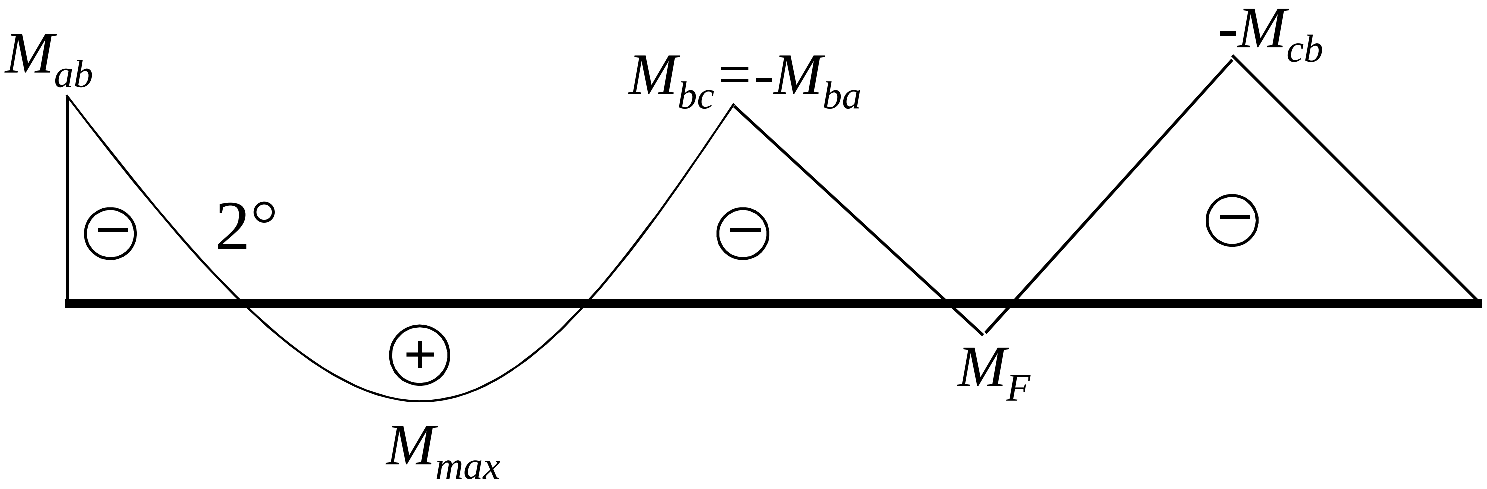


7. Draw shear force and bending moment diagrams.



Slope-Deflection Method: Beams Example 2

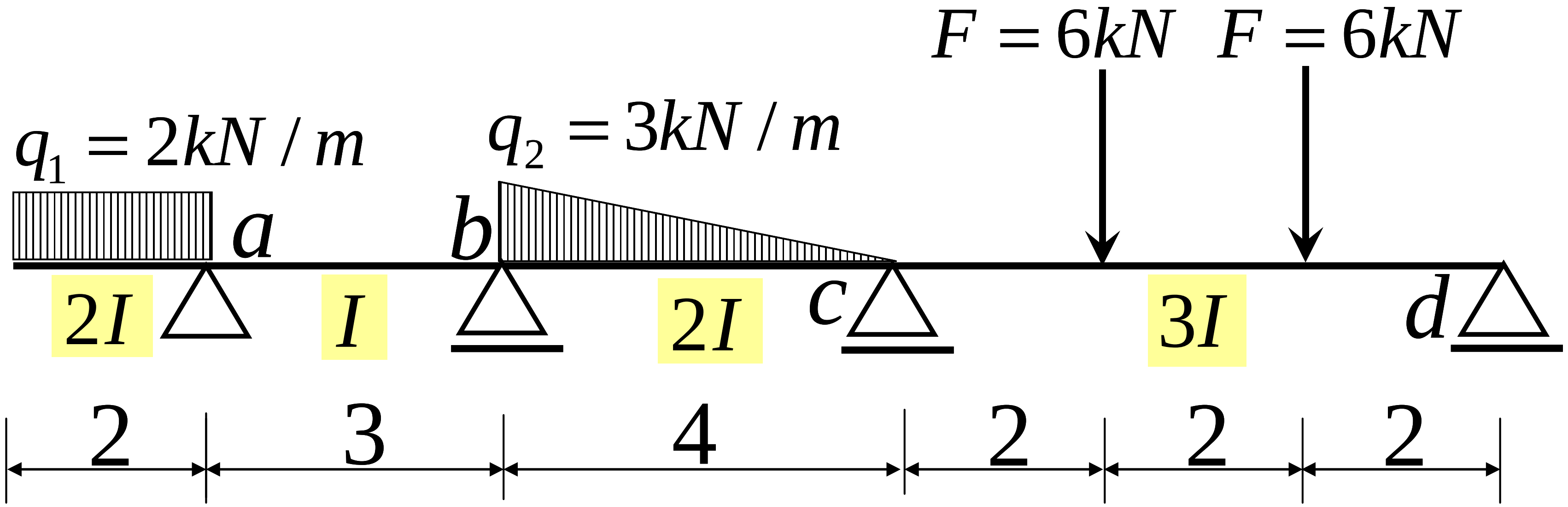




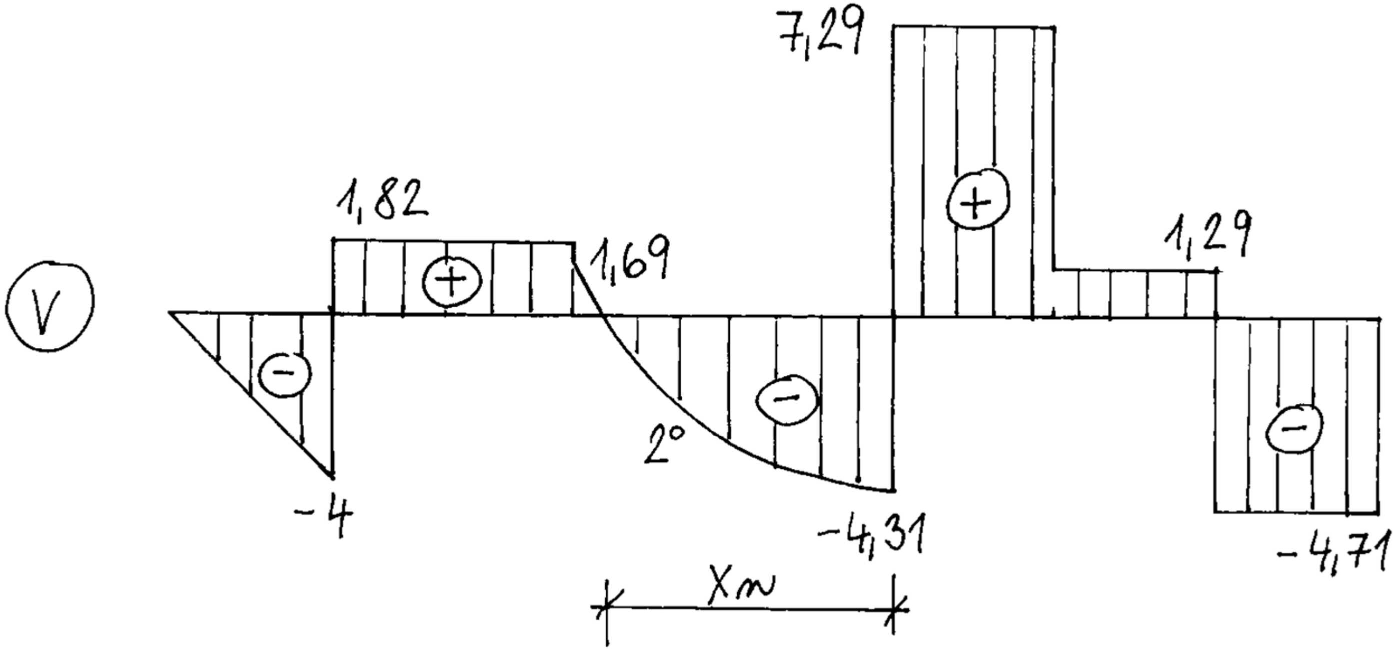
$$M_{max} = M_{ab} + R_{ab} \cdot x_n - q \cdot \frac{x_n^2}{2}$$

$$M_F = M_{bc} + R_{bc} \cdot \frac{l_{bc}}{2}$$

Slope-Deflection Method: Beams Example 3

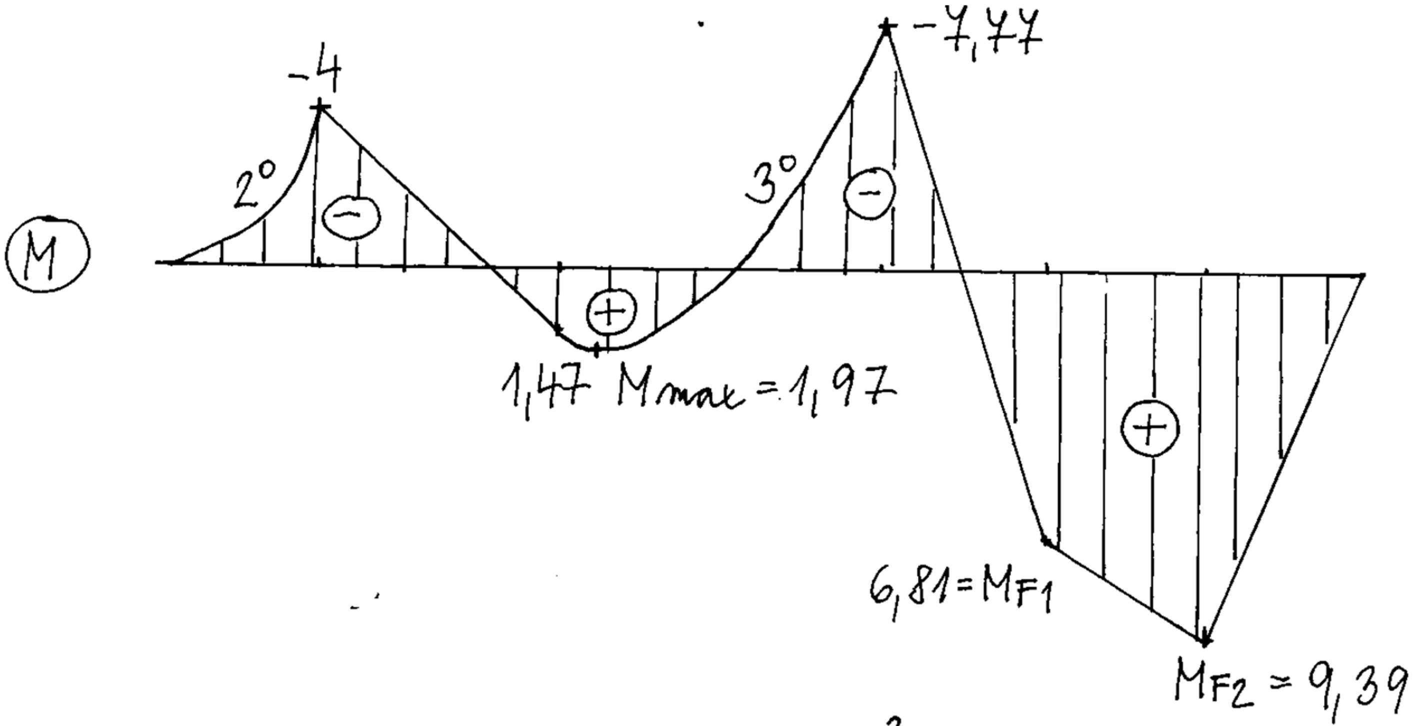


Slope-Deflection Method: Beams Example 3



$$X_m = \sqrt{\frac{|V_{cb}| \cdot 2 \cdot l_{bc}}{q_2}} = \sqrt{\frac{4,31 \cdot 2 \cdot 4}{3}} = 3,39\,m$$

Slope-Deflection Method: Beams
Example 3



$$M_{max} = -M_{cb} + R_{cb} \cdot X_m - \frac{q_2 \cdot X_m^3}{6 \cdot l_{bc}}$$

$$M_{max} = -7,77 + 4,31 \cdot 3,39 - \frac{3 \cdot 3,39^3}{6 \cdot 4} = 1,97 \text{ kNm}$$

$$M_{F1} = M_{cd} + R_{cd} \cdot 2 = -7,77 + 7,29 \cdot 2 = 6,81 \text{kNm}$$

$$M_{F2} = M_{cd} + R_{cd} \cdot 4 - F \cdot 2 = -7,77 + 7,29 \cdot 4 - 6 \cdot 2 = 9,39 \text{kNm}$$