## Lecture Notes 12

# Bonds and the Term Structure of Interest Rates: Pricing, Yields, and (No) Arbitrage 

I. Readings and Suggested Practice Problems
II. Bonds Prices and Yields (Revisited)
III. The Term Structure of Interest Rates (The Yield Curve)
IV. Theories of the Term Structure
V. Additional Readings

Buzz Words: YTM, IRR, Current Yield, Discount/Premium relative to Par, Default Risk, Credit Ratings, Forward Rates, Expectations Theory, Liquidity Premium Theory

## I. Readings and Suggested Practice Problems

A. BKM, Chapter 14.

We covered the essentials of this chapter in Lecture Notes 3. Still, a review is useful before discussing the term structure of interest rates and bond portfolio management.
You are NOT required to read the After-Tax Returns discussion on p. 434.

Suggested Problems: 3, 4, $31 b, c, d, e, g, h, k, l$.
B. BKM, Chapter 15.

Suggested Problems: 8, 13, 22.

## II. Bond Prices and Yields (Revisited)

A. Review of terminology

- Parfface value is the amount repaid at maturity
- Coupon payments $=$ coupon rate $(\%) \times$ Par value
(U.S. bonds typically have semiannual payments)


## B. Yield to Maturity (YTM)

## Definition

Yield to Maturity (YTM) is the constant interest rate (discount rate) that makes the present value of the bond's cash flows equal to its price.

YTM is sometimes referred to as the Internal Rate of Return (IRR).

## Example

12 year, 1,000 par, 10\% annual coupon, selling at 1071.61:

$$
\begin{aligned}
& 1,071.61=\sum_{t=1}^{12} \frac{100}{(1+y)^{1}}+\frac{1000}{(1+y)^{12}}=100 \times \operatorname{APVF}(y, 12)+\operatorname{PV}(1,000, y, 12) \\
& \Rightarrow y=9 \%
\end{aligned}
$$

On most calculators this can be solved with the following key strokes: $-1,071.61 \underline{\text { PV }} 100 \underline{\text { PMT }} 1,000 \underline{\mathbf{F V}} 12 \underline{\mathbf{n}}$, then I/YR displays: $9 \%$

## Usefulness and Interpretation of YTM

- Applicable to riskless (government) or defaultable (corporate) bonds.
- It is a convenient yardstick to compare bonds.

In particular, a YTM of a zero-coupon bond with certain payoff at maturity $T$ is the bond's annualized $T$-year (holding-period) return.

- YTM is a summary measure of the uncertain interest-rate environment given a particular cash-flow pattern. Hence, the YTM is in general different than the realized holding period return.
As noted, the YTM is indeed the (geometric) average annual return on a zero coupon bond (pure discount bond) if held to maturity. But for a coupon bond held to maturity, the realized average return will depend on the rate at which coupons can be reinvested.
(Also note that we can always compute the YTM of a coupon bond with a given maturity $T$, but this YTM is not, in general, the same YTM as of a pure discount bond with maturity $T$.)


## C. The holding period return (HPR)

## Example (continued)

The current bond price is $\mathrm{P}_{0}=1,071.61$.
Assume that in one year the YTM stays at $9 \%$.
In one year:

$$
\mathrm{P}_{1}=100 \times \operatorname{APVF}(9 \%, 11)+\operatorname{PV}(1,000,9 \%, 11)=1,068.05
$$

Year 1 total holding period return is:

$$
\mathrm{HPR}=\frac{\left(P_{1}+\text { Coupon }\right)-P_{0}}{P_{0}}
$$

$$
\begin{aligned}
& =\frac{\text { Coupon }}{P_{0}}+\frac{P_{1}-P_{0}}{P_{0}} \\
& \text { = "current yield" (defined as annual coupon/price) }+ \\
& \text { capital gains (i.e., price appreciation) } \\
& =100 / 1,071.61+(1,068.05-1,071.61) / 1,071.61 \\
& =9.33 \%-0.33 \%=9 \%
\end{aligned}
$$

## D. Zero-Coupon Bonds and Coupon Bonds

1. Zero-Coupon Bonds are also referred to as Zeros, as Pure Discount Bonds, or simply as Discount Bonds.

If the coupon rate is zero, the entire return comes from price appreciation.

Zero coupon bonds avoid reinvestment risk (uncertainty about rates at which coupon receipts can be reinvested).
2. Coupon Bonds and Zeros

A coupon bond can be viewed as a portfolio of zeros:
10-year, $10 \%$ annual coupon rate, 1,000 par bond $=$
1-year, 100 par zero

+ 2-year, 100 par zero
+10 -year, 100 par zero
+10 -year, 1,000 par zero
On the time-line we consider each CF separately:

| 0 | 1 | 2 | $\ldots$ | 10 |
| :--- | :--- | :--- | :--- | :--- |
|  | 100 | 100 |  | 100 |
|  |  |  |  | 1,000 |

## 3. Creation of zeros (Stripping)

Stripping is the process of spinning off each coupon and principal repayment as a separate zero.

Prior to mid-1970's there was little perceived need for zeros because interest rates were relatively stable.

Prior to 1982, zeros were fashioned by investment banks (e.g. TIGRs - Treasury Investment Growth Receipts, are U.S. Treasury bonds reissued by Merrill Lynch as a series of zero coupon bonds. Similarly CATs - were issued by Salomon Brothers).
1982. U.S. Treasury starts STRIPS program (Separate Trading of Registered Interest and Principal Securities)
1987. Treasury allows rebundling of components to recreate original bond ("reconstitutions")

## E. Corporate debt

## 1. Issuance and control

The indenture is the bond contract.
Typical indenture provisions: callability, sinking funds, convertibility, subordination, dividend restrictions.

Adherence is monitored by bond trustee.

## 2. Default risk and ratings

Debt is rated for safety by rating agencies such as Standard \& Poor's or Moody's.

Ratings:

Standard \& Poor's

AAA, AA $\begin{aligned} & \text { Very high } \\ & \text { quality }\end{aligned}$
A, BBB High quality
B B, B Speculative
CCC, D Very poor

- "Investment Grade:" above speculative
- "Junk:" anything speculative or worse


## III. The Term Structure of Interest Rates

The term structure of interest rates refers to the relation between the interest rate and the maturity or horizon of the investment

The term structure can be described using the Yield Curve.

## A. Yield Curve

1. The yield curve is just the yield to maturity (YTM) on a $t$-year (pure) discount bond graphed as a function of $t$.
2. Discount bonds of different maturities can have different yields to maturity.
3. The slope of the yield curve depends on the difference between yields on longer and shorter maturity bonds.
4. Recall from the definition of the YTM that it is not correct to use the YTM on a $T$-year coupon bond as the yield on a $T$-year zero coupon bond: these are not the same.
5. We focus on the nominal yield curve (you may think about the nominal rate as: "real" rate + expected inflation rate).

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[The Recent Curve Here]

## [The 3/1999 Upward-Slopping Curve Here]

## Examples

a. The above recent yield curve, and the inverted 2000 curve.
b. The more typical upward-slopping curve in 1999 (above) and in 1997 (below)

| 3 |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 14:10 | CURRENT MARKET CURRENT SECURITIES $\qquad$ |  |  |  | (BLOOMBERG GENERIC) |  |  |  |
| 3 Mo | 1/29/98 | 5.03 | 5.02 | 5.15 | $\begin{aligned} & 2 / 05 / 98 \\ & 5 / 07 / 98 \end{aligned}$ |  | $\begin{aligned} & 5.08 \\ & 5.06 \end{aligned}$ | $\begin{aligned} & 5.07 \\ & 5.05 \end{aligned}$ |
| 6 Mo | 4/30/98 | 5.06 | 5.05 | 5.25 |  |  |  |  |
| 1 Yr | 10/15/98 | 5.06 | 5.05 | 5.32 |  |  |  |  |
| 2 Yr | $5^{5}$ 8 $10 / 99$ | 100-00+ | 100-01 | 5.61 |  |  | 5.69 | 5.68 |
| 3 Yr | 6 8/00 | 100-24+ | 100-25 | 5.69 | $53_{4}$ | 11/00 |  |  |
| 5 Yr | $5{ }^{3} 410 / 02$ | 100-02+ | 100-03+ | 5.72 |  |  |  |  |
| 10 Yr | $6^{1} 88 / 07$ | 102-03 | 102-04 | 5.84 | $6^{1} 8$ | 8/07 | $\begin{aligned} & 5.88 \\ & 6.14 \end{aligned}$ | $\begin{array}{r} 5.86 \\ 6.13 \\ 76.50 \end{array}$ |
| 30 Yr | $63_{8}^{6} 8 / 27$ | 102-25+ | 102-26+ | 6.17 | $6^{18}$ | 11/27 |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

c. In those examples in Lecture Notes 2, where we used the same rate $r$ to discount cash flows occurring at different dates in the future, we implicitly assumed a flat curve.
d. Also check your market tracking data to examine how the short-end and the long-end of the yield curve evolved on a weekly basis.

## e. Using the yield curve to price a stream of cash flow

When the yields for the dates $1,2, \ldots, T$ are
$y_{1}, y_{2}, \ldots, y_{T}$, then the value of a stream of cash flow $C_{1}, C_{2}, \ldots, C_{T}$, is given by

$$
\begin{gathered}
\frac{C_{1}}{1+y_{1}}+\frac{C_{2}}{\left(1+y_{2}\right)^{2}}+\ldots+\frac{C_{T}}{\left(1+y_{T}\right)^{T}}= \\
=d_{1} C_{1}+d_{2} C_{2}+\ldots+d_{T} C_{T}
\end{gathered}
$$

where

$$
d_{t}=\frac{1}{\left(1+y_{t}\right)^{t}}
$$

is the discount factor for time $t$.

## B. Spot and Forward Rates

## 1. Definition

Note: An interest depends on the starting date, the ending date and the contracting date (when the decision is locked in).
$r_{t}$ One-year spot rate at time $t-1$. The return between time $t-1$ and $t$, contracted at time $t-1 .(t>0)$.
$f_{t}$ One-year forward rate. The return between time $t-1$ and $t$, contracted today.

Note: spot and forward rates may be for more than one year. A spot rate is not locked in until the investment starts (at some time $t$ ). A forward rate is locked in immediately (today). Of course $r_{1}=f_{1}$.

## Example

0

4

As of time 0: $\underline{\mathbf{0}}$

$$
\begin{array}{ll}
1 & 2 \\
& f_{3}=6 \% \\
& r_{3}=?
\end{array}
$$

As of time 1:0
1
2
$f_{3}=7 \%$
$r_{3}=$ ?
As of time 2: 0
1

$$
\begin{aligned}
& \underline{\mathbf{2}}_{f_{3}=5 \%}{ }^{3}=5 \%
\end{aligned}
$$

## 2. Relation to Yield to Maturity: An Example

Forward rates can be calculated from yields!
$y_{t}$ the yield to maturity (locked in today, time 0 ) for a zero-coupon bond with maturity $t$.


## Example

$$
\begin{array}{lll}
0 & 1 & 2 \\
r_{1}=8 \% & f_{2}=? & \\
& y_{2}=9 \% &
\end{array}
$$

If we buy-and-hold a 2-year zero, the 2-year return is $\left(1+y_{2}\right)^{2}$ If we buy a 1 -year zero and contract (today) to reinvest the proceeds, our return is $\left(1+r_{1}\right)\left(1+f_{2}\right)$
The implied forward rate is the $f_{2}$ that makes these equivalent:
$(1.09)^{2}=(1.08)\left(1+f_{2}\right) \Rightarrow f_{2}=10.01 \%$

## 3. The General Case

The implied forward rate:

$$
\left(1+y_{t-1}\right)^{t-1}\left(1+f_{t}\right)=\left(1+y_{t}\right)^{t}
$$

Similarly, can contract for a rate between $t-2$ and $t-1$,

$$
\left(1+y_{t-2}\right)^{t-2}\left(1+f_{t-1}\right)=\left(1+y_{t-1}\right)^{t-1}
$$

and substitute above:

$$
\left(1+y_{t-2}\right)^{t-2}\left(1+f_{t-1}\right)\left(1+f_{t}\right)=\left(1+y_{t}\right)^{t}
$$

## The relation between yield to maturity and forward rates

Can be obtained repeating the steps above for $t-3$, then $t-4$ $\ldots .$. until time 0 , where $f_{1}=r_{1}$ :

$$
\left(1+r_{1}\right)\left(1+f_{2}\right)\left(1+f_{3}\right) \ldots\left(1+f_{t-1}\right)\left(1+f_{t}\right)=\left(1+y_{t}\right)^{t}
$$

## Remark

The relation $\left(1+y_{t-1}\right)^{t-1}\left(1+f_{t}\right)=\left(1+y_{t}\right)^{t}$ can be taken as the definition of the forward rate (as in BKM), and then it follows that $f_{t}$, the one-year forward rate, must be the return between time $t-1$ and $t$, contracted at time 0 (today). Otherwise, arbitrage opportunities will arise. The two definitions of the forward rates (as the contracting rate, or the implied rate) are of course equivalent in markets that are arbitrage free.

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4. Another example for the relation between discount bond yields and forward rates

$\left(1+y_{2}\right)^{2}=\left(1+y_{1}\right)\left(1+f_{2}\right):$
$1.0562^{2}=1.0533\left(1+f_{2}\right)$
$\Rightarrow f_{2}=5.91 \%$
$\left(1+y_{3}\right)^{3}=\left(1+y_{2}\right)^{2}\left(1+f_{3}\right):$
$1.0570^{3}=1.0562^{2}\left(1+f_{3}\right)$
$\Rightarrow f_{3}=5.86 \%$
or,
$\left(1+y_{3}\right)^{3}=\left(1+y_{1}\right)\left(1+f_{2}\right)\left(1+f_{3}\right): 1.0570^{3}=1.0533 \times 1.0591 \times\left(1+f_{3}\right) \Rightarrow f_{3}=5.86 \%$
$\left(1+y_{4}\right)^{4}=\left(1+y_{3}\right)^{3}\left(1+f_{4}\right): \quad 1.0573^{4}=1.0570^{3}\left(1+f_{4}\right) \quad \Rightarrow f_{4}=5.82 \%$
or,
$\left(1+y_{4}\right)^{4}=\left(1+y_{2}\right)^{2}\left(1+f_{3}\right)\left(1+f_{4}\right): 1.0573^{4}=1.0562^{2} \times 1.0586 \times\left(1+f_{4}\right) \Rightarrow f_{4}=5.82 \%$

## C. Arbitrage and the Term Structure

## 1. An Arbitrage Opportunity

a. Definition:

An investment that generates a strictly positive cash inflow today and does not require any cash outflows in the future,
or an investment that does not require any cash outflows today and generates a strictly positive cash inflow (with some probability) in the future is referred to as an arbitrage opportunity.
b. In well functioning markets, arbitrage opportunities can not exist (for long...) since any investor who prefers more to less wants to invest as much as possible in the arbitrage opportunity, thereby eliminating it immediately.

## 2. No Arbitrage and the Law of One Price

The absence of arbitrage implies that any two assets with the same stream of riskless cash flows must have the same price.

This implication is known as the law of one price.

Otherwise, could buy the lower priced asset and sell the higher priced asset and earn an arbitrage profit:
(1) Positive cash flow today,
(2) Zero cash flows in the future.

## 3. Arbitrage and forward rates

If $y_{1}, y_{2}$, and $f_{2}$ are quoted market rates (recall $y_{1}=r_{1}=f_{1}$ ) and $\left(\mathbf{1}+y_{2}\right)^{\mathbf{2}} \neq\left(\mathbf{1}+y_{1}\right)\left(\mathbf{1}+f_{2}\right)$ then arbitrage is possible.

## Example

$y_{1}=10 \%, y_{2}=11 \%$, and $f_{2}=14 \%$,
so $(1+0.11)^{2}<(1+0.10)(1+0.14)$
To earn arbitrage profit (whenever the law of one price is violated) you follow the obvious strategy and sell, or short, the relatively overvalued asset and buy the relatively undervalued asset. With riskless bonds this translates to borrow at the lower rate and lend at the higher rate.

Borrow $\$ 100$ for two years at $y_{2}=11 \%$
(or short today $\$ 100$ worth of a two-year T-note with YTM 11\%)

Invest the borrowed $\$ 100$ for one year at $y_{1}=10 \%$ and the second year invest the proceeds at $f_{2}=14 \%$. (that is, buy a $\$ 100$ worth of a one-year T-note with YTM $10 \%$, and contract to invest in one year from now the amount of $\$ 110$ at a rate of $14 \%$ ):

There is no cash outflow at the present, and two years from now get:

$$
\mathrm{FV}=100(1.10)(1.14)-100(1.11)^{2}=\$ 2.19
$$

(This assumes that you can borrow and lend freely at the given rates, and there are no transaction costs.)

Although $\$ 2.19$ may not seem a lot, you can arbitrage with $\$ 100,000,000$ and the profit becomes $\$ 2,190,000$.

If enough investors will try to exploit the arbitrage opportunity, prices will adjust so that the arbitrage opportunity will disappear.

## How?

One possible scenario is that there would not be enough of those who will be willing to buy the relatively overvalued two-year notes from the arbitrageurs (there would not be enough investors willing to lend to the Treasury at a cheaper rate than the arbitrageurs are willing to) so prices will drop and $y_{2}$ will rise;
and there may be too few of those who want to sell to the arbitrageurs the relatively undervalued one-year notes and $y_{1}$ or $f_{2}$ or both will fall so that eventually the two-year investment and the one-year rolled-over investment would be equivalent when contracting at time $t=0$.

## D. Forward rates and expected future spot rates

Given the term structure:

| 0 | 1 | 2 |
| :---: | :--- | :---: |
| $r_{1}=8 \%$ | $r_{2}=?$ |  |
| $y_{2}=9 \%$ |  |  |

We can find the forward rate:
0
1
2
$r_{1}=8 \% \quad f_{2}=10.01 \%$

But what can we say about investors' $E\left(r_{2}\right)$, the expectations (held at time 0 ) of $r_{2}$, how are these related to forward rates?
-- Here must resort to theory!

## IV. Theories of the Term Structure

A. The (Pure) Expectations Hypothesis: $f_{2}=E\left(r_{2}\right)$

The forward rate is a "prediction" of the future spot rate.
Also can restate this as $\left(1+y_{2}\right)^{2}=\left(1+r_{1}\right)\left(1+\boldsymbol{E}\left(\boldsymbol{r}_{2}\right)\right)$

## 1. Explanation

- Suppose that $E\left(r_{2}\right) \ll f_{2}$, e.g., $E\left(r_{2}\right)=0$

$$
\begin{array}{ll}
0 & 1 \\
r_{1}=8 \% & E\left(r_{2}\right)=0 \% \\
y_{2}=9 \% &
\end{array}
$$

Investors with two-year horizons can:
(a) buy a two-year bond
(or invest one-year spot and one-year forward).
The expected 2 -yr return is

$$
(1.09)^{2}=(1.08)(1.1001)=1.188
$$

or
(b) invest for one year and take whatever $r_{2}$ happens to be at time 1 .
The expected $2-\mathrm{yr}$ return is $(1.08)(1+0)=1.08$.

Given their beliefs, $(a)$ is more attractive, so they will buy the 2-year bond:

Price of 2-year bond $\uparrow, y_{2} \downarrow$, therefore $f_{2} \downarrow$

- Suppose that $E\left(r_{2}\right) \gg f_{2}$, e.g., $E\left(r_{2}\right)=20 \%$

$$
\begin{array}{ll}
0 & 1 \\
r_{1}=8 \% & E\left(r_{2}\right)=20 \% \\
y_{2}=9 \% &
\end{array}
$$

Investors with two-year horizons can:
(a) buy a two-year bond (or invest one-year spot and one-year forward). The $2-y r$ return is

$$
(1.09)^{2}=(1.08)(1.1001)=1.188
$$

or
(b) invest for one year and take whatever $r_{2}$ happens to be at time 1 .
The expected $2-\mathrm{yr}$ return is $(1.08)(1.20)=1.296$
Given their beliefs, (b) is more attractive, so they will sell the 2-year bond:

Price of 2-year $\downarrow, y_{2} \uparrow$, therefore $f_{2} \uparrow$.
The adjustment stops when $f_{2}$ and $E\left(r_{2}\right)$ are approximately equal.

NOTE: the above is not an arbitrage. The choice (a) vs. (b) involves risk.
2. If the expectations hypothesis holds, then we have the following information in the yield curve:

- an upward sloping yield curve (implies that the forward rates are higher than the current spot rates and therefore) implies that the market is expecting higher spot rates in the future.
- a downward sloping (inverted) yield curve (implies that the forward rates are lower than the current spot rates and therefore) implies that the market is expecting lower spot rates in the future.

3. The Expectation Hypothesis is an equilibrium theory of the term structure:

When $f_{2}=E\left(r_{2}\right)$, the theory tell us that there would be no buying or selling pressure, and hence prices and yields would be in equilibrium (when expectations are revised, perhaps on a continuous basis, the yield curve changes its shape accordingly).
B. Liquidity Preference: $f_{2}>E\left(r_{2}\right)$
i.e., $f_{2}=E\left(r_{2}\right)+L_{2}$
where $L_{2}$ is the liquidity premium at 2 years horizon.
$\left(\operatorname{In} f_{2}=E\left(r_{2}\right)+L_{2}, f_{2}\right.$ is known, but $E\left(r_{2}\right)$ and $L_{2}$ aren't).

The explanation is based on willingness of investors to substitute one maturity for another (at some price)

## C. Market Segmentation Hypothesis

There is no substitution: e.g., the market for 5-year debt is determined entirely by 5 -year issuers and investors, who ignore the rates on 4- or 6-year bonds.

## D. Preferred Habitat Theory

As the Market Segmentation Hypothesis, this theory says that yields on different bonds are determined by the supply and demand for that security and that securities with similar maturities are not close substitutes, but nevertheless investors still can substitute to some extent.

## V. Additional Readings

