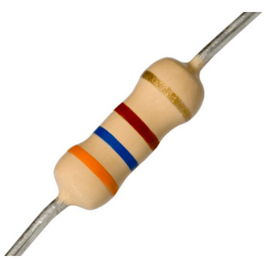
Physics 120/220



Voltage Divider
Capacitor
RC circuits

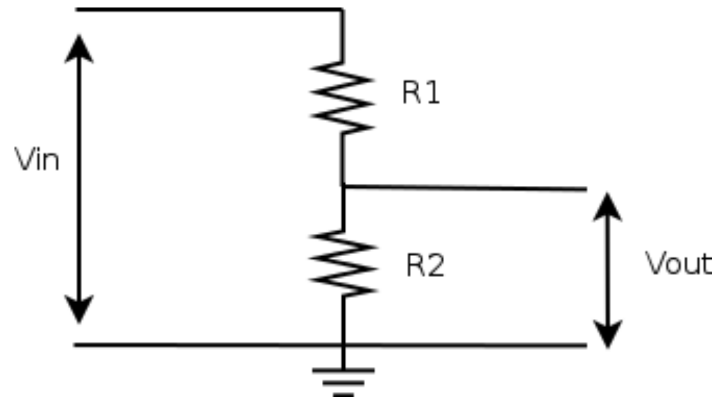
Prof. Anyes Taffard

Voltage Divider

The figure is called a voltage divider.

It's one of the most useful and important circuit elements we will encounter.

It is used to generate a particular voltage for a large fixed V_{in}.



Current (R_1 & R_2) $I = \dfrac{V_{in}}{R_1 + R_2}$

Output voltage:

$$V_{out} = IR_2 = \frac{R_2}{R_1 + R_2} V_{in}$$

$$\therefore V_{out} \le V_{in}$$

Voltage drop is proportional to the resistances

V_{out} can be used to drive a circuit that needs a voltage lower than V_{in}.

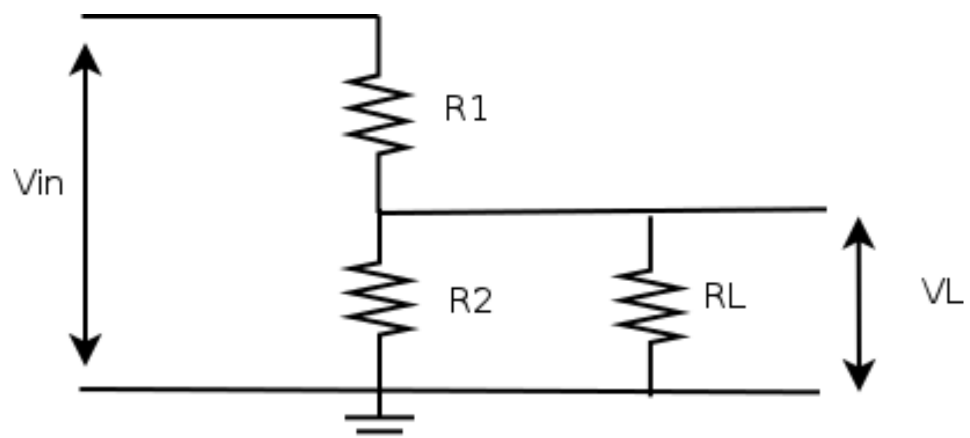
Voltage Divider (cont.)

Add load resistor R_L in parallel to R_2.

You can model R_2 and R_L as one resistor (parallel combination), then calculate V_{out} for this new voltage divider

If $R_L >> R_2$, then the output voltage is still: $$V_L = \frac{R_2}{R_1 + R_2} V_{in}$$

However, if R_L is comparable to R_2, V_L is reduced. We say that the circuit is "loaded".



Ideal voltage and current sources

<u>Voltage source:</u> provides fixed V_{out} regardless of current/load resistance.

Has *zero* internal resistance (perfect battery).

Real voltage source supplies only finite max I.

(a) *ideal voltage source* (b) *real voltage source including internal resistance r*

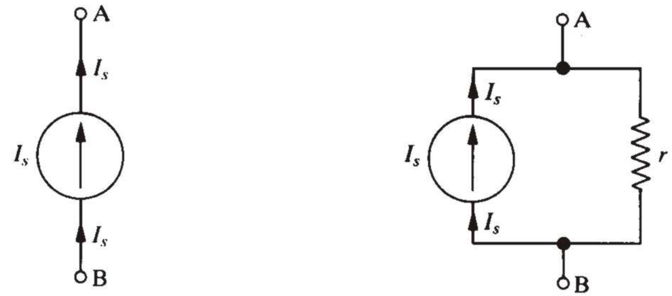
<u>Current source:</u> provides fixed I_{out} regardless of voltage/load resistance.

Has *infinite* resistance.

Real current source have limit on voltage they can provide.

<u>Voltage source</u>

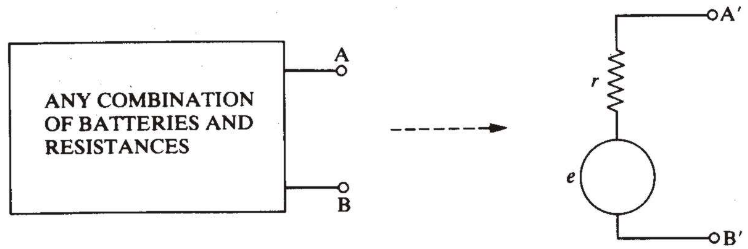
- More common
- In almost every circuit
- Battery or Power Supply (PS)



(c) *ideal current source* (d) *real current source including shunt resistance r*

Thevenin's theorem

Thevenin's theorem states that any two terminals network of R & V sources has an equivalent circuit consisting of a single voltage source V_{TH} and a single resistor R_{TH}.



To find the Thevenin's equivalent V_{TH} & R_{TH}:

- For an "open circuit" ($R_L \rightarrow \infty$), then

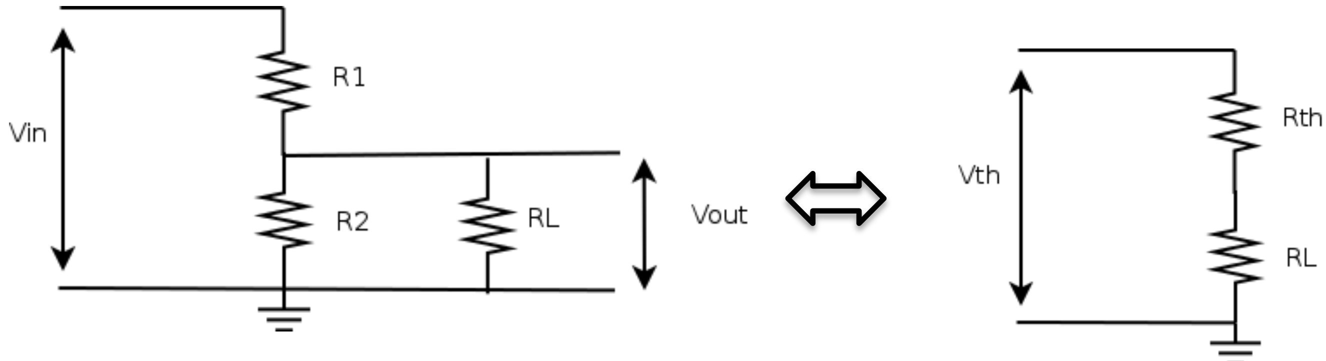
$$V_{Th} = V_{open\ circuit}$$

  - Voltage drops across device when disconnected from circuit – no external load attached.

$$R_{Th} = \frac{V_{open\ circuit}}{I_{short\ circuit}}$$

- For a "short circuit" ($R_L \rightarrow 0$), then
  - $I_{short\ circuit}$ = current when the output is shorted directly to the ground.

Thevenin's theorem (cont)



Thevenin equivalent

Open circuit voltage: $V_{TH} = V_{out} = V_{in} \frac{R_2}{R_1 + R_2}$ ← Lower leg of divider ← Total R

Short circuit current: $I_{\text{short circuit}} = \frac{V_{in}}{R_1}$

Thevenin equivalent:

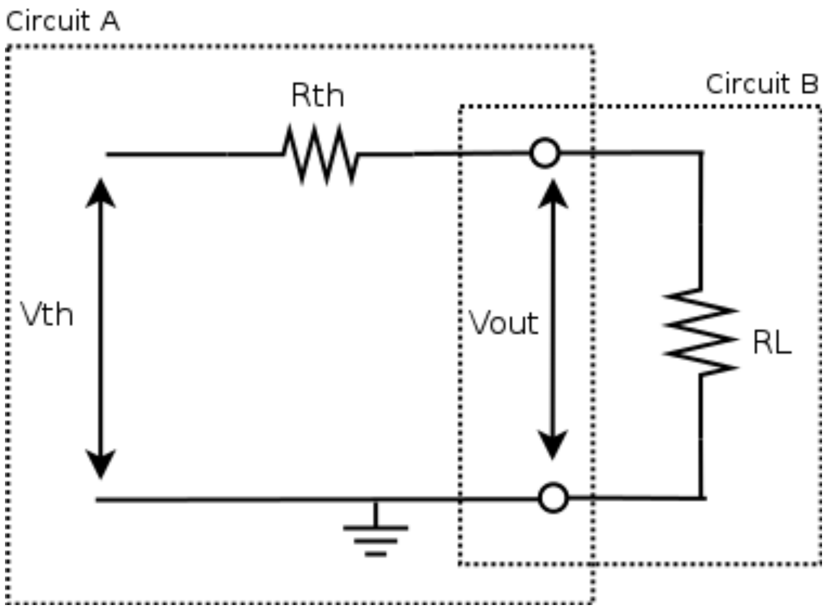
Voltage source: $$V_{TH} = V_{in} \frac{R_2}{R_1 + R_2}$$ V open-circuit – no external load

in series with: $$R_{TH} = \frac{R_1 R_2}{R_1 + R_2}$$ "like" R_1 in parallel with R_2

R_{Th} is called the <u>output impedance</u> (Z_{out}) of the voltage divider

Thevenin's theorem (cont)

Very useful concept, especially when different circuits are connected with each other. Closely related to the concepts of input and output impedance (or resistance).

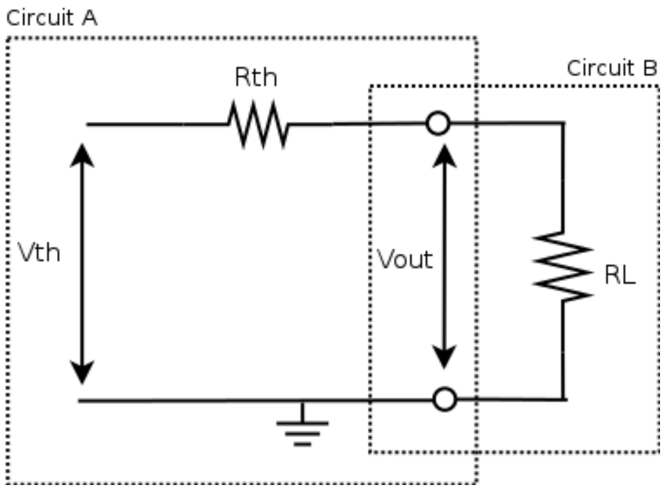


Circuit A, consisting of V_{TH} and R_{TH}, is fed to the second circuit element B, which consists of a simple load resistance R_L.

Avoiding circuit loading

The combined equivalent circuit (A+B) forms a voltage divider:

$$V_{out} = V_{TH}\frac{R_L}{R_{TH}+R_L} = \frac{V_{TH}}{1+\left(R_{TH}/R_L\right)}$$



R_{TH} determines to what extent the output of the 1st circuit behave as an ideal voltage source.

To approximate ideal behavior and avoid loading the circuit, the ratio R_{TH}/R_L should be kept small.

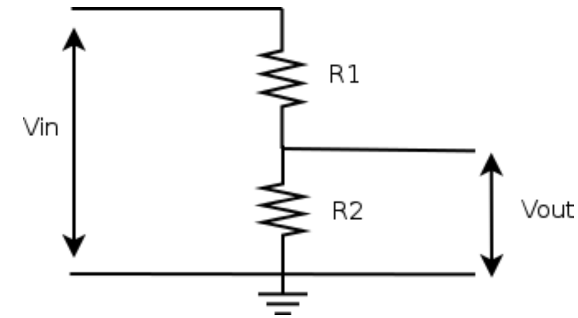
10X rule of thumb: $R_{TH}/R_L = 1/10$

The output impedance of circuit A is the Thevenin equivalent resistance R_{TH} (also called source impedance).

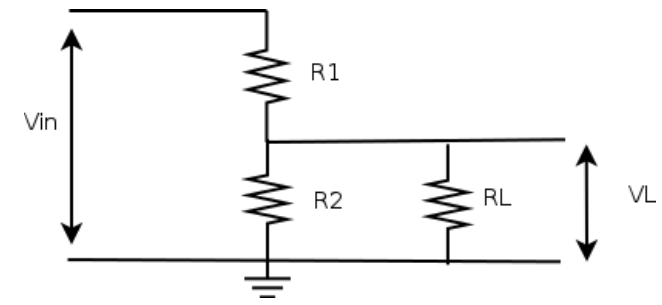
The input impedance of circuit B is its resistance to ground from the circuit input. In this case, it is simply R_L.

Example: voltage divider

V_{in}=30V, R_1=R_2=R_{load}=10k



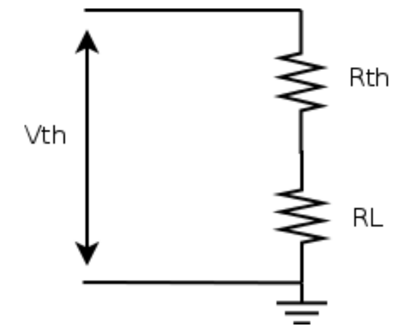
a) Output voltage w/ no load [Answ 15V]



b) Output voltage w/ 10k load [Answ 10V]

Example (cont.)

c) Thevenin equivalent circuit [V_{TH}=15V, R_{TH}=5k]



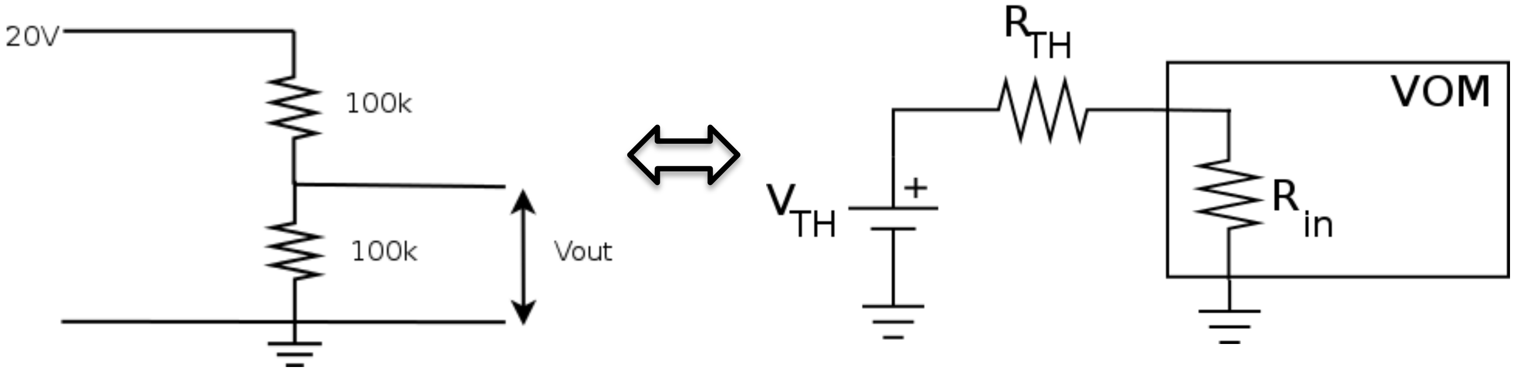
d) Same as b) but using the Thevenin equivalent circuit [Answ 10V]

e) Power dissipated in each of the resistor [Answ P_{R1}=0.04W, P_{R2}=P_{RL}=0.01W]

Example: impedance of a Voltmeter

We want to measure the internal impedance of a voltmeter.

Suppose that we are measuring V_{out} of the voltage divider:



- R_{TH}: 2 100k in parallel, 100k/2 = 50k
- $V_{TH} = 20 \frac{100k}{2 \times 100k} = 10V$

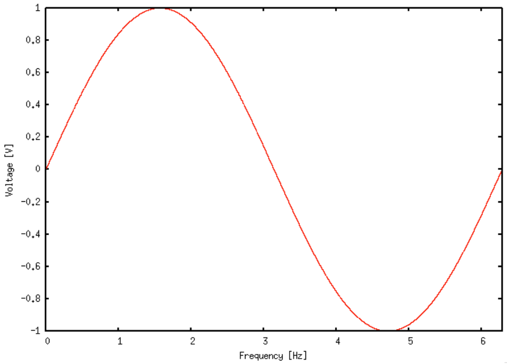
- Measure voltage across R_{in} (V_{out})= 8V, thus 2V drop across R_{TH}
- The relative size of the two resistances are in proportion of these two voltage drops, so R_{in} must be 4 (8/2) R_{TH} , so R_{in}= 200k

Terminology

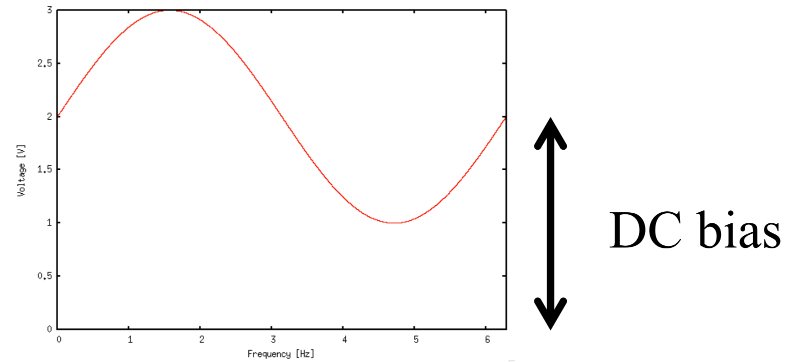
Terminology (cont)

<u>Offset = bias</u>

A DC voltage shifts an AC voltage up or down.



AC signal



AC signal with DC offset

<u>Gain:</u>

$$A_V = \frac{V_{out}}{V_{in}}$$

Voltage gain

Unity gain: $V_{out} = V_{in}$

$$A_I = \frac{I_{out}}{I_{in}}$$

Current gain

Terminology

When dealing with AC circuits we'll talk about V & I vs time or A vs f.

Lower case symbols:

- i: AC portion of current waveform
- v: AC portion of voltage waveform. $V(t) = V_{DC} + v$

Decibels:

To compare ratio of two signals: $dB = 20\log_{10}\frac{\text{amplitude 2}}{\text{amplitude 1}}$

Often used for gain: eg ratio is 1.4≈√2 $20\log_{10}\sqrt{1.4} = 3\text{ dB}$

NB: 3dB ~ power ratio of ½

~ amplitude ratio of 0.7

Capacitors and RC circuits

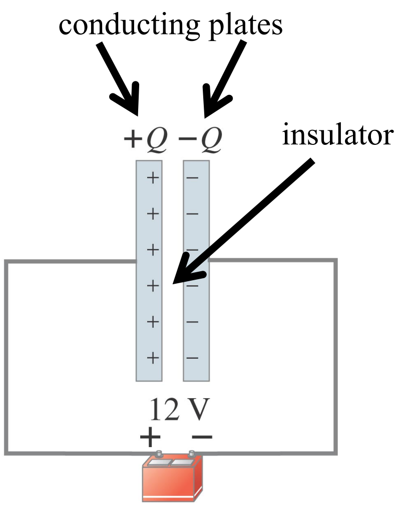
Capacitor: reminder

$$Q = CV$$

Q: total charge [Coulomb]

C: capacitance [Farad 1F = 1C/1V] $C = \frac{Q}{V} = \varepsilon_0 \frac{A}{d}$ [parallel-plate capacitor]

V: voltage across cap



Since $I = \frac{dQ}{dt}$ $\boxed{I = C\frac{dV}{dt}}$ I: rate at which charge flows or rate of change of the voltage

For a capacitor, no DC current flows through, but AC current does.

Large capacitances take longer to charge/discharge than smaller ones.

Typically, capacitances are

- μF (10^{-6})
- pF (10^{-12})

$$C_{eq} = C_1 + C_2 + C_3 \quad \text{[parallel]}$$

Same voltage drop across caps

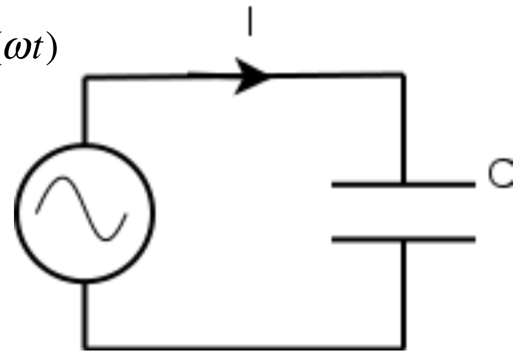
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \quad \text{[series]}$$

All caps have same Q

Frequency analysis of reactive circuit

$$I(t) = C\frac{dV(t)}{dt} = C\omega V_0\cos(\omega t)$$

$V(t) = V_0\sin(\omega t)$



ie the current is out of phase by 90° to wrt voltage (leading phase)

Considering the amplitude only: $I = \dfrac{V_0}{1/\omega C}$ $\omega = 2\pi f$

Frequency dependent resistance: $R = 1/\omega C = \dfrac{1}{2\pi f C}$

<u>Example:</u> C=1µF 110V (rms) 60Hz power line

$$I_{rms} = \frac{110}{1/\left(2\pi\times 60\times 10^{-6}\right)} = 41.5mA(rms)$$

$$I_{rms} = \frac{I}{\sqrt{2}}$$

Impedance of a capacitor

Impedance is a generalized resistance.

It allows rewriting law for capacitors so that it resembles Ohm's law.

Symbol is Z and is the ratio of voltage/current.

Recall: $I = C\dfrac{dV}{dt}$

$$V(t) = V_0\cos(\omega t) = \mathrm{Re}\left[V_0 e^{j\omega t}\right]$$

$$I = C\frac{d}{dt}\left(V_0 e^{j\omega t}\right) = j\omega C V_0 e^{j\omega t}$$

$$I = \frac{V_0 e^{j\omega t}}{-j/\omega C}$$

The actual current is: $I = \mathrm{Re}\left[\dfrac{V_0 e^{j\omega t}}{Z_c}\right]$ $\quad \boxed{Z_c = {-j}/{\omega C}}$

Z_c is the impedance of a capacitor at frequency ω.

As ω (or f) increases (decreases), Z_c decreases (increases)

The fact that Z_c is complex and negative is related to the fact the the voltage across the cap lags the current through it by 90^o.

Ohm's law generalized

Ohm's law for impedances: $V(t) = ZI(t)$ $\tilde{V} = \tilde{Z}\tilde{I}$ using complex notation

$$Z_{eq} = Z_1 + Z_2 + Z_3 \quad \text{[series]}$$

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \quad \text{[parallel]}$$

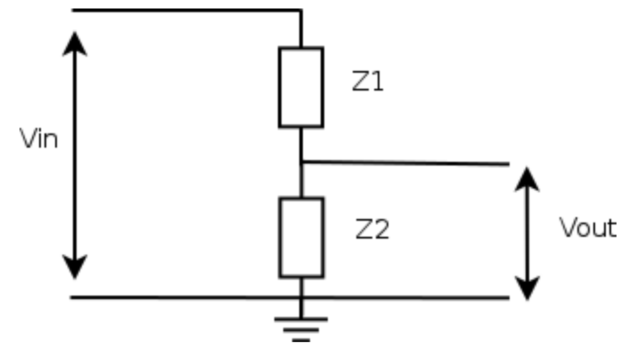
Resistor: $Z_R = R$ in phase with I

Capacitor: $Z_c = {}^{-j}\!/_{\omega C} = {}^{1}\!/_{j\omega C}$ lags I by 90°

Inductor: $Z_L = j\omega L$ leads I by 90° (use mainly in RF circuits)

Can use Kirchhoff's law as before but with complex representation of V & I.

Generalized voltage divider: $$\tilde{V}_{out} = \tilde{V}_{in} \frac{\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}$$

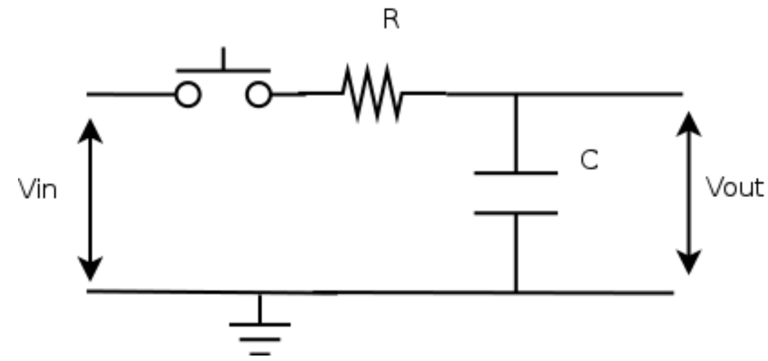


RC circuit

Capacitor is uncharged. At t=0, the RC circuit is connected to the battery (DC voltage)

The voltage across the capacitor increases with time according to:

$$I = C\frac{dV}{dt} = \frac{V_i - V}{R} \quad \rightarrow \quad V = V_i + Ae^{-t/RC}$$

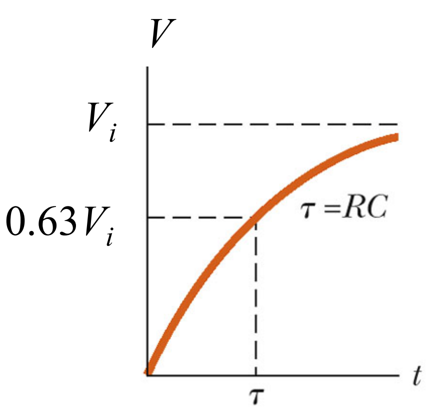


A is determined by the initial condition:

@ t=0, V=0 thus A=-V_i

$$V = V_i\left(1 - e^{-t/RC}\right)$$

when t=RC 1/e=0.37



Rate of charge/discharge is determined by RC constant

@ 1RC 63% of voltage

@ 5RC 99% of voltage

<u>Time constant RC:</u>

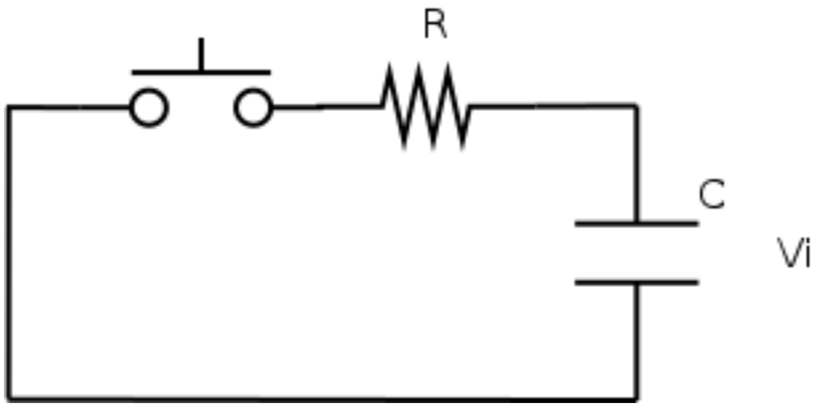
For R Ohms and C in Farads, RC is in seconds

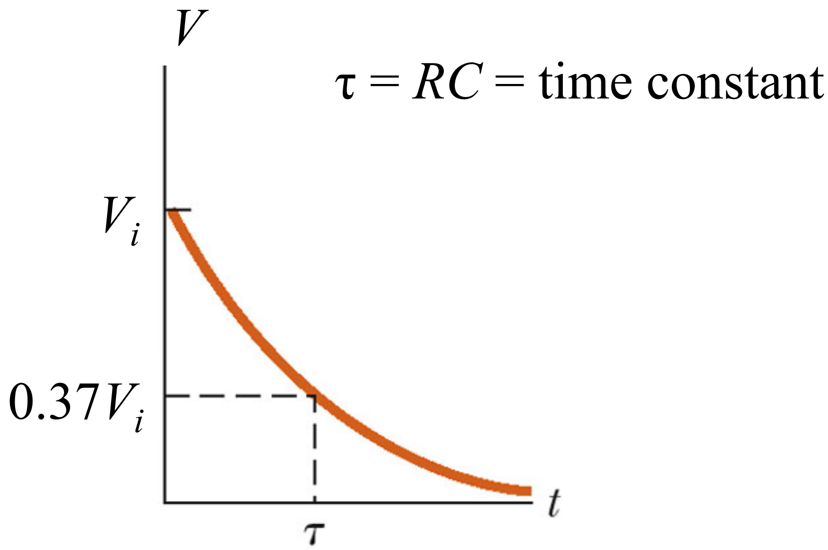
For MΩ and μF, RC is seconds

For kΩ and μF, RC is ms

RC circuit (cont.)

Consider a circuit with a charge capacitor, a resistor, and a switch





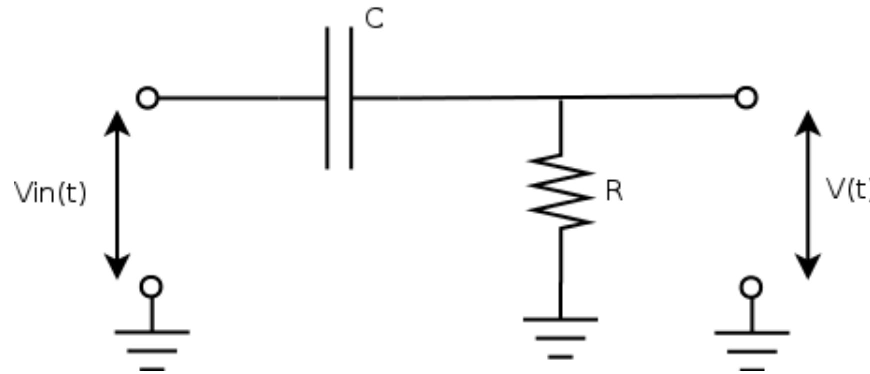
Before switch is closed, $V = V_i$ and $Q = Q_i = CV_i$

After switch is closed, capacitor discharges and voltage across capacitor decreases exponentially with time

$$C\frac{dV}{dt} = I = -\frac{V}{R} \quad \rightarrow \quad \boxed{V = V_i e^{-t/RC}}$$

Differentiator

Consider the series *RC* circuit as a *voltage divider*, with the output corresponding to the voltage across the *resistor*:



V across C is V_{in}-V

$$I = C\frac{d}{dt}\left(V_{in} - V\right) = \frac{V}{R}$$

If we choose R & C small enough so that

$$\frac{dV}{dt} << \frac{dV_{in}}{dt}$$

then, $$V(t) = RC\frac{d}{dt}V_{in}(t)$$

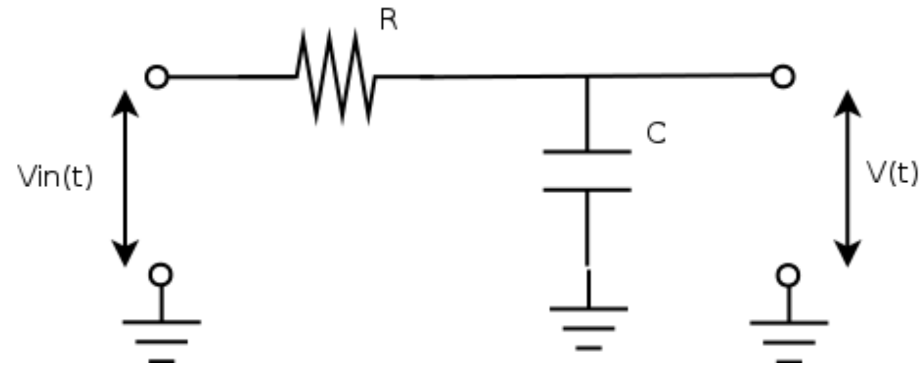
Thus the output differentiate the input waveform!

Simple rule of thumb: differentiator works well if $V_{\mathrm{out}} << V_{\mathrm{in}}$

Differentiators are handy for detecting leading edges & trailing edges in pulse signals.

Integrator

Now flip the order of the resistor and capacitor, with the output corresponding to the voltage across the *capacitor*:



V across R is V_{in}-V

$$I = C\frac{dV}{dt} = \frac{V_{in} - V}{R}$$

If RC is large, then $V \ll V_{in}$ and

$$C\frac{dV}{dt} \cong \frac{V_{in}}{R} \quad \rightarrow \quad V(t) = \frac{1}{RC}\int^{t} V_{in}(t)dt + cst$$

Thus the output integrate the input!

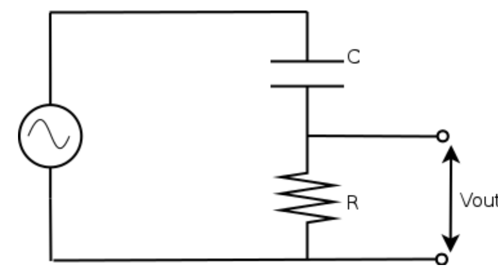
Simple rule of thumb: integrator works well if $V_{out} << V_{in}$

Integrators are used extensively in analog computation (eg analog/digital conversion, waveform generation etc…)

High-pass filter

Let's interpret the differentiator *RC* circuit as a *frequency-dependent voltage divider* ("frequency domain"):

Using complex Ohm's law: $\tilde{I} = \frac{\tilde{V}_{in}}{\tilde{Z}_{total}} = \frac{\tilde{V}_{in}}{R - \left(j/\omega C \right)} = \frac{\tilde{V}_{in}\left[R + \left(j/\omega C \right) \right]}{R^2 + 1/\omega^2 C^2}$



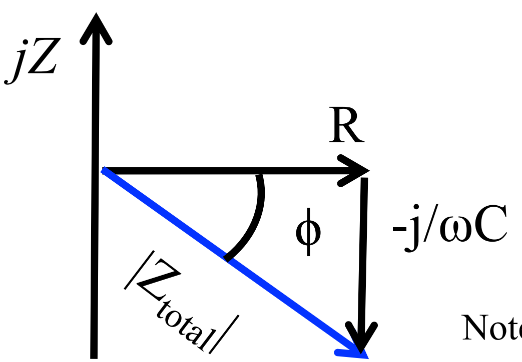
Voltage divider made of R & C

Voltage across R is: $\tilde{V}_{out} = \tilde{I}R = R\frac{\tilde{V}_{in}\left[R + \left(j/\omega C \right) \right]}{R^2 + 1/\omega^2 C^2}$

If we care only about the amplitude: $V_{out} = \left(\tilde{V}_{out} \tilde{V}^*_{out} \right)^{1/2} = V_{in} \frac{R}{\left[R^2 + 1/\omega^2 C^2 \right]^{1/2}}$

- Thus V_{out} increases with increasing f

Impedance of a series RC combination: $\tilde{Z}_{total} = R - j/\omega C$



$\phi = \tan^{-1}\left(\frac{-1/\omega C}{R} \right)$

$|Z_{total}| = \sqrt{R^2 + 1/\omega^2 C^2}$

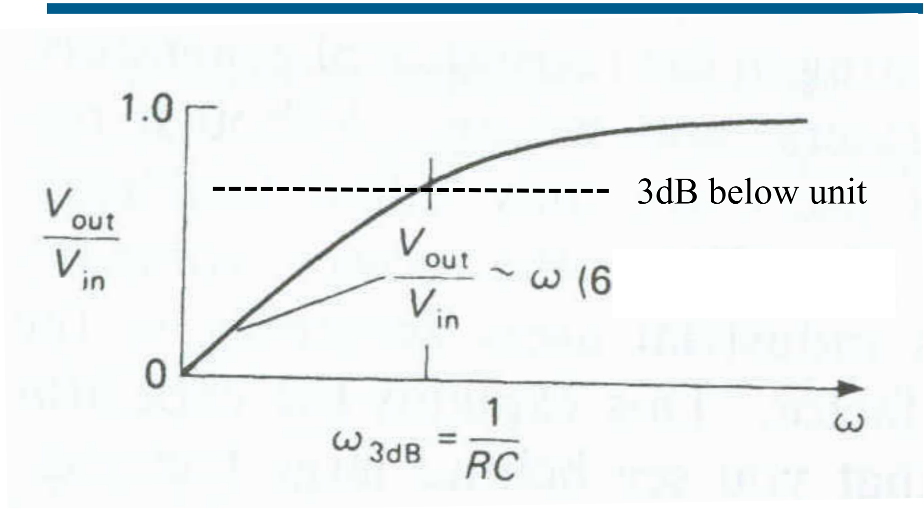
Note phase of output signal

$V_{out} = V_{in}\frac{R}{|Z_{total}|}$

impedance of lower-leg of divider

Magnitude of impedance of R & C

High-pass filter frequency response



High-pass filter frequency response curve

Output ~ equal to input at high frequency

when ω ~ 1/RC [rad]

Goes to zero at low frequency.

A high-pass filter circuit attenuates low frequency and "passes" the high frequencies.

The frequency at which the filter "turns the corner" (ie $V_{out}/V_{in}=1/\sqrt{2}=0.7$) is called the 3dB point:

$$f_{3dB} = \frac{1}{2\pi RC}[Hz]$$

Use this in lab

otherwise factor 2π off

occurs when $Z_c=R$

NB: 3dB ~ power ratio of ½

~ amplitude ratio of 0.7

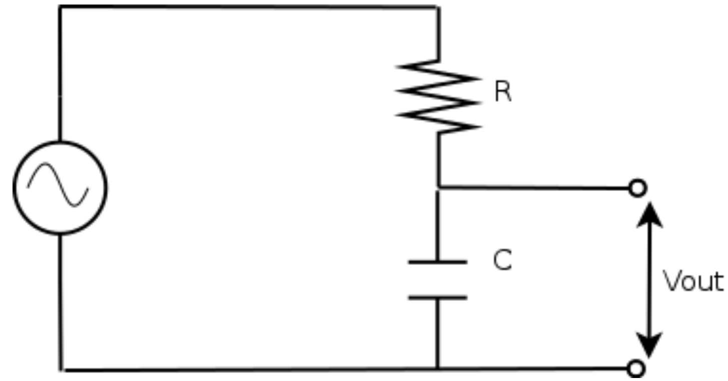
Low-pass filter

Now simply switch the order of the resistor and capacitor in the series circuit (same order as the integrator circuit earlier):

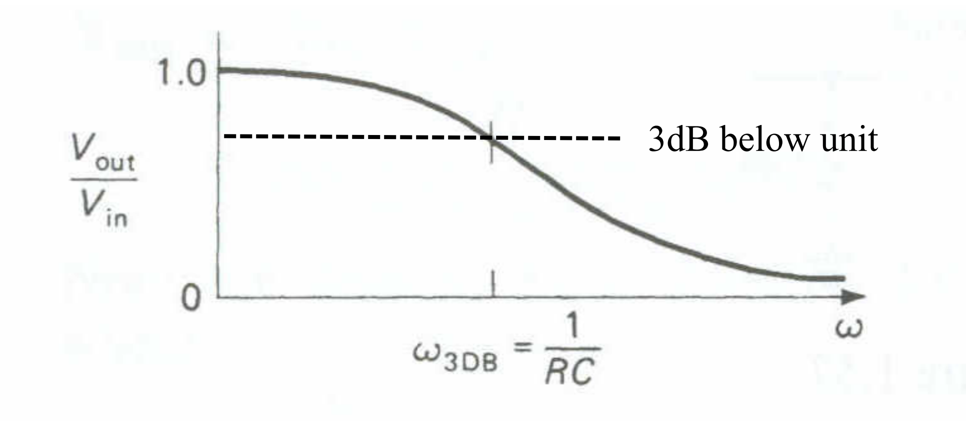
$$V_{out} = V_{in} \frac{1/\omega C}{\left[R^2 + 1/\omega^2 C^2\right]^{1/2}}$$

impedance of lower-leg of divider

Magnitude of impedance of R & C



A low-pass filter circuit attenuates high frequency and "passes" the low frequencies.



Low-pass filter frequency response curve