TRIGONOMETRIC FUNCTIONS (18.014, FALL 2015)

These are notes for Lecture 21, in which trigonometric functions were defined.

1. Definition of sine and cosine

We define the sine function as the unique function satisfying a certain differential equation and certain initial conditions. We prove existence and uniqueness in the following theorem.

Theorem 1.1. *There is exactly one function* $f : \mathbb{R} \to \mathbb{R}$ *that is twice-differentiable, satisfies* $f(0) = 0, f'(0) = 1$*, and satisfies the differential equation*

$$f''(x) = -f(x) \text{ for } x \in \mathbb{R}.$$

Proof. **Existence:** First we construct one function f with the desired properties. Problem Set 8 outlines one approach to this construction; here we take a more advanced approach using infinite series. Define f by the power series

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

The ratio of absolute values of consecutive terms is

$$\left| \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{x^{2n-1}} \right| = \frac{|x|^2}{(2n)(2n+1)},$$

which tends to 0 as $n \to \infty$ for any x, so by the ratio test this power series converges absolutely on the entire real line and indeed defines a function $f : \mathbb{R} \to \mathbb{R}$. Then by Theorem 11.9 we can differentiate term-by-term to obtain

$$f'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

and

$$f''(x) = 0 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots .$$

Thus $f''(x) = -f(x)$ and we can also see that $f(0) = 0, f'(0) = 1$, so f is as desired.

Uniqueness: Suppose that f and g are two functions with the desired properties, and let $h = f - g$. Then we have that $h(0) = h'(0) = 0$ and $h'' = -h$. Let $j(x) = h(x)^2 + h'(x)^2$. Then we can compute that

$$j'(x) = 2h(x)h'(x) + 2h'(x)h''(x) = 0,$$

so j is a constant function. Since $j(0) = 0^2 + 0^2 = 0$, this means $j(x) = 0$ for all x. Since $h(x)^2 \geq 0$ and $h'(x)^2 \geq 0$, this implies that $h(x) = 0$ for all x, so $f = g$ as desired. □

Definition. The *sine function* $\sin : \mathbb{R} \to \mathbb{R}$ is the unique function satisfying the conditions in Theorem 1.1. The *cosine function* $\cos : \mathbb{R} \to \mathbb{R}$ is the derivative of the sine function.

2. Easy properties of sin and cos

Proposition 2.1. *The sine and cosine functions satisfy the following properties:*

(a) $\frac{d}{dx}\sin x = \cos x$*,* $\frac{d}{dx}\cos x = -\sin x$

(b) $\sin^2 x + \cos^2 x = 1$

(c) $\sin(-x) = -\sin x$*,* $\cos(-x) = \cos x$

(d) $\sin(x+y) = \sin x \cos y + \cos x \sin y$*,* $\cos(x+y) = \cos x \cos y - \sin x \sin y$*.*

Proof. (a) This follows from the definition of the cosine function and the differential equation satisfied by the sine function.

(b) We compute the derivative:

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = 2\sin x \cos x - 2\cos x \sin x = 0.$$

Thus $\sin^2 x + \cos^2 x$ is constant, and $\sin^2 0 + \cos^2 0 = 0^2 + 1^2 = 1$ so it is equal to 1 for all x.

(c) The function $f(x) = -\sin(-x)$ satisfies all the conditions in Theorem 1.1, so it is equal to $\sin x$. Differentiating this identity gives the second identity.

(d) For any fixed x, the function $f(z) = \sin x \cos(z-x) + \cos x \sin(z-x)$ satisfies all the conditions in Theorem 1.1, so it is equal to $\sin z$. Replacing z by $x+y$ gives the first identity, and differentiating with respect to x then gives the second identity.

□

3. Definition of π

We will need the following lemma:

Lemma 3.1. *There exists* $x > 0$ *such that* $\cos x = 0$*.*

Proof. Because $\sin^2 x + \cos^2 x = 1$, $|\sin x| \leq 1$ for all $x \in \mathbb{R}$. Also, by the Mean Value Theorem applied to the interval $[1,5]$, there exists some $c \in (1,5)$ such that

$$\cos c = \frac{\sin 5 - \sin 1}{5-1}.$$

Combining these we obtain that $|\cos c| \leq \frac{1}{2}$. Then using the cosine double-angle formula (a consequence of Proposition 2.1), we have that

$$\cos(2c) = 2\cos^2 c - 1 \leq 2 \cdot \frac{1}{4} - 1 = -\frac{1}{2} < 0.$$

Since $\cos 0 = 1 > 0$ and $\cos(2c) < 0$, by the intermediate value theorem there exists $x \in (0, 2c)$ with $\cos x = 0$. □

Once we know that $\cos x = 0$ for some positive x, it is a simple continuity argument to show that there is a minimum such x. We choose π so that that minimum x is $\frac{\pi}{2}$.

Definition. The real number π is defined by

$$\pi = 2\min\{x > 0 \mid \cos x = 0\}.$$

Since $\cos x > 0$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\sin x$ is strictly increasing on this interval. Since $\sin x = \pm\sqrt{1-\cos^2 x}$, this implies that $\sin\frac{\pi}{2} = 1$. It is now straightforward to compute $\sin(\frac{n\pi}{2})$ and $\cos(\frac{n\pi}{2})$ for any integer n using the sine and cosine addition rules. For example,

$$\sin(2\pi) = 2\sin\pi\cos\pi = 4\sin\frac{\pi}{2}\cos\frac{\pi}{2}\left(2\cos^2\frac{\pi}{2} - 1\right) = 0$$

and

$$\cos(2\pi) = 2\cos^2\pi - 1 = 2\left(\cos^2\frac{\pi}{2} - 1\right)^2 - 1 = 1.$$

We can now see that sin and cos are periodic functions with period 2π, since the addition formulas give that

$$\sin(x+2\pi) = \sin x\cos(2\pi) + \cos x\sin(2\pi) = \sin x$$

and

$$\cos(x+2\pi) = \cos x\cos(2\pi) - \sin x\sin(2\pi) = \cos x.$$

4. Inverse trigonometric functions

The sine and cosine functions take on the same value many times; e.g.

$$0 = \sin 0 = \sin\pi = \sin(2\pi) = \sin(3\pi) = \cdots.$$

Because of this, we define inverse functions $\sin^{-1}$ and $\cos^{-1}$ by restricting to some interval where the function takes on every value in $[-1,1]$ exactly once.

In the case of sin, we know that $\sin(-\frac{\pi}{2}) = -1, \sin\frac{\pi}{2} = 1$, and that sin is strictly increasing between these two values (since $\cos x$ is positive there). Thus we can define

$$\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

as the inverse function to sin on that interval. Similarly we define

$$\cos^{-1} : [-1,1] \to [0,\pi].$$

By the theorem on differentiating inverse functions, these two functions are differentiable at all points except -1 and 1, with derivatives

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}\cos^{-1} x = \frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}},$$

where we've used the facts that cos is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and that sin is positive on $(0,\pi)$. The fact that these derivatives are negatives of each other is consistent with the easily checked identity

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

We can now check that our definition of π agrees with the definition of π as the area of a circle of unit radius: this area is computed by the integral

$$\int_{-1}^{1} 2\sqrt{1-x^2},$$

and we can compute

$$\frac{d}{dx}(\sin^{-1} x + x\sqrt{1-x^2}) = 2\sqrt{1-x^2},$$

so by the fundamental theorem of calculus, the area is equal to

$$\sin^{-1} x + x\sqrt{1-x^2}\Big|_{-1}^{1} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi.$$