4/5/2012: Second midterm practice A

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The formula $\int_0^x f''(x)\, dx = f'(x) - f'(0)$ holds.

Solution:
Apply the fundamental theorem to the derivative.

2) T F The area of the upper half disc is the integral $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

Solution:
The circle has the equation $x^2 + y^2 = 1$. Solving for y gives $y = \sqrt{1-x^2}$.

3) T F If the graph of the function $f(x) = x^2$ is rotated around the interval $[0, 1]$ we obtain a solid with volume $\int_0^1 \pi x^4\, dx$.

Solution:
Yes, the cross section area is $A(x) = \pi x^4$. Integrate this from 0 to 1.

4) T F The function $f(x) = e^x$ is the only anti derivative of e^x.

Solution:
We can scale this function too. The function $2e^x$ for example has also the same anti derivative $2e^x$.

5) T F If f has a critical point at 1, then $F(x) = \int_0^x f(t)\, dt$ has an inflection point at 1.

Solution:
By the fundamental theorem of calculus, $F' = f$ and $F'' = f'$. If f has the critical point 1, then $f'(1) = 0$ and so $F''(1) = 0$ which by definition means that F has an inflection point. An inflection point for F is a point, where $F'' = f'$ changes sign. That means $f' = 0$.

6) T F Catastrophes are parameter values c for a family of functions $f_c(x)$, for which a local minimum of f_c disappears.

Solution:
This is the definition of a catastrophe.

7) T F The volume of a cylinder of height and radius 1 minus the volume of a cone of height and radius 1 is half the volume of a sphere of radius 1.

Solution:
This was Archimedes insight and you have discovered yourself in a homework.

8) T F Rolle's theorem tells that if $0 < c < 1$ is a critical point of f and $f(0) = f(1)$, then the critical point is in the interval $[0, 1]$.

Solution:
There could also be critical points outside the interval.

9) T F Rolle also introduced the notation $|x|^{1/3}$ for roots.

Solution:
Yes, this was mentioned in class and on the handout.

10) T F Integrals are linear: $\int_0^x f(t) + g(t)\; dt = \int_0^x f(t)\; dt + \int_0^x g(t)\; dt$.

Solution:
You have verified this in a homework.

11) T F The function $\text{Li}(x) = \int_2^x dt/\log(t)$ has an anti-derivative which is a finite construct of trig functions.

Solution:
No, it is known that this logarithmic integral has no elementary anti-derivative.

12) T F There is a region enclosed by the graphs of x^5 and x^6 which is finite and positive.

Solution:
Yes, there is a finite region enclosed. It is between $x = 0$ and $x = 1$.

13) T F The integral $\int_{-1}^{1} 1/x^4\; dx = -1/(5x^5)|_{-1}^{1} = -1/5 - 1/5 = -2/5$ is defined and negative.

Solution:
It is not defined at 0 and can not even be saved using the Cauchy principal value.

14) T F Gabriel's trumpet has finite volume but infinite surface area.

Solution:
Yes, we have seen this in class.

15) T F A function $f(x)$ is a probability density function, if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\; dx = 1$.

Solution:
This is the definition of a probability density function.

16) T F The mean of a function on an interval $[a, b]$ is $\int_a^b f(x)\; dx$.

Solution:
For the mean, we would have to divide by $(b - a)$.

17) T F The cumulative probability density function is an antideriative of the probability density function.

Solution:
Yes, by definition. It is a particular antiderivative which has the property that $\lim_{x\to-\infty} F(x) = 0$.

18) T F The integral $\int_{-\infty}^{\infty}(x^2 - 1)\; dx$ is finite.

Solution:
No way. The function does not even go to zero at infinity. $\int_{-R}^{R}(x^2-1)\; dx = (x^3/3-x)|_{-R}^{R} = 2R^3/3 - 2R$ does not converge for $R \to \infty$.

19) T F The total prize is the derivative of the marginal prize.

Solution:
Its reversed. The marginal prize is the derivative of the total prize.

20) T F The acceleration is the anti-derivative of the velocity.

Solution:
It is reversed. The acceleration is the derivative of the velocity.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their anti derivatives. Of course only 6 of the 30 functions will appear.

Function	Antiderivative Enter 1-30
$\cos(3x)$	
$\sin(3x)$	
$3x$	

Function	Antiderivative Enter 1-30
$1/(3x)$	
$\tan(3x)$	
$1/(1+9x^2)$	

1) $\sin(3x)$
2) $-\sin(3x)/3$
3) $\sin(3x)/3$
4) $-3\sin(3x)$
5) $3\sin(3x)$

6) $\cos(3x)$
7) $-\cos(3x)/3$
8) $\cos(3x)/3$
9) $-3\cos(3x)$
10) $3\cos(3x)$

11) $\log(x)/3$
12) $1/(3-x)$
13) $1/(3x)$
14) $\log(x/3)$
15) $-1/(3x^2)$

16) $3x^2$
17) $x^2/2$
18) $3x^2/2$
19) 3
20) x^2

21) $\arctan(3x)/3$
22) $3\arctan(3x)$
23) $1/(1+9x^2)$
24) $3/(1+9x^2)$
25) $-3/(1+x^2)$

26) $1/\cos^2(3x)$
27) $\log(\cos(3x))$
28) $-\log(\cos(3x))/3$
29) $\log(\cos(3x))/3$
30) $3/\cos^3(3x)$

Solution:
The magic numbers are $3, 7, 18, 11, 28, 21$.

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following formulations is a Riemann sum approximating the integral $\int_0^3 f(x)\,dx$ of $f(x) = x^2$ over the interval $0, 3$.

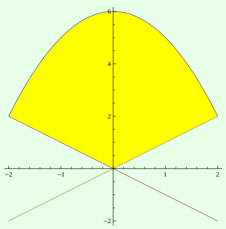
Sum	Check if this is the Riemann sum.
$n\sum_{k=0}^{n-1}(3k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{n-1}(3k/n)^2$	
$n\sum_{k=0}^{3n-1}(k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{3n-1}(k/n)^2$	

Solution:
It is the fourth entry.

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the three curves $y = 6 - x^2, y = -x$ and $y = x$.

Solution:
Make a picture:



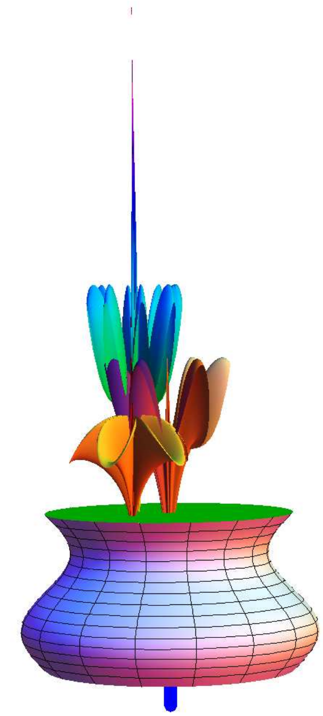
We can look at the region on the right of the y-axis and have therefore the area between the graph of $f(x) = x$ and $f(x) = 6 - x^2$. These graphs intersect at $x = 2$. The integral is

$$\int_0^2 (6 - x^2 - x)\, x\, dx = 22/3 .$$

Since we have only computed half, we have to multiply this with 2. The area is 44/3.

Problem 5) Volume computation (10 points)

Jeanine grows some magical plants in a pot which is a rotationally symmetric solid for which the radius at position x is $5 + \sin(x)$ and $0 \leq x \leq 2\pi$. Find the volume of the pot.



Solution:
The area of the cross section at hight x is $A(x) = \pi(5 + \sin(x))^2$. The volume is

$$\int_0^{2\pi} \pi(5 + \sin(x))^2 \, dx = \pi \int_0^{2\pi} 25 + 10\sin(x) + \sin^2(x)\, dx = (50\pi + \pi)\pi = 51\pi^2 .$$

To find the anti-derivative of $\sin^2(x)$ we use the **double angle formula** $1 - 2\sin^2(x) = \cos(2x)$ leading to $\sin^2(x) = [1 - \cos(2x)]/2$ so that $\int \sin^2(x)\, dx = x/2 - \sin(2x)/4$.

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x^{1/5} + x^4 + 1/x \, dx$.

b) (5 points) $\int_1^3 2x + \sin(x-1) + \cos(x+2)\ dx$

Solution:
a) We get $((5/6)x^{6/5} + x^5/5 + \log(x)|_1^2 = 161/30 + (5/3)2^(1/5) + \log(2)$.
b) We get $x^2 - \cos(x-1) + \sin(x+2)|_1^3 = 9 - \cos(2) - \sin(3) + \sin(5)$.

Problem 7) Anti-derivatives (10 points)

Find the following anti-derivatives

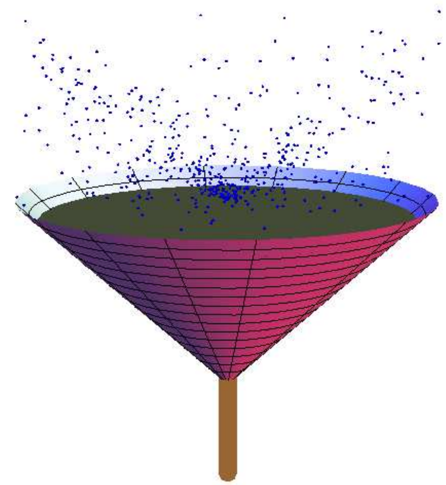
a) (5 points) $\int \frac{3}{\sqrt{1-x^2}} + x^4 + \frac{1}{1+x^2}\ dx$

b) (5 points) $\int \frac{1}{x-2} + \frac{1}{x+4} + \frac{2}{x-1}\ dx$

Solution:
a) $3\arcsin(x) + x^5/5 + \arctan(x)$.
b) $\log(x-2) + \log(x+4) + 2\log(x-1)$.

Problem 8) Related rates (10 points)

A coffee machine has a filter which is a cone of radius z at height z. Coffee drips out at a rate of 1 cubic centimeter per second. How fast does the water level sink at height $z = 10$?



Solution:
We first compute the area $A(z) = z^2\pi$ and then the volume $V(z) = \int_0^z s^2\pi\ ds = z^3\pi/3$. Therefore, $1 = d/dt V(z(t)) = z^2\pi z'(t)$ so that $z'(t) = 1/(z^2\pi) = 1/(100\pi)$.

Problem 9) Implicit differentiation (10 points)

Find the derivatives $y' = dy/dx$ of the following implicitly defined functions:

a) (5 points) $x^5 + 3x + y = e^y$.

b) (5 points) $\sin(x^2 - 2y) = y - x$.

Solution:
a) $5x^2 + 3 + y' = e^y y'$. Solve for $y' = (5x^4+3)/(e^y - 1)$.
b) $\cos(x^2-2y)(2x-2y') = y' - 1$. Solve for $y' = [1 + 2x(\cos(x^2-2y)]/(1+2\cos(x^2-2y)$.

Problem 10) Improper integrals (10 points)

Evaluate the following improper integrals or state that they do not exist

a) (3 points) $\int_1^\infty 1/\sqrt{x}\, dx$.

b) (2 points) $\int_0^1 \sqrt{x}\, dx$.

c) (3 points) $\int_0^\infty 2xe^{-x^2}\, dx$.

d) (2 points) $\int_0^\infty \frac{1}{x}\, dx$

Solution:
a) $\int_1^R 1/\sqrt{x}\, dx = 2\sqrt{x}|_1^R$ does not have a limit for $R \to \infty$.
b) $\int_0^1 \sqrt{x}\, dx = x^{3/2}(2/3)|_0^1 = 2/3$.
c) $-e^{-x^2}|_0^\infty = 1$.
d) $\log(x)|_a^R = \log(R) - \log(a)$ does not exist on both ends. For $a \to 0$ and for $R \to \infty$.