Dept. of Math. Sci., WPI
MA 1034 Analysis 4
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Homework Assignment 4

Due Friday, April 25, 2003

1. If $f(x, y) = 3x^2y^3 - \sin x$, find f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxy}, and f_{xyx}.

   Solution. We have:

$$
\begin{aligned}
f_x &= 6xy^3 - \cos x \\
f_y &= 9x^2y^2 \\
f_{xx} &= 6y^3 + \sin x \\
f_{xy} &= 18xy^2 \\
f_{yy} &= 18x^2y \\
f_{xxy} = f_{xyx} &= 18y^2
\end{aligned}
$$

2. Suppose that

$$
f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}
$$

   (a) Use the definitions of partial derivatives to compute $f_x(0, y)$, $f_y(x, 0)$, $f_{xy}(0, 0)$, and $f_{yx}(0, 0)$. Are the mixed partials at $(0, 0)$ equal?

   (b) Compute $f_x(x, y)$ and $f_y(x, y)$ for $(x, y) \neq (0, 0)$. Are the values $f_x(0, y)$ and $f_y(x, 0)$ the same as the ones found in part (a)?

   Solution.

   (a) We have

$$
f_x(0, y) = \lim_{x \to 0} \frac{f(x, y) - f(0, y)}{x} = \lim_{x \to 0} \frac{y(x^2 - y^2)}{x^2 + y^2} = -y,
$$

   and

$$
f_y(x, 0) = \lim_{y \to 0} \frac{f(x, y) - f(x, 0)}{y} = \lim_{y \to 0} \frac{x(x^2 - y^2)}{x^2 + y^2} = x.
$$

   Hence,

$$
f_{xy} = -1 \qquad \text{and} \qquad f_{yx} = 1.
$$

   (b) A standard calculation gives

$$
f_x(x, y) = \frac{y(x^4 - y^4 + 4x^2y^2)}{(x^2 + y^2)^2}
$$

   and

$$
f_y(x, y) = \frac{x(x^4 - y^4 - 4x^2y^2)}{(x^2 + y^2)^2}.
$$

   Substituting $x = 0$ and $y = 0$ in the two formulas respectively gives us the same answer as part (a).

3. Show that $f(x, y) = \sqrt{x^2 + y^2}$ is not differentiable at the origin by showing that:

(a) there is no $\vec{m}$ as needed in Definition 11.4.1.

(b) $f_x(0,0)$ does not exist and using part (c) of remark 11.4.2

SOLUTION.

(a) Let $\vec{m} = \langle m_1, m_2 \rangle$. If f is differentiable at $(0,0)$ then m_1 and m_2 can be chosen so that
$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - m_1x - m_2y}{\sqrt{x^2+y^2}} = 0.$$
In our case, we have
$$\frac{f(x,y) - f(0,0) - m_1x - m_2y}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2} - m_1x - m_2y}{\sqrt{x^2+y^2}} = 1 - \frac{m_1x + m_2y}{\sqrt{x^2+y^2}}.$$
We observe that
$$\lim_{(x,y)\to(0,0)} \frac{m_1x + m_2y}{\sqrt{x^2+y^2}}$$
does not exist (e.g. by taking sequences $(x_n, y_n) = (1/n, 0)$, $(x_n, y_n) = (0, 1/n)$, and $(x_n, y_n) = (1/n, 1/n)$) unless $m_1 = m_2 = 0$. But in the latter case,
$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - m_1x - m_2y}{\sqrt{x^2+y^2}} = 1,$$
not zero as reqired.

(b) A standard calculation gives
$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0} \frac{|x|}{x}$$
and the latter limit does not exist (e.g. by taking sequences $(x_n, y_n) = (1/n, 0)$, and $(x_n, y_n) = (-1/n, 0)$).

4. Consider the function $f(x,y) = \sqrt[3]{xy}$.

(a) Show that $f_x(0,0) = 0 = f_y(0,0)$.

(b) Find $\nabla f(0,0)$.

(c) Show that f is not differentiable at $(0,0)$.

(d) Is f continuous at $(0,0)$? Explain.

SOLUTION.

(a) We have
$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} == \lim_{x\to 0} \frac{0-0}{x} = 0,$$
and similarly, $f_y(0,0) = 0$.

(b) Formally, $\nabla f(0,0) = \langle f_x(0,0), f_y(0,0) \rangle = \langle 0, 0 \rangle$. Observe however, that the function is not differentiable at the origin (see part (c)).

(c) We look at
$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \nabla f(0,0)\cdot\langle x, y\rangle}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{xy}}{\sqrt{x^2+y^2}},$$
and see that this limit does not exist, (e.g. by taking sequences $(x_n, y_n) = (1/n, 0)$, and $(x_n, y_n) = (1/n, 1/n)$).

(d) Of course it is, as a composition of continuous functions, the cube root (which is everywhere continuous) and a polynomial.

5. Show that

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

is not differentiable at $(0,0)$.

SOLUTION. By taking sequences $(x_n, y_n) = (1/n, 0)$, and $(x_n, y_n) = (1/n, 1/n)$, we see that f is not even continuous at $(0,0)$, let alone differentiable.

6. Find a point (a,b) for which the function

$$f(x,y) = \begin{cases} (x-y)^2 \sin \frac{1}{x-y}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

is differentiable at (a,b), but f_x and f_y are not continuous at (a,b).

SOLUTION. The point $(0,0)$ is such a point (in fact, every point of the form (a,a) will work). We have

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0,$$

and similarly $f_y(0,0) = 0$.Further,

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \nabla f(0,0) \cdot \langle x, y\rangle}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{\sqrt{x^2+y^2}} \sin \frac{1}{x-y} \\
&= \lim_{(x,y)\to(0,0)} (x-y) \frac{(x-y)}{\sqrt{x^2+y^2}} \sin \frac{1}{x-y} \\
&= 0,
\end{aligned}$$

because the first factor, $(x-y)$, goes to 0, and everything else is bounded. This proves the differentiability at the origin.

On the other hand, the partial derivatives are not continuous. Ideed, for $x \neq y$ we have

$$f_x(x,y) = 2(x-y) \sin \frac{1}{x-y} - \cos \frac{1}{x-y}$$

and

$$f_y(x,y) = -2(x-y) \sin \frac{1}{x-y} + \cos \frac{1}{x-y}$$

For both partial derivatives, if we let $(x,y) \to (0,0)$, the limit does not exist.

7. If

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

show that

(a) f is continuous at $(0,0)$.

(b) $f_x(0,0)$ and $f_y(0,0)$ both exist.

(c) f is not differentiable at $(0,0)$.

Solution.

(a) We have

$$\frac{xy}{\sqrt{x^2+y^2}} = x\frac{y}{\sqrt{x^2+y^2}}$$

As $(x,y) \to (0,0)$, the first term in this product goes to 0, while the second is bounded. Thus, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, and the function is continuous.

(b) We have

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} == \lim_{x\to 0}\frac{0-0}{x} = 0,$$

and similarly, $f_y(0,0) = 0$.

(c) We look at

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\nabla f(0,0)\cdot\langle x,y\rangle}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2},$$

and see that this limit does not exist, (e.g. by taking sequences $(x_n,y_n) = (1/n,0)$, and $(x_n,y_n) = (1/n,1/n)$).

8. If

$$f(x,y) = \begin{cases} \frac{x^2y}{\sqrt{x^6+2y^2}}, & \text{if } (x,y) \neq (0,0),\\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

show that f is not continuous at $(0,0)$, but has a directional derivative in every direction at $(0,0)$.

Solution. Let $\vec{u} = \langle u_1, u_2\rangle$. We have

$$D_{\vec{u}}f(0,0) = \lim_{t\to 0}\frac{f(u_1t,u_2t)-f(0,0)}{t} = \lim_{t\to 0}\frac{1}{t}\frac{u_1^2u_2t^3}{\sqrt{u_1^6t^6+2u_2^2t^2}} = \lim_{t\to 0}\frac{u_1u_2^2t}{\sqrt{u_1^6t^4+2u_2^2}} = 0.$$

In particular, we see that the directional derivatives at the origin exists and is equal to zero in every direction.

At the same time, the function is not continuous, as can be seen by choosing a sequence $(x_n,y_n) = (1/n,1/n^3)$.

9. If

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0),\\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

show that $D_{\vec{u}}f(0,0)$ exists only if $\vec{u} = \langle 1,0\rangle$ or $\vec{u} = \langle 0,1\rangle$.

Solution. Let $\vec{u} = \langle u_1, u_2\rangle$. We have

$$D_{\vec{u}}f(0,0) = \lim_{t\to 0}\frac{f(u_1t,u_2t)-f(0,0)}{t} = \lim_{t\to 0}\frac{1}{t}\frac{u_1u_2}{u_1^2+u_2^2} = \lim_{t\to 0}\frac{u_1u_2}{t}.$$

This limit exists if and only if $u_1u_2 = 0$. Combining this with the fact that $u_1^2+u_2^2 = 1$, we see that we must have $\vec{u} = \langle 1,0\rangle$ or $\vec{u} = \langle 0,1\rangle$.

10. Find the unit vector in the direction in which $f(x, y) = y^2 \sin x$ increases most rapidly at the point $(0, -2)$. What is the maximum rate of change of f at $(0, -2)$?

SOLUTION. We have

$$\nabla f(x, y) = \langle y^2 \cos x, 2y \sin x \rangle$$

and

$$\nabla f(0, -2) = \langle 4, 0 \rangle.$$

The norm of the gradient, $\|\nabla f(0, -2)\| = 4$ gives us the maximum rate of change of f at $(0, -2)$, whereas, the direction is given by the unit vector

$$\frac{\nabla f(0, -2)}{\|\nabla f(0, -2)\|} = \langle 1, 0 \rangle.$$