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1-3 Study Guide and Intervention

Continuity, End Behavior, and Limits

Continuity A function $f(x)$ is **continuous** at $x = c$ if it satisfies the following conditions.

(1) $f(x)$ is defined at c; in other words, $f(c)$ exists.

(2) $f(x)$ approaches the same function value to the left and right of c; in other words, $\lim_{x \to c} f(x)$ exists.

(3) The function value that $f(x)$ approaches from each side of c is $f(c)$; in other words, $\lim_{x \to c} f(x) = f(c)$.

Functions that are not continuous are **discontinuous.** Graphs that are discontinuous can exhibit **infinite discontinuity**, **jump discontinuity**, or **removable discontinuity** (also called **point discontinuity**).

Example **Determine whether each function is continuous at the given x-value. Justify using the continuity test. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.**

a. $f(x) = 2|x| + 3$; $x = 2$

(1) $f(2) = 7$, so $f(2)$ exists.

(2) Construct a table that shows values for $f(x)$ for x-values approaching 2 from the left and from the right.

x	y = f(x)
1.9	6.8
1.99	6.98
1.999	6.998

x	y = f(x)
2.1	7.2
2.01	7.02
2.001	7.002

The tables show that y approaches 7 as x approaches 2 from both sides.

It appears that $\lim_{x \to 2} f(x) = 7$.

(3) $\lim_{x \to 2} f(x) = 7$ and $f(2) = 7$.

The function is continuous at $x = 2$.

b. $f(x) = \frac{2x}{x^2 - 1}$; $x = 1$

The function is not defined at $x = 1$ because it results in a denominator of 0. The tables show that for values of x approaching 1 from the left, $f(x)$ becomes increasingly more negative. For values approaching 1 from the right, $f(x)$ becomes increasingly more positive.

x	y = f(x)
0.9	−9.5
0.99	−99.5
0.999	−999.5

x	y = f(x)
1.1	10.5
1.01	100.5
1.001	1000.5

The function has infinite discontinuity at $x = 1$.

Exercises

Determine whether each function is continuous at the given x-value. Justify your answer using the continuity test. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.

1. $f(x) = \begin{cases} 2x + 1 \text{ if } x > 2 \\ x - 1 \text{ if } x \le 2 \end{cases}$; $x = 2$

$\lim_{x \to 2^-} f(x) = 1$ and $\lim_{x \to 2^+} f(x) = 5$, so the function is not continuous; it has jump discontinuity.

2. $f(x) = x^2 + 5x + 3$; $x = 4$ **$f(4) = 39$**

$\lim_{x \to 4^-} f(x) = 39$ and $\lim_{x \to 4^+} f(x) = 39$, so the function is continuous.

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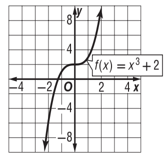
1-3 Study Guide and Intervention *(continued)*

Continuity, End Behavior, and Limits

End Behavior The **end behavior** of a function describes how the function behaves at either end of the graph, or what happens to the value of $f(x)$ as x increases or decreases without bound. You can use the concept of a limit to describe end behavior.

> Left-End Behavior (as x becomes more and more negative): $\lim_{x \to -\infty} f(x)$
>
> Right-End Behavior (as x becomes more and more positive): $\lim_{x \to \infty} f(x)$
>
> The $f(x)$ values may approach negative infinity, positive infinity, or a specific value.

Example **Use the graph of $f(x) = x^3 + 2$ to describe its end behavior. Support the conjecture numerically.**



As x decreases without bound, the y-values also decrease without bound. It appears the limit is negative infinity: $\lim_{x \to -\infty} f(x) = -\infty$.

As x increases without bound, the y-values increase without bound. It appears the limit is positive infinity: $\lim_{x \to \infty} f(x) = \infty$.

Construct a table of values to investigate function values as $|x|$ increases.

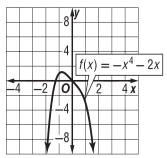
x	−1000	−100	−10	0	10	100	1000
f(x)	−999,999,998	−999,998	−998	2	1002	1,000,002	1,000,000,002

As $x \to -\infty$, $f(x) \to -\infty$. As $x \to \infty$, $f(x) \to \infty$. This supports the conjecture.

Exercises

Use the graph of each function to describe its end behavior. Support the conjecture numerically.

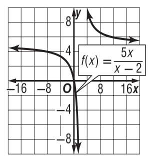
1.



$\lim_{x \to -\infty} f(x) = -\infty$; $\lim_{x \to \infty} f(x) = -\infty$

See students' work.

2.



$\lim_{x \to -\infty} f(x) = 5$; $\lim_{x \to \infty} f(x) = 5$

See students' work.

Lesson 1-3

Answers

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1-3 Practice

Continuity, End Behavior, and Limits

Determine whether each function is continuous at the given x-value(s). Justify using the continuity test. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.

1. $f(x) = -\frac{2}{3x^2}$; at $x = -1$
   Yes; the function is defined at $x = -1$, the function approaches $-\frac{2}{3}$ as x approaches -1 from both sides; $f(-1) = -\frac{2}{3}$.

2. $f(x) = \frac{x - 2}{x + 4}$; at $x = -4$
   No; the function is infinitely discontinuous at $x = -4$.

3. $f(x) = x^3 - 2x + 2$; at $x = 1$
   Yes; the function is defined at $x = -1$, the function approaches 1 as x approaches 1 from both sides; $f(1) = 1$.

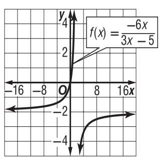
4. $f(x) = \frac{x + 1}{x^2 + 3x + 2}$; at $x = -1$ and $x = -2$
   No; the function has a removable discontinuity at $x = -1$ and infinite discontinuity at $x = -2$.

Determine between which consecutive integers the real zeros of each function are located on the given interval.

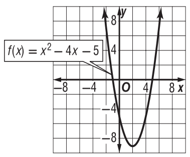
5. $f(x) = x^3 + 5x^2 - 4$; $[-6, 2]$
   [–5, –4], [–1, 0], [0, 1]

6. $g(x) = x^4 + 10x - 6$; $[-3, 2]$
   [–3, –2], [0, 1]

Use the graph of each function to describe its end behavior. Support the conjecture numerically.

7.


$\lim_{x \to -\infty} f(x) = -2$; $\lim_{x \to \infty} f(x) = -2$
See students' work.

8.


$\lim_{x \to -\infty} f(x) = \infty$; $\lim_{x \to \infty} f(x) = \infty$
See students' work.

9. **ELECTRONICS** Ohm's Law gives the relationship between resistance R, voltage E, and current I in a circuit as $R = \frac{E}{I}$. If the voltage remains constant but the current keeps increasing in the circuit, what happens to the resistance? **Resistance decreases and approaches zero.**

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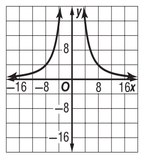
1-3 Word Problem Practice

Continuity, End Behavior, and Limits

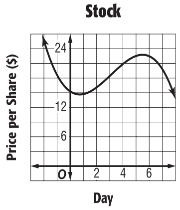
1. **HOUSING** According to the U.S. Census Bureau, the approximate percent of Americans who owned a home from 1900 to 2000 can be modeled by $h(x) = -0.0009x^4 - 0.09x^3 + 1.54x^2 - 4.12x + 47.37$, where x is the number of decades since 1900. Graph the function on a graphing calculator. Describe the end behavior.
   $\lim_{x \to -\infty} f(x) = -\infty$;
   $\lim_{x \to \infty} f(x) = -\infty$

2. **GEOMETRY** The height of a rectangular prism with a square base and a volume of 250 cubic units can be modeled by $f(x) = \frac{250}{x^2}$, where x is the length of one side of the base.
   a. Determine whether the function is continuous at $x = 5$. Justify the answer using the continuity test.
   Yes; because $f(5) = 10$, the function is defined when $x = 5$, $\lim_{x \to 5} f(x) = 10$.
   b. Is the function continuous? Justify the answer using the continuity test. If discontinuous, explain your reasoning and identify the type of discontinuity as *infinite*, *jump*, or *removable*.
   No; because $f(0)$ does not exist, $f(x)$ is discontinuous at $x = 0$; infinite.
   c. Graph the function to verify your conclusion from part **b**.



3. **TRIP** The per-person cost of a guided climbing expedition can be modeled by $f(x) = \frac{600}{x + 25}$, where x is the number of people on the trip.
   a. Graph the function using a graphing calculator. Use the graph to identify and describe any points of discontinuity. **infinite discontinuity at $x = -25$**
   b. Are there any points of discontinuity in the relevant domain? Explain.
   No, x will not be negative because the fewest number of people is 0.

4. **STOCK** The average price of a share of a certain stock x days after a company restructuring is modeled by $f(x) = -0.15x^3 + 1.4x^2 - 1.8x + 15.29$.



Use the graph to describe the end behavior of the function. Support your conjecture numerically.
$\lim_{x \to -\infty} f(x) = \infty$; $\lim_{x \to \infty} f(x) = -\infty$
See students' work.

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1-3 Enrichment

Reading Mathematics

The following selection gives a definition of a continuous function as it might be defined in a college-level mathematics textbook. Notice that the writer begins by explaining the notation to be used for various types of intervals. Although a great deal of the notation is standard, it is a common practice for college authors to explain their notations. Each author usually chooses the notation he or she wishes to use.

> Throughout this book, the set S, called the domain of definition of a function, will usually be an interval. An interval is a set of numbers satisfying one of the four inequalities $a < x < b$, $a \le x < b$, $a < x \le b$, or $a \le x \le b$. In these inequalities, $a \le b$. The usual notations for the intervals corresponding to the four inequalities are (a, b), $[a, b)$, $(a, b]$, and $[a, b]$, respectively.
>
> An interval of the form (a, b) is called open, an interval of the form
>
> $[a, b)$ or $(a, b]$ is called half-open or half-closed, and an interval of the form $[a, b]$ is called closed.
>
> Suppose I is an interval that is either open, closed, or half-open. Suppose $f(x)$ is a function defined on I and x_0 is a point in I. We say that the function $f(x)$ is continuous at the point x_0 if the quantity $|f(x) - f(x_0)|$ becomes small as $x \in I$ approaches x_0.

Use the selection above to answer these questions.

1. What happens to the four inequalities in the first paragraph when $a = b$?

Only the last inequality can be satisfied.

2. What happens to the four intervals in the first paragraph when $a = b$?

The first interval is ∅ and the others reduce to the point $a = b$.

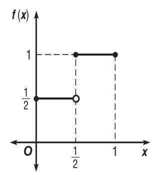
3. What mathematical term makes sense in this sentence?
If $f(x)$ is not ____ at x_0, it is said to be discontinuous at x_0. **continuous**

4. What notation is used in the selection to express the fact that a number x is contained in the interval I?

$x \in I$

5. In the space at the right, sketch the graph of the function $f(x)$ defined as follows.

$$f(x) = \begin{cases} \frac{1}{2} \text{ if } x \in \left[0, \frac{1}{2}\right) \\ 1 \text{ if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$



6. Is the function given in Exercise 5 continuous on the interval [0, 1]? If not, where is the function discontinuous?

No; it is discontinuous at $x = \frac{1}{2}$.

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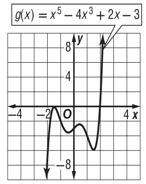
1-4 Study Guide and Intervention

Extrema and Average Rates of Change

Increasing and Decreasing Behavior Functions can increase, decrease, or remain constant over a given interval. The points at which a function changes its increasing or decreasing behavior are called critical points. A critical point can be a **relative minimum**, **absolute minimum**, **relative maximum**, or **absolute maximum**. The general term for minimum or maximum is extremum or **extrema**.

Example **Estimate to the nearest 0.5 unit and classify the extrema for the graph of $f(x)$. Support the answers numerically.**

Analyze Graphically



It appears that $f(x)$ has a relative maximum of 0 at $x = -1.5$, a relative minimum of -3.5 at $x = -0.5$, a relative maximum of -2.5 at $x = 0.5$, and a relative minimum of -6 at $x = 1.5$. It also appears that $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$, so there appears to be no absolute extrema.

Support Numerically

Choose x-values in half-unit intervals on either side of the estimated x-value for each extremum, as well as one very small and one very large value for x.

x	−100	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	100
$f(x)$	-1×10^{10}	−7	−0.09	−2	−3.5	−3	−2.47	−4	−5.91	1	1×10^{10}

Because $f(-1.5) > f(-2)$ and $f(-1.5) > f(-1)$, there is a relative maximum in the interval $(-2, -1)$ near -1.5.

Because $f(-0.5) < f(-1)$ and $f(-0.5) < f(0)$, there is a relative minimum in the interval $(-1, 0)$ near -0.5.

Because $f(0.5) > f(0)$ and $f(0.5) > f(1)$, there is a relative maximum in the interval $(0, 1)$ near 0.5.

Because $f(1.5) < f(1)$ and $f(1.5) < f(2)$, there is a relative minimum in the interval $(1, 2)$ near 1.5.

$f(-100) < f(-1.5)$ and $f(100) > f(1.5)$, which supports the conjecture that f has no absolute extrema.

Exercises

Use a graphing calculator to approximate to the nearest hundredth the relative or absolute extrema of each function. State the x-value(s) where they occur.

1. $f(x) = 2x^6 + 2x^4 - 9x^2$

abs. min. of −5.03 at $x = -0.97$ and at $x = 0.97$; rel. max. of 0 at $x = 0$

2. $f(x) = x^3 + 9x^2$

rel. min. of 0 at $x = 0$; rel. max. of 108 at $x = -6$

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1-4 Study Guide and Intervention *(continued)*

Extrema and Average Rates of Change

Average Rate of Change The **average rate of change** between any two points on the graph of f is the slope of the line through those points. The line through any two points on a curve is called a **secant line**.

The average rate of change on the interval $[x_1, x_2]$ is the slope of the secant line, m_{sec}.

$$m_{sec} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example **Find the average rate of change of $f(x) = 0.5x^3 + 2x$ on each interval.**

a. [−3, −1]

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(-1) - f(-3)}{-1 - (-3)}$$ Substitute −3 for x_1 and −1 for x_2.

$$= \frac{[0.5(-1)^3 + 2(-1)] - [0.5(-3)^3 + 2(-3)]}{-1 - (-3)}$$ Evaluate $f(-1)$ and $f(-3)$.

$$= \frac{-2.5 - (-19.5)}{-1 - (-3)} \text{ or } \frac{17}{2}$$ Simplify.

b. [−1, 1]

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(-1)}{1 - (-1)}$$ Substitute −1 for x_1 and 1 for x_2.

$$= \frac{2.5 - (-2.5)}{1 - (-1)} \text{ or } \frac{5}{2}$$ Evaluate and simplify.

Exercises

Find the average rate of change of each function on the given interval.

1. $f(x) = x^4 + 2x^3 - x - 1$; $[-3, -2]$
−28

2. $f(x) = x^4 + 2x^3 - x - 1$; $[-1, 0]$
0

3. $f(x) = x^3 + 5x^2 - 7x - 4$; $[-3, -1]$
−14

4. $f(x) = x^3 + 5x^2 - 7x - 4$; $[1, 3]$
26

5. $f(x) = x^4 + 8x - 3$; $[-4, 0]$
−56

6. $f(x) = -x^4 + 8x - 3$; $[0, 1]$
7

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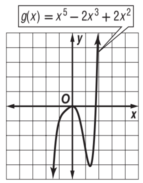
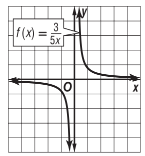
1-4 Practice

Extrema and Average Rates of Change

Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. Support the answer numerically.

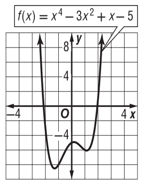
1. 

increasing on (−∞, 0); decreasing on (0, 1.5); increasing on (1.5, ∞); See students' work.

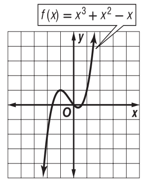
2. 

decreasing on (−∞, 0); decreasing on (0, ∞); See students' work.

Estimate to the nearest 0.5 unit and classify the extrema for the graph of each function. Support the answers numerically.

3. 

rel. min. of −8.5 at x = −1.5; rel. max. of −5 at x = 0; rel. min. of −6 at x = 1; See students' work.

4. 

rel. max. of 1 at x = −1; rel. min. of 0 at x = 0.5; See students' work.

5. GRAPHING CALCULATOR Approximate to the nearest hundredth the relative or absolute extrema of $h(x) = x^5 - 6x + 1$. State the x-values where they occur.

rel. max. (−1.05, 6.02); rel. min. (1.05, −4.02)

Find the average rate of change of each function on the given interval.

6. $g(x) = x^4 + 2x^2 - 5$; $[-4, -2]$
−132

7. $g(x) = -3x^3 - 4x$; $[2, 6]$
−160

8. PHYSICS The height t seconds after a toy rocket is launched straight up can be modeled by the function $h(t) = -16t^2 + 32t + 0.5$, where $h(t)$ is in feet. Find the maximum height of the rocket. **16.5 ft**

Lesson 1-4

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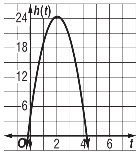
1-4 Word Problem Practice

Extrema and Average Rates of Change

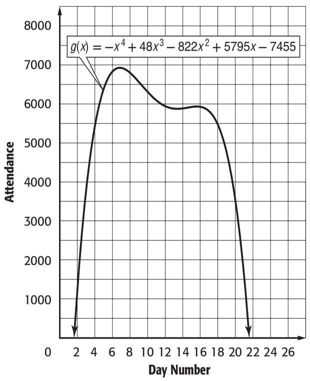
1. **FLARE** A lost boater shoots a flare straight up into the air. The height of the flare, in meters, can be modeled by $h(t) = -4.9t^2 + 20t + 4$, where t is the time in seconds since the flare was launched.

   a. Graph the function.



   b. Estimate the greatest height reached by the flare. Support the answer numerically.
   24.4 m; See students' work.

2. **RECREATION** The daily attendance at a state fair is modeled by $g(x) = -x^4 + 48x^3 - 822x^2 + 5795x - 7455$, where x is the number of days since opening. Estimate to the nearest unit the relative or absolute extrema and the x-values where they occur.



rel. max. (7, 6897); rel. min. (13, 5857); rel. max. (16, 5909)

3. **RECREATION** For the function in Exercise 2, find the average rate of change for each time interval.

   a. Day 2 to Day 6
   1395

   b. Day 13 to Day 15
   19

   c. Day 18 to Day 20
   −921

4. **BOXES** A box with no top and a square base is to be made by taking a piece of cardboard, cutting equal-sized squares from the corners and folding up each side. Suppose the cardboard piece is square and measures 18 inches on each side.

   a. Write a function $v(x)$ where v is the volume of the box and x is the length of the side of a square that was cut from each corner of the cardboard.
   $v(x) = 4x^3 - 72x^2 + 324x$

   b. What value of x maximizes the volume? What is the maximum volume?
   3 in.; 432 in^3

   c. What is the relative minimum of the function? Explain what this minimum means in the context of the problem.
   (9, 0); When squares with 9-inch sides are cut from each corner, the volume of the box is 0 because no material remains.

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1-4 Enrichment

"Unreal" Equations

There are some equations that cannot be graphed on the real-number coordinate system. One example is the equation $x^2 - 2x + 2y^2 + 8y + 14 = 0$. Completing the squares in x and y gives the equation $(x - 1)^2 + 2(y + 2)^2 = -5$.

For any real numbers x and y, the values of $(x - 1)^2$ and $2(y + 2)^2$ are nonnegative. So, their sum cannot be −5. Thus, no real values of x and y satisfy the equation; only imaginary values can be solutions.

Determine whether each equation can be graphed on the real-number plane. Write *yes* or *no*.

1. $(x + 3)^2 + (y - 2)^2 = -4$
   no
2. $x^2 - 3x + y^2 + 4y = -7$
   no
3. $(x + 2)^2 + y^2 - 6y + 8 = 0$
   yes
4. $x^2 + 16 = 0$
   no
5. $x^4 + 4y^2 + 4 = 0$
   no
6. $x^2 + 4y^2 + 4xy + 16 = 0$
   no

In Exercises 7 and 8, for what values of *k* :

a. will the solutions of the equation be imaginary?

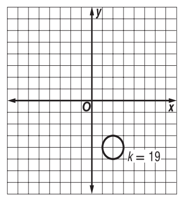
b. will the graph be a point?

c. will the graph be a curve?

d. Choose a value of *k* for which the graph is a curve. Then sketch the curve on the axes provided.

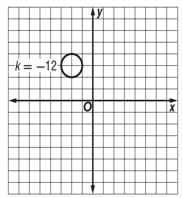
7. $x^2 - 4x + y^2 + 8y + k = 0$

   a. $k > 20$; b. $k = 20$;
   c. $k < 20$;
   d.



8. $x^2 + 4x + y^2 - 6y - k = 0$

   a. $k < -13$; b. $k = -13$;
   c. $k > -13$;
   d.



9. Why would it make no sense to discuss extrema and average rate of change for the graphs in Exercises 7 and 8?
   Sample answer: They are not functions.

Lesson 1-4

Answers