11. Describing Angular or Circular Motion

Introduction

Examples of angular motion occur frequently. Examples include the rotation of a bicycle tire, a merry-go-round, a toy top, a food processor, a laboratory centrifuge, and the orbit of the Earth around the Sun. More complicated examples involving rotational motion combined with linear motion include a rolling billiard ball, the tire of a bicycle that is ridden, a rolling pin in the kitchen, etc. However, mostly we will consider only the case of pure rotational motion since it is simpler. Also, again for reasons of simplicity, we will look only at angular motion that has a fixed radius. Keep in mind that there are two kinds of angular motion: **rotational** motion (e.g., the Earth rotates on its axis once every 24 hours) and **revolutionary** motion (e.g., the Earth revolves about the Sun and makes one complete cycle in a year).

Measuring Angular Distance θ

1. The Degree Measure of θ (example: 35°, where degree is represented by the symbol °)

A. This is the most common way of measuring angular motion. One complete cycle or revolution is divided up into 360 equal bits called degrees.

B. The 360 is arbitrary and is kept for historical reasons but any other number could have been chosen, for example, 100.

C. Important: Notice that the size of the degree is unrelated to the size of the circle. For a circle of radius r, there are 360° in one revolution or complete trip around the circle.

D. θ is the angular measure of distance traveled. θ for angular motion is the analog of the arc distance traveled S for linear motion. (We have used x, y, s, h, etc. for the linear distance before; these all mean the same.)

E. A related measure of angular distance is the cycle. One complete cycle equals 360°. You should be able to convert from degrees to cycles and vice versa.

Examples: 35°=0.097 cycles. 0.75 cycle = 270°

$$35\,° * \frac{1\ \text{cycle}}{360.\,°}$$

```
0.0972222 cycle
```

```
0.75 * 360.
```

```
270.
```

Often angular speeds are given in cycles/sec. Example: Con Ed supplies power at 60 cycles/sec.

F. The **rev** short for **revolution** measure is also used in some applications.

One rev = One cycle, so rev and cycle are basically interchangeable.

G. Each degree is subdivided into **minutes** or **min** so that one degree = 60 minutes. Each minute is subdivided further into **seconds** or **sec** so that one minute = 60 seconds. This use of the terms minutes and seconds has (almost) nothing to do with the use of these terms to measure time.

Example: 48.32° = 48° 19.2 min.

$$0.32\,° * \frac{60\,\text{min}}{1\,°}$$

19.2 min

The Radian Measure of Angular Distance θ

A. You probably are familiar with the concept of one revolution (1 rev in shorthand) being one complete turn around a circle which is also 360°. It is easy enough to convert from degrees to revolutions by setting up a proportionality. Suppose you want the angle θ in degrees or ° measure corrresponding to 2.5 rev so you can write the proportionality

$$\frac{\theta}{360\,°} = \frac{2.5\text{ rev}}{1\text{ rev}} \quad \text{or} \quad \theta = 360° \times 2.5 = 900°$$

since the rev unit cancels out. You can do the inverse conversion to (for example) find the angle θ in rev measure corresponding to 60°

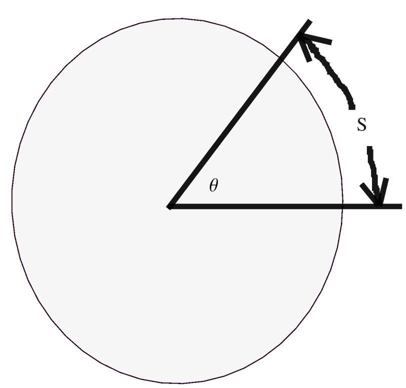
$$\frac{\theta}{1\text{ rev}} = \frac{60\,°}{360\,°} \quad \text{or} \quad \theta = \frac{1}{6}\text{ rev } \simeq 0.17\text{ rev}$$

B. The **radian** or **rad** is another way measuring angular motion. The radian measure is used quite often in scientific applications as it is the unit of angular measure in the S.I. system.

C. The radian measure of an angle θ is defined

$$\theta = \frac{S}{r}\text{ rad}$$

where θ is the measure of the angle in radians (or rad for short hand), r is the radius of the circle measured in meters and S is the arc distance in meters along the part of the circumference associated with the angle θ.



D. For one complete revolution, the distance along the circumference is $S=2\pi r$, so the angle $\theta=2\pi r/r = 2\pi$ radians associated with one rev which is also 360°.

E. The radius r cancels out so the angular measure $\theta = 2\pi$ rad in one revolution is NOT dependent on the size of the circle. The radian measure has this in common with the degree measure.

F. Conversion of a given angle θ from being measured in degrees to being measured in radians is done with the conversion 360° = 2π rad

EXAMPLE: For our angle 35°, we find the angle $\theta = 0.61$ rad by setting up the proportionality

$$\frac{\theta}{2\pi\,\text{rad}} = \frac{35°}{360°}$$

and using *Mathematica* we get after canceling the degrees

```
35. * (2 π / 360) rev
0.610865 rev
```

You should be able to quickly convert between the degree, radian, cycle, and revolution measures of an angle.

F. Unit-checking: The right-hand side of $\theta = \frac{S}{r}$ rad has meters in both the numerator and the denominator. These cancel out, which is good since there are no meter units in θ measured in radians. However, while $\theta = \frac{S}{r}$ rad defines the radian, this unit is nowhere to be found on the right-hand side of the equation unless you put it in by hand. Conclusion: Do not worry ***too much*** about radian units when working with units in kinematic equations. (See below for more comments.)

Angular Velocity ω

Recall that the ***average*** linear velocity $\langle v \rangle$ was defined as

$$\langle v \rangle = \frac{x_f - x_0}{\Delta t}$$

where x_0 is the position of the object at t_0, x_f is the position of the object at a later time t_f and $\Delta t = t_f - t_0$. Since the angular measure θ is the angular analog of the linear distance measure, it is natural to define the average angular velocity $\langle \omega \rangle$ as

$$\langle \omega \rangle = \frac{\theta_f - \theta_0}{\Delta t}$$

where θ_0 is the initial angular position of the object when Δt=0 and θ_f is the final angular position of the object after a time Δt of angular motion. Actually, we most often use the above equation in the form

$$\theta_f = \theta_0 + \langle \omega \rangle t \tag{1}$$

where Δt = t in the case where the initial time is 0. The ***instantaneous*** angular velocity ω at a particular time can be defined similar to the instantaneous linear velocity

$$\omega = \underset{\Delta t \to 0}{\text{Limit}} \frac{\theta_f - \theta_0}{\Delta t}$$

but we will not have much use for this.

Units: θ is measured in radians so ω has units of rad/sec.

Angular Acceleration α

Finally, recall the definition of linear acceleration a for the motion of an object with a changing linear velocity from v_0 to a final linear velocity v_f over a time Δt

$$\langle a \rangle = \frac{v_f - v_0}{\Delta t}$$

The angular analog of ***linear*** acceleration is the ***angular*** acceleration α, which is useful for situations where the angular velocity α is changing. Suppose the original angular velocity ω_0 changes to a final angular velocity ω_f over a time Δt. Then the average angular acceleration $\langle \alpha \rangle$ is defined

$$\langle \alpha \rangle = \frac{\omega_f - \omega_0}{\Delta t}$$

Actually, we will usually use this definition in the form

$$\omega_f = \omega_0 + \langle \alpha \rangle t \tag{2}$$

It is possible to define an instantaneous angular acceleration similar to the instantaneous linear acceleration, but for the problems we are interested in this is not necessary.

Units: The definition of $\langle \alpha \rangle$ involves ω in the numerator which has units radians/sec and time t in the denominator, so $\langle \alpha \rangle$ has units of rad/sec^2.

◄ | ►

An Equation for the Average Angular Velocity $\langle\omega\rangle$

When the angular velocity is not constant, there is a simple formula for the average angular velocity $\langle\omega\rangle$ in terms of the original angular velocity ω_0 and the final angular velocity ω_f after a time Δt

$$\langle\omega\rangle = \frac{\omega_o + \omega_f}{2} \tag{3}$$

Equation (3) holds provided the angular acceleration is constant. The derivation of equation (3) is similar to that for the formula for linear motion $\langle v\rangle = \frac{v_0+v_f}{2}$ and the derivation of equation (3) is left as homework.

Two More Kinematic Equations for Angular Motion

You might guess that since we have these analogies

Measure	Linear Motion	Angular Motion
distance	x	θ
velocity	v	ω
acceleration	a	α

between linear and angular motion that there are two angular analogs for the two other kinematic equations for linear motion

$\triangle x = v_0\, t + \frac{1}{2}\, a\, t^2$ and $v_f^2 = v_0^2 + 2\, a \triangle x$

and you would be right. The two additional equations are

$$\theta = \omega_0\, t + \frac{1}{2}\, \alpha\, t^2 \tag{4}$$

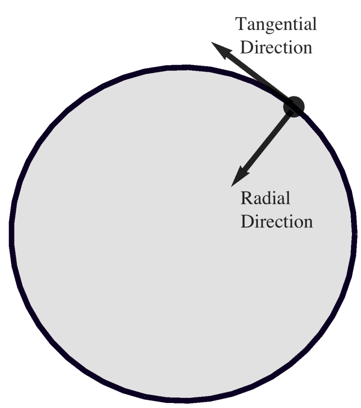
$$\omega_f^2 = \omega_0^2 + 2\,\alpha\,\theta \tag{5}$$

The derivations of these two equations are similar to the derivations in the case of linear motion and will be left as an exercise for you.

Important Note: When using the kinematic equations, including (4) and (5) above, it is a good idea to always use the radian measure for θ, ω, and α. If you are given ω in, say, rev/sec, convert to rad/sec before proceeding. Above all do NOT mix degree, radian, and rev units in the kinematic equations.

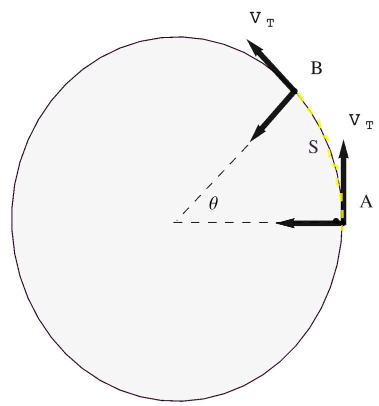
The Radial - Tangential Coordinate System

When dealing with linear motion, it was found useful to define an XY coordinate system that simplified the problem to be solved. For example, it was sometimes useful to have a coordinate system fixed with respect to the moving object while for some other problems it was useful to have a coordinate system fixed with respect to the laboratory or the Earth. For angular motion, it is useful to have a coordinate system that is moving along with the object in angular motion. By convention, the axes of this coordinate system are called not XY but Tangential and Radial. The Radial direction runs along the radius, as you might guess, and the positive Radial direction is inward. The **Tangential direction** is perpendicular to the **Radial direction**, and as you might guess, the Tangential direction is tangent to the circle of motion at the location of the moving object. A picture might help clarify things. This example is of a disk rotating counter-clockwise.



The Tangential Velocity v^T

Just as for linear motion, the tangential velocity can be calculated using $v^T = \frac{S}{\triangle t}$ where S is the distance traveled along the circumference when the point of interest moves from point A to point B (as in the figure below.) It was also assumed it took a time $\triangle$t to move from A to B.



For the moment, assume the disk is rotating at a constant angular velocity ω. Because of this, the length of the tangential velocity v^T vector is constant. The length of v^T is just the distance traveled S divided by the time of travel $\triangle$t, thus

$$v^T = \frac{S}{\triangle t}$$

Using the connection S = r θ between S the distance traveled on the circumference and θ the angular measure we get from the above equation

$$v^T = \frac{(r\,\theta)}{\Delta t} = r\,\frac{\theta}{\Delta t} = r\,\omega$$

Note that we used the definition of the angular velocity $\omega = \frac{\theta}{\Delta t}$ to get the last relation above.

Thus we have found an important relationship between a linear quantity, v^T, and an angular quantity ω

$$v^T = r\,\omega$$

Notice that the form of the above relation is similar to the relation S = rθ, which connects the linear measure of distance S with the angular measure of distance θ. Both relations involve the radius r in the same way. You might guess that there is a similar relationship $a^T = r\alpha$ between the linear tangential acceleration a^T and the angular acceleration α and shortly we will see that this is true.

◀ | ▶

The Tangential Acceleration a^T

You might think there is no tangential acceleration a^T in the above example since ω the angular velocity is constant, and you would be correct. However, suppose ω is NOT constant and instead changes from ω_0 to ω_f over a time $\triangle$t. Then the tangential velocity will change from $v_0^T = r\,\omega_0$ to $v_f^T = r\,\omega_f$ and thus the velocities change in the definition the tangential acceleration

$a^T = \left(v_f^T - v_0^T\right) / \triangle t.$

We easily get

$a^T = \frac{(r\omega_f - r\omega_0)}{\triangle t} = r \times \frac{(\omega_f - \omega_0)}{\triangle t} = r\,\alpha$

or, more simply:

$$a^T = r\,\alpha \tag{6}$$

◀ | ▶

Summary: Connection between Tangential (Linear) Quantities and Angular Quantities

You get from angular quantities to linear quantities by multiplying by the radius r

$$\boxed{\begin{aligned} S &= r\,\theta \\ v^T &= r\,\omega \\ a^T &= r\,\alpha \end{aligned}} \qquad (7)$$

and this is easy enough to remember.

Notation:

1. Quite often we will write just v instead of v^T because there will be only one kind of linear velocity in the angular problems we consider.

2. The tangential acceleration a^T is another matter, since we will shortly talk about another acceleration in the Radial direction, a^R. So the superscript T will be kept in the Tangential acceleration a^T.

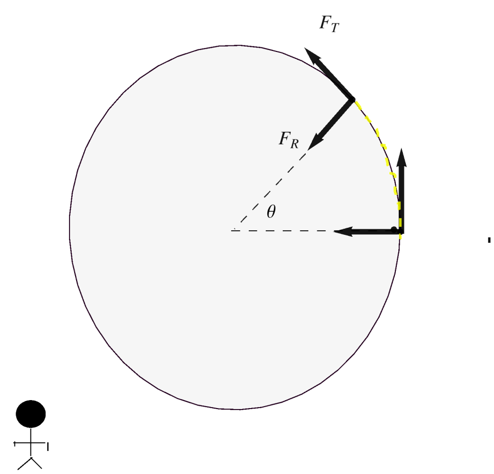
3. You might have trouble remembering the above equations (do you divide or multiply by r?). However, looking at the units involved helps. For example, S is measured in meters and θ has no meters involved, so you must multiply θ by r to get the meter units on both sides of the equation.

Newton's 2nd Law of Motion:

Remember that $\vec{F} = M\,\vec{a}$ is shorthand for two equations $F_x = M\,a_x$ and $F_y = M\,a_y$. For angular motion the XY coordinates are fixed to the particle in angular motion. So the XY coordinates are moving with respect to the ground. It makes sense when describing angular motion to use the tangential T direction and radial R direction instead of XY. For one thing, the force F^R holding the particle in angular motion is in the inward radial direction. The force in the tangential direction F^T speeds up or slows down the angular motion. So instead of the XY coordinate system fixed with respect to the ground we use the RT coordinates fixed to the object moving in angular motion. Thus we write

$$F_R = M\,a_R \quad \text{and} \quad F_T = M\,a_T \tag{8}$$

and for the moment we will focus on the first, or Radial, equation.



Important Aside: *Until stated explicitly near the end of these notes, always assume that the motion is being observed by a person on the ground (represented by a stick figure in the above diagram). Newton's 2nd Law holds for such an observer.*

Centripetal force is another term for the force in the Radial direction F_R. Centripetal means "center seeking," which is toward the center of the circle. But what is F_R physically?

Well, it depends on the circumstance:

1. If you tie a pail to a rope and swing it over your head, then the centripetal force is the tension force in the rope that holds the pail in a circular motion.
2. The orbital motion of the Earth about the Sun is due to the centripetal force we call gravity.
3. The electrons are held in their orbits about the nucleus of an atom by the electric force.

What about the centripetal acceleration a^R? Is there anything special about it besides its being in the Radial direction?

The Centripetal Acceleration a^R

There is a very useful formula for the acceleration in the Radial direction a^R

$$a^R = \frac{v^2}{r}$$

Here v is the linear tangential velocity v^T and r is the radius of the angular motion. We will derive this result shortly. You might already be familiar with the above equation in connection with something called the "centrifugal force" but that might lead to some misconceptions as will be seen shortly.

Observation: It looks like the larger the radius r of the angular motion, the smaller the Radial acceleration a^R, but this is WRONG because the tangential velocity also depends upon r. Remember v = r ω, so using this in the above equation to eliminate v results in

$a^R = \frac{(r\,\omega)^2}{r}$ = rω^2 or simply

$$a^R = \omega^2 r$$

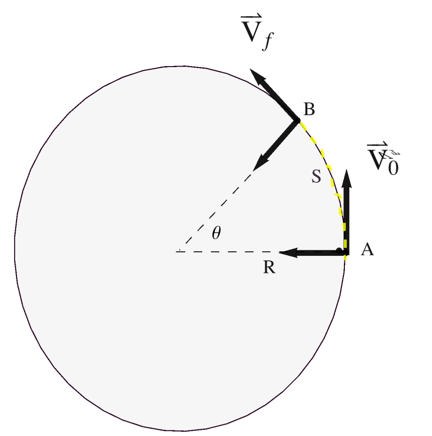
The angular velocity $\omega = \frac{\theta}{\triangle t}$ depends upon the angle θ, and the angle θ is independent of r, so ω=θ/t is ***also independent*** of r. Looking at the above equation it is easy to see that a^R INCREASES as r increases.

Derivation of the Formula for the Radial Acceleration a^R

There is an acceleration a^R in the Radial direction because the tangential velocity V is changing its **direction**. Recall that linear acceleration is a vector quantity defined by

$$\vec{a} = \frac{\vec{v}_f - \vec{v}_0}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t} \text{ or } |\vec{a}| = \frac{|\Delta \vec{v}|}{\Delta t}$$

where $\Delta \vec{v}_f = \vec{v}_f - \vec{v}_0$. A picture of the angular motion going from A to B might help:

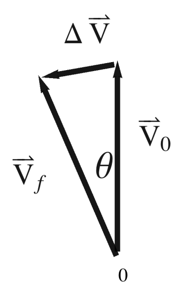


Observations:

1. There is no tangential force F^T for now, so the angular velocity ω is constant and the length of the tangential force vector stays the same in going from A to B, $|\vec{v}_f| = |\vec{v}_0|$.

2. But this does NOT mean that $\Delta\vec{v} = \vec{v}_f - \vec{v}_0 = 0$. You can see that $\vec{v}_0$ is in a different **direction** from $\vec{v}_f$ in the diagram above, and as a result, $\Delta\vec{v} \neq 0$.

3. Remember a vector is the same provided its length and direction are the same. So you can move a vector anywhere you want provided you keep its length and direction the same. In particular move the vector $\vec{v}_0$ parallel to itself until the tail of the vector $\vec{v}_0$ is at the tail of vector $\vec{v}_f$. After this is done, a vector diagram of **just** $\vec{v}_0$, $\vec{v}_f$, and $\Delta\vec{v}$ appears below.



The $\Delta\vec{v}$ makes sense because $\Delta\vec{v} = \vec{v}_f - \vec{v}_0$ can be written $\vec{v}_0 + \Delta\vec{v} = \vec{v}_f$. This means that $\Delta\vec{v}$ added to $\vec{v}_0$ yields $\vec{v}_f$, as in the diagram.

4. What is needed in the formula for the acceleration is $|\Delta\vec{v}|$, which can be obtained by geometry

using similar triangles. In the original diagram, look at the triangle formed from 0AB. (*Aside: OK so OAB is not a triangle since one side is curved; however, draw a straight line between A and B to make a triangle and for $\Delta t \to 0$ this approximation gets better.*) In any case, the triangle OAB has two sides of equal length r which is the radius of the circle with the angle θ between these two sides. 0AB is similar to the triangle in the smaller diagram above involving only velocities $\vec{v}_0$, $\vec{v}_f$, and $\Delta\vec{v}$ since this triangle also has two sides of equal length $|\vec{v}_f| = |\vec{v}_0| = v$ with the same angle θ between them. (*Aside: You can convince yourself that θ in the triangle OAB is the same as the θ in the smaller diagram involving just the velocities provided you notice that the tangential velocity **is always pependicular** to the radius. The radius and the tangent are tied together so if the radius goes through an angle θ so does the tangent. By the way, the two triangles are similar since they both are iscocles triangles with the same angle θ between the equal sides.)* Similar triangles tells us that the ratio of side opposite the angle θ to the length of the equal sides is the same in the two similar triangles, that is

$$\frac{S}{r} = \frac{|\Delta\vec{v}|}{|\vec{v}_0|} = \frac{|\Delta\vec{v}|}{v} \text{ and thus } \left|\Delta\vec{v}\right| = \frac{S\,v}{r}$$

(*Aside: This entire step #4 is a little hard to explain in writing so come to class and I will try to convince you by giving a truly "hand waiving argument". Or a least I can try to answer your questions.*)

5. Since the magnitude of the tangential velocity is not changing $|\vec{v}_f| = |\vec{v}_0| =$ v, we get the result

$$|\vec{a}| = \frac{|\Delta \vec{v}_f|}{\Delta t} = \frac{v}{r} \times \frac{s}{\Delta t} = \frac{v^2}{r}$$

keeping in mind that v=s/Δt is the tangential velocity too. So we get simply $a=v^2 / r$ for the centripetal acceleration a in the radial direction. v is the tangential velocity and the radius of the circle r. This is obviously a lot of work for such a simple result but it is a useful result. For example, the formula explains how the centrifuge works in the biology or chemistry laboratory.

◄ | ►

The Radial Part of Newton's 2nd Law: Centripetal Force

Recall that $F_R = M a^R$. We call the radial force F_R the centripetal force $F_{\text{centripetal}}$ and the radial acceleration is $a^R = \frac{v^2}{r}$, so we have

$$F_{\text{centripetal}} = M \frac{v^2}{r}$$

which is the radial part of Newton's 2nd Law. Another useful form is obtained from

$v = \omega r$

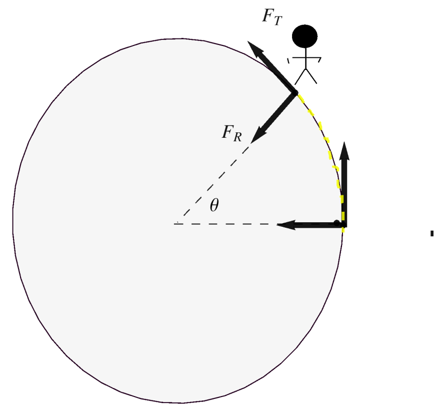
$$F_{\text{centripetal}} = M \omega^2 r$$

It is important to keep in mind that the centripetal force $F_{\text{centripetal}}$ is a real force because it is produced by something (for example, the tension in a rope, gravity, electric force, etc.). The centripetal force cause the mass to move in a circle since the natural tendency of the mass is to go in a straight line. If you somehow could stop the centripetal force (by for example, cutting the rope connected to the moving bucket) then the bucket would continue in a straight line tangent to the circle at the point where the rope was cut.

Also notice that the right hand side $M \omega^2 r$ of Newton's 2nd law, involves the mass M. So all other things (that is, ω and r) being equal two masses with different M will have a different radial acceleration. This is the principle behind the gas centrifuges in Iran which are being discussed in the news in the news. In this case, two isotopes of Uranium U^{235} and U^{238} are separated or "enriched" either for nuclear power generation or nuclear weapons. Two atoms are isotopes if they have same number of protrons 92 but different number of neutrons; 3 in this case. It is the U^{235} isotope (the atom that is slightly less massive) that is useful in fission devices and the U^{238} after separation is put to other uses.

The Centrifugal "Force"

Suppose we look at way things are viewed by an observer fixed to the rotating object as indicated below:



Question: How does the angular motion appear to the person sitting on the moving mass? In particular, what does Newton's 2nd Law $\sum F_R = M a_R$ look like for that person?

Importance of this Question: We would like to apply Newton's 2nd Law to rotating frames of reference. After all, we live on a rotating coordinate system because the Earth is rotating about its axis. (The Earth is also revolving about the Sun, which makes the Earth's coordinate doubly "rotating.")

Answer: The mass does not appear to be moving so a_R=0 for this person. Newton's 2nd law for the radial part looks like $\sum F_R = M a_R$ = 0. So the sum of the forces should be zero $\sum F_R$ =0 . However, the person on the mass knows of only one force, the $F_{\text{centripetal}}$, so there is something wrong with Newton's 2nd Law for the person viewing things on the mass. Actually this should not be a surprise, since this person is in a ***non-inertial reference frame**, in* which Newton's 1st Law does not hold. This frame of reference is accelerated by centripetal acceleration with respect to the ground. (For example, say you are on a merry-go-round and drop a mass. Once you release the mass, there is no horizontal force to keep it moving in a circle, and the mass flies off the merry-go-round in addition to falling downward.)

Resolution of the Problem: a_R=0 and nothing much can be done about this. However, if you rewrite the radial part of Newton's 2nd Law (which was OK for the observer on the ground) as

$$\sum F_R = F_{\text{centripetal}} - M\,\frac{v^2}{r} = 0$$

Then ***define*** the **centrifugal** "force" as

$$F_{\text{centrifugal}} = -\,M\,\frac{v^2}{r}$$

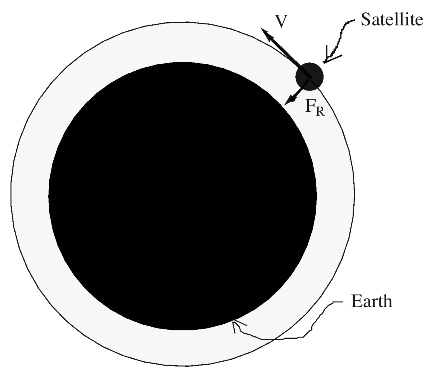
and then Newton's 2nd Law for the observer in the rotating frame of reference appears

$$F_{\text{centripetal}} + F_{\text{centrifugal}} = 0$$

So the observer on the rotating object says the centripetal force (the tension in the rope, the gravity force, etc.) is balanced by the centrifugal "force." You have all felt the centrifugal force when you are in a car going around the curve (even though it is a fictional force). None-the-less the centrifugal force is not a real force because there is nothing that causes it. The force you feel when you are in the car is the centripetal force of the car door on you making you go in a circle (your natural motion is in a straight line tangent to the circle of motion).

Bottom Line: You can apply Newton's 2nd law if you are in a rotating coordinate system provided you add a centrifugal force.

Application: A Satellite in Orbit Around the Earth



The radial force in this case is the force of gravity on the satellite by the Earth

$$F_R = \frac{G m M}{r^2}$$

where m is the mass of the satellite, M is the mass of the Earth, G is the constant of Universal Gravitation, and r is the radius of the satellites orbit. r is the radius of the Earth plus the height of the satellite above the Earth's surface. The radial component of Newton's 2nd Law for an observer in outer space appears

$$F_R = m \frac{v^2}{r}$$

where the radial acceleration is $a^R = v^2 / r$. Keep in mind that the right-hand side of the above equation is actually just the $m a_R$ part of Newton's 2nd Law. Combine the two equations above to eliminate F_R:

$$m \frac{V^2}{r} = \frac{G m M}{r^2}$$

The mass m of the satellite cancels out and solving for the velocity (actually, the speed) of the satellite

$$v = \sqrt{\frac{G M}{r}}$$

One Use of This Equation: Suppose we want to know how fast v a satellite has to be moving in order to remain a height h above the surface of the Earth. If r_E is the radius of the Earth then $r=r_E+h$.

Notice (perhaps surprisingly) that this result for the velocity v of the satellite does NOT depend upon the mass m of the satellite.