6

Simplifying Through Substitution

In previous chapters, we saw how certain types of first-order differential equations (directly integrable, separable, and linear equations) can be identified and put into forms that can be integrated with relative ease. In this chapter, we will see that, sometimes, we can start with a differential equation that is not one of these desirable types, and construct a corresponding separable or linear equation whose solution can then be used to construct the solution to the original differential equation.

6.1 Basic Notions

There are many first-order differential equations, such as

$$\frac{dy}{dx} = (x + y)^2 \quad ,$$

that are neither linear nor separable, and which do not yield up their solutions by direct application of the methods developed thus far. One way of attempting to deal with such equations is to replace y with a cleverly chosen formula of x and "u" where u denotes another unknown function of x. This results in a new differential equation with u being the function of interest. If the substitution truly is clever, then this new differential equation will be separable or linear (or, maybe, even directly integrable), and can be be solved for u in terms of x using methods discussed in previous chapters. Then the function of real interest, y, can be determined from the original 'clever' formula relating u, y and x.

Here are the basic steps to this approach, described in a little more detail and illustrated by being used to solve the above differential equation:

1. Identify what is hoped will be good formula of x and u for y,

$$y = F(x, u) \quad .$$

   This 'good formula' is our substitution for y. Here, u represents another unknown function of x (so "$u = u(x)$"), and the above equation tells us how the two unknown functions y and u are related. (Identifying that 'good formula' is the tricky part. We'll discuss that further in a little bit.)

Let's try a substitution that reduces the right side of our differential equation,

$$\frac{dy}{dx} = (x+y)^2 \quad ,$$

to u^2. This means setting $u = x + y$. Solving this for y gives our substitution,'

$$y = u - x \quad .$$

2. Replace *every* occurrence of y in the given differential equation with that formula of x and u, *including the y in the derivative*. Keep in mind that u is a function of x, so the $^{dy}/_{dx}$ will become a formula of x, u, and $^{du}/_{dx}$ (it may be wise to first compute $^{dy}/_{dx}$ separately).

   Since we are using $y = u - x$ (equivalently, $u = x + y$), we have

$$(x+y)^2 = u^2 \quad ,$$

   and

$$\frac{dy}{dx} = \frac{d}{dx}[u - x] = \frac{du}{dx} - \frac{dx}{dx} = \frac{du}{dx} - 1 \quad .$$

   So, under the substitution $y = u - x$,

$$\frac{dy}{dx} = (x+y)^2$$

   becomes

$$\frac{du}{dx} - 1 = u^2 \quad .$$

3. Solve the resulting differential equation for u (don't forget the constant solutions!). If possible, get an explicit solution for u in terms of x. (This assumes, of course, that the differential equation for u is one we can solve. If it isn't, then our substitution wasn't that clever, and we may have to try something else.)

   Adding 1 to both sides of the differential equation just derived for u yields

$$\frac{du}{dx} = u^2 + 1 \quad ,$$

   which we recognize as being a relatively easily solved separable equation with no constant solutions. Dividing through by $u^2 + 1$ and integrating,

$$\frac{1}{u^2+1}\frac{du}{dx} = 1$$

$$\hookrightarrow \qquad \int \frac{1}{u^2+1}\frac{du}{dx}\,dx = \int 1\,dx$$

$$\hookrightarrow \qquad \arctan(u) = x + c$$

$$\hookrightarrow \qquad u = \tan(x+c) \quad .$$

4. If you get an explicit solution $u = u(x)$, then just plug that formula $u(x)$ into the original substitution to get the explicit solution to the original equation,

$$y(x) = F(x, u(x)) \quad .$$

If, instead, you only get an implicit solution for u, then go back to the original substitution, $y = F(x,u)$, solve that to get a formula for u in terms of x and y (unless you already have this formula for u), and substitute that formula for u into the solution obtained to convert it to the corresponding implicit solution for y.

Our original substitution was $y = u - x$. Combining this with the formula for u just obtained, we get

$$y = u - x = \tan(x+c) - x$$

as a general solution to our original differential equation,

$$\frac{dy}{dx} = (x+y)^2 \quad .$$

The key to this approach is, of course, in identifying a substitution, $y = F(x,u)$, that converts the original differential equation for y to a differential equation for u that can be solved with reasonable ease. Unfortunately, there is no single method for identifying such a substitution. At best, we can look at certain equations and make good guesses at substitutions that are likely to work. We will next look at three cases where good guesses can be made. In these cases the suggested substitutions are guaranteed to lead to either separable or linear differential equations. As you may suspect, though, they are not guaranteed to lead to simple separable or linear differential equations.

6.2 Linear Substitutions

If the given differential equation can be rewritten so that the derivative equals some formula of $Ax + By + C$,

$$\frac{dy}{dx} = f(Ax + By + C) \quad ,$$

where A, B, and C are known constants, then a good substitution comes from setting

$$u = Ax + By + C \quad ,$$

and then solving for y. For convenience, we'll call this a *linear* substitution[1].

We've already seen one case where a linear substitution works — in the example above illustrating the general substitution method. Here is another example, one in which we end up with an implicit solution.

!►*Example 6.1:* *To solve*

$$\frac{dy}{dx} = \frac{1}{2x - 4y + 7} \quad ,$$

we use the substitution based on setting

$$u = 2x - 4y + 7 \quad .$$

[1] because $Ax + By + C = 0$ is the equation for a straight line

Solving this for y *and then differentiating yields*

$$y = \frac{1}{4}[2x - u + 7] = \frac{x}{2} - \frac{u}{4} + \frac{7}{4}$$

and

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x}{2} - \frac{u}{4} - \frac{7}{4}\right] = \frac{1}{2} - \frac{1}{4}\frac{du}{dx} \quad .$$

So, the substitution based on $u = 2x - 4y + 7$ *converts*

$$\frac{dy}{dx} = \frac{1}{2x - 4y + 7}$$

to

$$\frac{1}{2} - \frac{1}{4}\frac{du}{dx} = \frac{1}{u} \quad .$$

This differential equation for u *looks manageable, especially since it contains no* x*'s. Solving for the derivative in this equation, we get*

$$\frac{du}{dx} = -4\left[\frac{1}{u} - \frac{1}{2}\right] = -4\left[\frac{2}{2u} - \frac{u}{2u}\right] = -4\left[\frac{2-u}{2u}\right] \quad ,$$

which simplifies to

$$\frac{du}{dx} = 2\left[\frac{u-2}{u}\right] \quad . \tag{6.1}$$

Again, this is a separable equation. This time, though, the differential equation has a constant solution,

$$u = 2 \quad . \tag{6.2}$$

To find the other solutions to our differential equation for u, *we multiply both sides of equation (6.1) by* u *and divide through by* $u - 2$, *obtaining*

$$\frac{u}{u-2}\frac{du}{dx} = 2 \quad .$$

After noticing that

$$\frac{u}{u-2} = \frac{u-2+2}{u-2} = \frac{u-2}{u-2} + \frac{2}{u-2} = 1 + \frac{2}{u-2} \quad ,$$

we can integrate both sides of our last differential equation for u,

$$\int \frac{u}{u-2}\frac{du}{dx}\,dx = \int 2\,dx$$

$$\hookrightarrow \qquad \int \left[1 + \frac{2}{u-2}\right] du = 2x + c$$

$$\hookrightarrow \qquad u + 2\ln|u-2| = 2x + c \quad . \tag{6.3}$$

Sadly, the last equation is not one we can solve to obtain an explicit formula for u *in terms of* x. *So we are stuck with using it as an implicit solution of our differential equation for* u.

Together, formula (6.2) and equation (6.3) give us all the solutions to the differential equation for u. *To obtain all the solutions to our original differential equation for* y, *we must recall the original (equivalent) relations between* u *and* y,

$$u = 2x - 4y + 7 \qquad \text{and} \qquad y = \frac{x}{2} - \frac{u}{4} + \frac{7}{4} \quad .$$

The latter with the constant solution $u = 2$ *(formula (6.2)) yields*

$$y = \frac{x}{2} - \frac{2}{4} + \frac{7}{4} = \frac{x}{2} + \frac{5}{4} \quad .$$

On the other hand, it is easier to combine the first relation between u *and* y *with the implicit solution for* u *in equation (6.3),*

$$u = 2x - 4y + 7 \qquad \text{with} \qquad u + 2\ln|u-2| = 2x + c \quad ,$$

obtaining

$$[2x - 4y + 7] + 2\ln|[2x - 4y + 7] - 2| = 2x + c \quad .$$

After a little algebra, this simplifies to

$$\ln|2x - 4y + 5| = 4y + C \quad .$$

which does not include the "constant u*" solution above. So, for* $y = y(x)$ *to be a solution to our original differential equation, it must either be given by*

$$y = \frac{x}{2} + \frac{5}{4}$$

or satisfy

$$\ln|2x - 4y + 5| = 4y + C \quad .$$

Let us see what happens whenever we have a differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

(where A, B, and C are known constants), and we attempt the substitution based on setting

$$u = Ax + By + C \quad .$$

Solving for y and then differentiating yields

$$y = \frac{1}{B}[u - Ax - C] \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{B}\left[\frac{du}{dx} - A\right] \quad .$$

Under these substitutions,

$$\frac{dy}{dx} = f(Ax + By + C)$$

becomes

$$\frac{1}{B}\left[\frac{du}{dx} - A\right] = f(u) \quad .$$

After a little algebra, this can be rewritten as

$$\frac{du}{dx} = A + Bf(u) \quad ,$$

which is clearly a separable equation. Thus, we will always get a separable differential equation for u. Moreover, the ease with which this differential equation can be solved clearly depends only on the ease with which we can evaluate

$$\int \frac{1}{A + Bf(u)}\, du \quad .$$

6.3 Homogeneous Equations

We now consider first-order differential equations in which the derivative can be viewed as a formula of the ratio y/x. In other words, we are now interested in any differential equation that can be rewritten as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{6.4}$$

where f is some function of a single variable. Such equations are sometimes said to be *homogeneous*.[2] Unsurprisingly, the substitution based on setting

$$u = \frac{y}{x} \qquad \text{(i.e., } y = xu \text{)}$$

is often useful in solving these equations. We will, in fact, discover that this substitution will always transform an equation of the form (6.4) into a separable differential equation.

!►*Example 6.2:* *Consider the differential equation*

$$xy^2 \frac{dy}{dx} = x^3 + y^3 \quad .$$

Dividing through by xy^2 *and doing a little factoring yields*

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} = \frac{x^3\left[1 + \frac{y^3}{x^3}\right]}{x^3\left[\frac{y^2}{x^2}\right]} \quad ,$$

which simplifies to

$$\frac{dy}{dx} = \frac{1 + \left[\frac{y}{x}\right]^3}{\left[\frac{y}{x}\right]^2} \quad . \tag{6.5}$$

That is,

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \text{with} \quad f(\text{whatever}) = \frac{1 + \text{whatever}^3}{\text{whatever}^2} \quad .$$

So we should try letting

$$u = \frac{y}{x}$$

or, equivalently,

$$y = xu \quad .$$

On the right side of equation (6.5), replacing y *with* xu *is just the same as replacing each* y/x *with* u. *Either way, the right side becomes*

$$\frac{1 + u^3}{u^2} \quad .$$

[2] Warning: Later we will refer to a completely different type of differential equation as being "homogeneous".

On the left side of equation (6.5), the substitution $y = xu$ *is in the derivative. Keeping in mind that* u *is also a function of* x *, we have*

$$\frac{dy}{dx} = \frac{d}{dx}[xu] = \frac{dx}{dx}u + x\frac{du}{dx} = u + x\frac{du}{dx} \quad .$$

So,

$$\frac{dy}{dx} = \frac{1 + \left[\frac{y}{x}\right]^3}{\left[\frac{y}{x}\right]^2} \quad \overset{y=xu}{\Longrightarrow} \quad u + x\frac{du}{dx} = \frac{1+u^3}{u^2} \quad .$$

Solving the last equation for du/dx *and doing a little algebra, we see that*

$$\frac{du}{dx} = \frac{1}{x}\left[\frac{1+u^3}{u^2} - u\right] = \frac{1}{x}\left[\frac{1+u^3}{u^2} - \frac{u^3}{u^2}\right] = \frac{1}{x}\left[\frac{1+u^3-u^3}{u^2}\right] = \frac{1}{xu^2} \quad .$$

How nice! Our differential equation for u *is the very simple separable equation*

$$\frac{du}{dx} = \frac{1}{xu^2} \quad .$$

Multiplying through by u^2 *, integrating, and doing a little more algebra:*

$$\int u^2 \frac{du}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$\hookrightarrow \qquad \frac{1}{3}u^3 = \ln|x| + c$$

$$\hookrightarrow \qquad u^3 = 3\ln|x| + 3c$$

$$\hookrightarrow \qquad u = \sqrt[3]{3\ln|x| + 3c} \quad .$$

Combining this with our substitution $y = xu$ *gives*

$$y = xu = x\left[\sqrt[3]{3\ln|x| + 3c}\right] = x\sqrt[3]{3\ln|x| + C}$$

as the general solution to our original differential equation.

In practice, it may not be immediately obvious if a given first-order differential equation can be written in form (6.4), but it is usually fairly easy to find out. First, algebraically solve the differential equation for the derivative to get

$$\frac{dy}{dx} = \text{“some formula of } x \text{ and } y\text{”} \quad .$$

With a little luck, you'll be able to do a little algebra (as we did in the above example) to see if that “formula of x and y” can be written as just a formula of y/x , $f(y/x)$.

If it's still not clear, then just go ahead and try the substitution $y = xu$ in that “formula of x and y”. If all the x's cancel out and you are left with a formula of u , then that formula, $f(u)$, is the right side of (6.4) (remember, $u = y/x$). So the differential equation can be written in the desired form. Moreover, half the work in plugging the substitution into the differential equation is now done.

On the other hand, if the x's do not cancel out when you substitute xu for y , then the differential equation cannot be written in form (6.4), and there is only a small chance that this substitution will yield an ‘easily solved’ differential equation for u .

!►Example 6.3: *Again, consider the differential equation*

$$xy^2\frac{dy}{dx} = x^3 + y^3 \quad ,$$

which we had already studied in the previous example. Solving for the derivative again yields

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} \quad .$$

Instead of factoring out x^3 *from the numerator and denominator of the right side, let's go ahead and try the substitution* $y = xu$ *and see if the* x*'s cancel out:*

$$\frac{x^3 + y^3}{xy^2} = \frac{x^3 + [xu]^3}{x[xu]^2} = \frac{x^3 + x^3u^3}{x^3u^2} = \frac{x^3\left(1 + u^3\right)}{x^3u^2}$$

The x*'s clearly do cancel out, leaving us with*

$$\frac{1 + u^3}{u^2} \quad .$$

Thus, (as we already knew), our differential equation can be put into form (6.4). What's more, getting our differential equation into that form and using $y = xu$ *will lead to*

$$\frac{1 + u^3}{u^2}$$

for the right side, just as we saw in the previous example.

When employing the substitution $y = xu$ to solve

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad ,$$

do not forget to treat u as a function of x ! Thus, when we differentiate y, we have

$$\frac{dy}{dx} = \frac{d}{dx}[xu] = \frac{dx}{dx}u + x\frac{du}{dx} = u + x\frac{du}{dx} \quad .$$

This is not a formula worth memorizing — I wouldn't even suggest remembering that $y = xu$ — it should be quite enough to remember that $u = u(x)$ with $u = {}^{y}/_{x}$.

However, it is worth noting that, if we plug these substitutions into

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad ,$$

we always get

$$u + x\frac{du}{dx} = f(u) \quad ,$$

which is the same as

$$\frac{du}{dx} = \frac{f(u) - u}{x} \quad .$$

This confirms that we will always get a separable equation, just as with linear substitutions. This time, the ease with which the differential equation for u can be solved depends on the ease with which we can evaluate

$$\int \frac{1}{f(u) - u}\,du \quad .$$

6.4 Bernoulli Equations

A *Bernoulli equation* is a first-order differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = f(x)y^n \tag{6.6}$$

where $p(x)$ and $f(x)$ are known functions of x only, and n is some real number. This looks much like the standard form for linear equations. Indeed, a Bernoulli equation is linear if $n = 0$ or $n = 1$ (and is also separable if $n = 1$). Consequently, our main interest is in solving such an equation when n is neither 0 nor 1 .

The above equation can be solved using a substitution, though good choice for that substitution might not be immediately obvious. You might suspect that setting $u = y^n$ would help, but it doesn't — unless, that is, it leads you to try a substitution based on

$$u = y^r$$

where r is some value yet to be determined. If you solve this for y in terms of u and plug the resulting formula for y into the Bernoulli equation, you will then discover, after a bit of calculus and algebra, that you have a linear differential equation for u if and only if $r = 1 - n$ (see problem 6.5). So the substitution that does work is the one based on setting

$$u = y^{1-n} \quad .$$

In the future, you can either remember this, re-derive it as needed, or know where to look it up.

!▶Example 6.4: *Consider the differential equation*

$$\frac{dy}{dx} + 6y = 30e^{3x}y^{2/3} \quad .$$

This is in form (6.6), with $n = 2/3$ *. Setting*

$$u = y^{1-n} = y^{1-2/3} = y^{1/3} \quad ,$$

we see that the substitution

$$y = u^3$$

is called for. Plugging this into our original differential equation, we get

$$\frac{dy}{dx} + 6y = 30e^{3x}y^{2/3}$$

$$\hookrightarrow \qquad \frac{d}{dx}\left[u^3\right] + 6\left[u^3\right] = 30e^{3x}\left[u^3\right]^{2/3}$$

$$\hookrightarrow \qquad 3u^2\frac{du}{dx} + 6u^3 = 30e^{3x}u^2 \quad .$$

Dividing this last equation through by $3u^2$ *gives*

$$\frac{du}{dx} + 2u = 10e^{3x} \quad ,$$

a relatively simple linear equation with integrating factor

$$\mu = e^{\int 2\,dx} = e^{2x} \quad .$$

Continuing as usual with such equations,

$$e^{2x}\left[\frac{du}{dx} + 2u = 10e^{3x}\right]$$

$$\hookrightarrow \qquad e^{2x}\frac{du}{dx} + 2e^{2x}u = 10e^{5x}$$

$$\hookrightarrow \qquad \frac{d}{dx}[e^{2x}u] = 10e^{5x} \quad .$$

Integrating both sides with respect to x *then yields*

$$e^{2x}u = \int 10e^{5x}\,dx = 2e^{5x} + c \quad ,$$

which tells us that

$$u = e^{-2x}\left[2e^{5x} + c\right] = 2e^{3x} + ce^{-2x} \quad .$$

Finally, after recalling the substitution that led to the differential equation for u *, we obtain our general solution to the given Bernoulli equation,*

$$y = u^3 = \left[2e^{3x} + ce^{-2x}\right]^3 \quad .$$

Additional Exercises

6.1. *In these problems, use linear substitutions (as described in section 6.2).*

a. *Find a general solution to each of the following:*

i. $\dfrac{dy}{dx} = \dfrac{1}{(3x+3y+2)^2}$ ***ii.*** $\dfrac{dy}{dx} = \dfrac{(3x-2y)^2+1}{3x-2y} + \dfrac{3}{2}$

iii. $\cos(4y-8x+3)\dfrac{dy}{dx} = 2 + 2\cos(4y-8x+3)$

b. *Solve the initial-value problem*

$$\frac{dy}{dx} = 1 + (y-x)^2 \qquad \text{with} \quad y(0) = \frac{1}{4} \quad .$$

6.2. *In these problems, use substitutions appropriate to homogeneous first-order differential equations (as described in section 6.3), do the following:*

a. *Find a general solution to each of the following:*

i. $x^2\dfrac{dy}{dx} - xy = y^2$ ***ii.*** $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$

iii. $\cos\left(\frac{y}{x}\right)\left[\frac{dy}{dx} - \frac{y}{x}\right] = 1 + \sin\left(\frac{y}{x}\right)$

b. *Solve the initial-value problem*

$$\frac{dy}{dx} = \frac{x-y}{x+y} \qquad \text{with} \quad y(0) = 3 \quad .$$

6.3. *In these problems, use substitutions appropriate to Bernoulli equations (as described in section 6.4).*

a. *Find a general solution to each of the following:*

i. $\frac{dy}{dx} + 3y = 3y^3$.

ii. $\frac{dy}{dx} - \frac{3}{x}y = \left(\frac{y}{x}\right)^2$

iii. $\frac{dy}{dx} + 3\cot(x)y = 6\cos(x)y^{2/3}$

b. *Solve the initial-value problem*

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{y} \qquad \text{with} \quad y(1) = 3 \quad .$$

6.4. *For each of the following, determine a substitution that simplifies the given differential equation, and, using that substitution, find a general solution. (Warning: The substitutions for some of the later equations will not be substitutions already discussed.)*

a. $\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2$

b. $3\frac{dy}{dx} = -2 + \sqrt{2x+3y+4}$

c. $\frac{dy}{dx} + \frac{2}{x}y = 4\sqrt{y}$

d. $\frac{dy}{dx} = 4 + \frac{1}{\sin(4x-y)}$

e. $(y-x)\frac{dy}{dx} = 1$

f. $(x+y)\frac{dy}{dx} = y$

g. $\left(2xy+2x^2\right)\frac{dy}{dx} = x^2 + 2xy + 2y^2$

h. $\frac{dy}{dx} + \frac{1}{x}y = x^2y^3$

i. $\frac{dy}{dx} = 2\sqrt{2x+y-3} - 2$

j. $\frac{dy}{dx} = 2\sqrt{2x+y-3}$

k. $x\frac{dy}{dx} - y = \sqrt{xy+x^2}$

l. $\frac{dy}{dx} + 3y = 28e^{2x}y^{-3}$

m. $\frac{dy}{dx} = (x-y+3)^2$

n. $\frac{dy}{dx} + 2x = 2\sqrt{y+x^2}$

o. $\cos(y)\frac{dy}{dx} = e^{-x} - \sin(y)$

p. $\frac{dy}{dx} = x\left[1 + 2\frac{y}{x^2} + \frac{y^2}{x^4}\right]$

6.5. *Consider a generic Bernoulli equation*

$$\frac{dy}{dx} + p(x)y = f(x)y^n$$

where $p(x)$ *and* $f(x)$ *are known functions of* x *and* n *is any real number other than* 0 *or* 1 *. Use the substitution* $u = y^r$ *(equivalently,* $y = u^{1/r}$ *) and derive that the above Bernoulli equation for* y *reduces to a linear equation for* u *if and only if* $r = 1 - n$ *. In the process, also derive the resulting linear equation for* u *.*