## Pre-Calculus Mathematics 12 - 1.1 - Functions and Relations

**Goals:**
**1.** Identify relations, functions, one-to-one functions, domains, ranges, vertical and horizontal line tests, restrictions
**2.** Recognize function types

**Definitions:**

**Domain** - the set of all possible (   ) values of a relation.

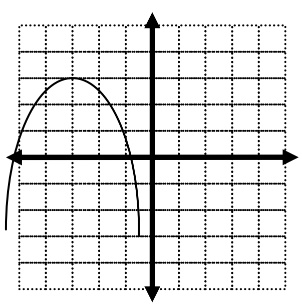
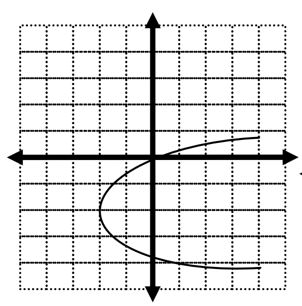
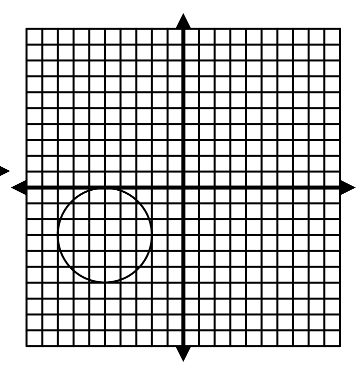
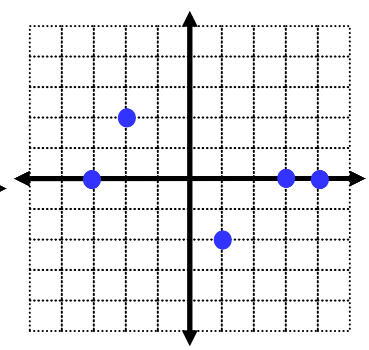
**Range** - the set of all possible (   ) values of a relation.

**Relation** - a set of ordered pair(s)

**Function** - a relation in which each domain (   ) value is paired with only one unique range (   ) value.

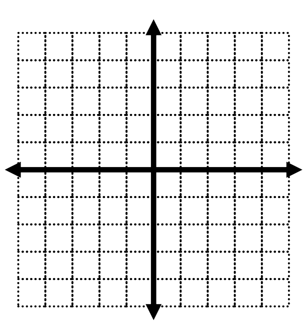
**Vertical line test** - an equation defines y as a function of x if and only if every vertical line in the coordinate plane intersects the graph of the equation only once.

**Example 1:** Determine the domain/range of the following graphs and whether they are a function/relation

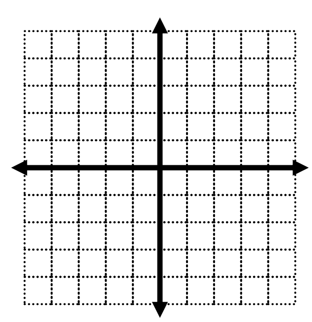





**Types of Functions:**

1. Constant function: - eg. $x$ = k



2. Linear: $y = mx + b \text{ or } f(x) = mx + b$



3. Quadratic Standard form

$$f(x) = a(x-h)^2 + k$$

vertex

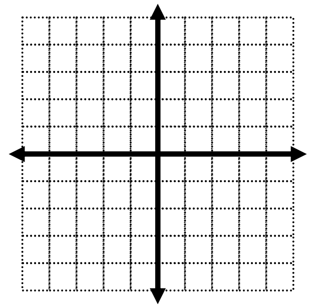
*Axis of symmetry:*

if a > 0, graph opens _______

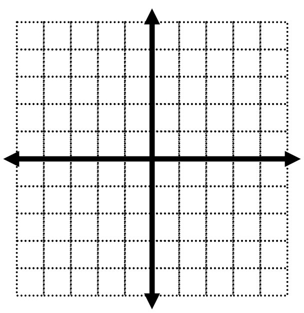
if a < 0, graph opens _______

General (expanded) form

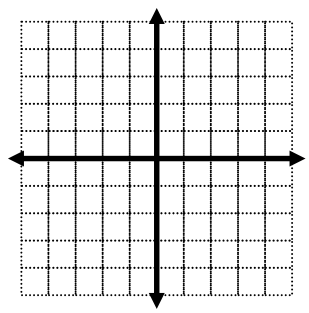
$$f(x) = ax^2 + bx + c \qquad a, b, c \in \mathbb{R} \quad a \neq 0$$



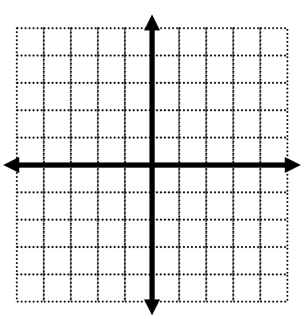
4. Cubic: $f(x) = ax^3 + bx^2 + cx + d$



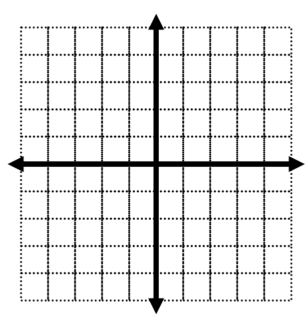
5. Absolute value: $f(x) = a|x-h| + k$



6. Radical: $f(x) = a\sqrt{x-h} + k$



7. Reciprocal: $f(x) = \dfrac{a}{x-h} + k$



**Restrictions on the domain of a functions:**

1. Cannot have a negative number inside an even root. $f(x)=\sqrt{3-x}$

2. Cannot have zero in a denominator $f(x)=\dfrac{8}{x-6}$

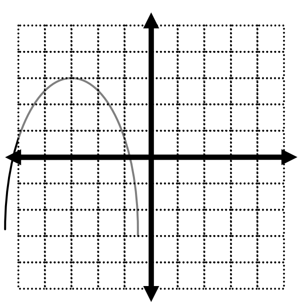
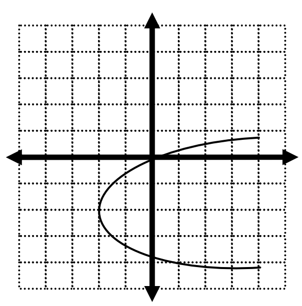
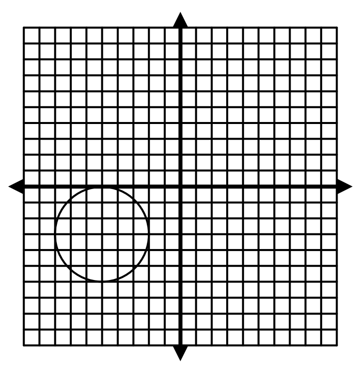
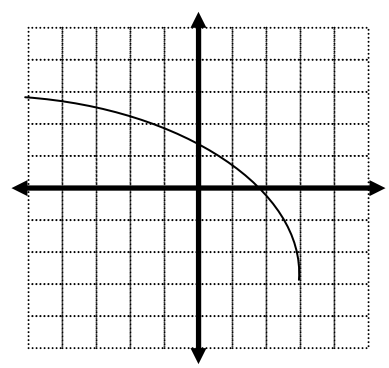
**One–to–One Function**

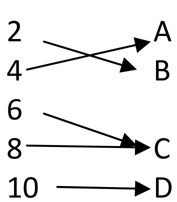
A one-to-one function is a function in which every single value of the domain is associated with only one value in the range, and vice-versa.

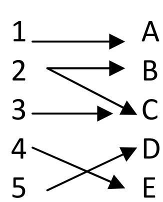
**Horizontal line test** - for a one-to-one function:

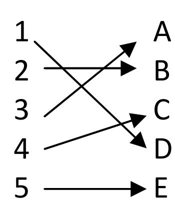
A function, f(x), is a one-to-one function of x if and only if every horizontal line in the coordinate plane intersects the function only once at most.

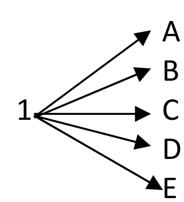
**Example 2:** Determine whether the following relations are functions, one-to-one functions or neither










Practice: Pg 9 & 10 # 1, 2 and 3

**Goal:**
**1.** Perform operations with functions both with and without the graph of the function

Functions are number generators. When you put a value for the domain in the function, you will get the resulting value in the range.

$$f(x) = x^2 + 5$$

And just like numbers, functions can be added, subtracted, multiplied and divided.

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Use the following functions, $f(x) = 2x - 1$ & $g(x) = x^2 - 4$ to determine:

a) Sum: $(f+g)(x) = f(x) + g(x)$

b) Difference: $(f-g)(x) = f(x) - g(x)$

Product: $(fg)(x) = f(x) \cdot g(x)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

**Note:** The domain of the new function must include the restrictions of the new functions as well as the restriction(s) of the original function(s)

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**Example 1:** Given functions below, determine each new combined function and its domain.

$f(x) = \frac{1}{x^2}$, $g(x) = \frac{2}{x+2}$, $h(x) = 2x^2 - 5x - 3$ $i(x) = x^2 - 9$, $j(x) = x^2$, $k(x) = \sqrt{x}$

a) $(gj)(-3)$

b) $\left(\frac{f}{g}\right)(-4)$

**Example 1 continued…** Given functions below, determine each new combined function and its domain.

$f(x)=\frac{1}{x^2}$, $g(x)=\frac{2}{x+2}$, $h(x)=2x^2-5x-3$ $i(x)=x^2-9$, $j(x)=x^2$, $k(x)=\sqrt{x}$

c) $(g-f)(x)$

d) $\left(\frac{gk}{i}\right)(x)$

e) $\left(\frac{1}{k}\right)\left(\frac{1}{k}\right)(x)$

f) $\left(\frac{f}{x}\right)(x)$

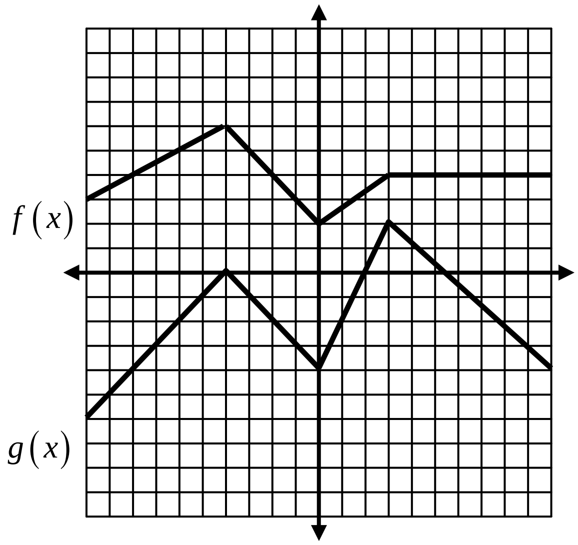
**Example 1 continued…** Given functions below, determine each new combined function and its domain.

$f(x)=\dfrac{1}{x^2}$, $g(x)=\dfrac{2}{x+2}$, $h(x)=2x^2-5x-3$ $i(x)=x^2-9$, $j(x)=x^2$, $k(x)=\sqrt{x}$

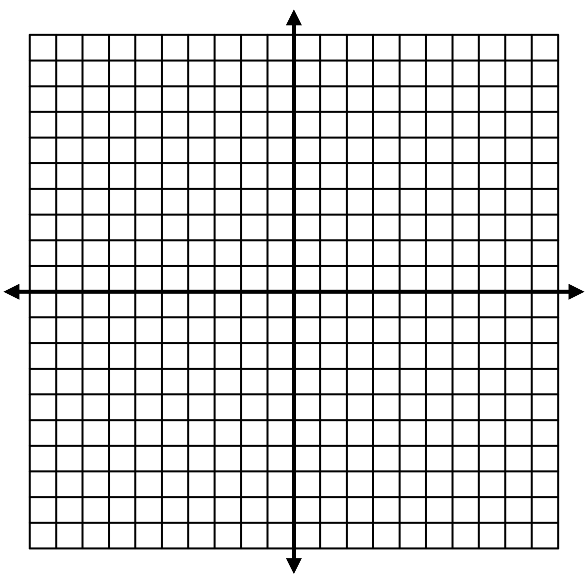
g) $\left(\dfrac{g}{f}\right)(x)-\left(\dfrac{h}{i}\right)(x)$

h) $\left[g\left(h-i\right)\right](x)$

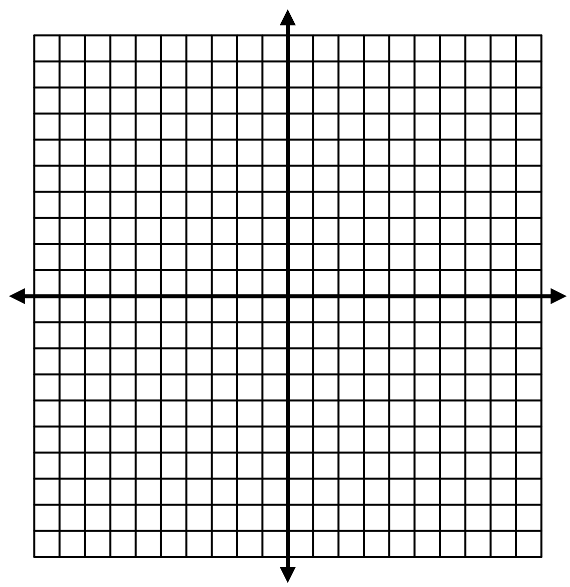
**Example 2:** Use the graph of $f(x)$ and $g(x)$ to graph the following functions.




a) $(f-g)(x)$



b) $\left(2f-\frac{1}{2}g\right)(x)$



Practice: Pg 13 -16 # 1, 2, 3 (a, b, c, d, i, j), 4 (a, d), 5b, 6c, 7d, 8e

**Goals:**
1. Perform the composite of two or more functions
2. Decompose a composite function

A function can be considered like a machine. It has an input value (x) and generates an output, y, or $f(x)$.

$$f(x)=3x-4$$

A function can also be "input" into another function. This would then generate a **composite function.**

Using the functions $f(x)=3x-4$ & $g(x)=x^2+2$

Determine: $f(g(x))$ $g(f(x))$

Notation: $f(g(x))=(f\circ g)(x)=f\circ g$

With composite functions, the output of one function becomes the input of another function(s).

When determining $(f\circ g)(x)$ the output of function $g(x)$ becomes the input of the function $f(x)$.

**Example 1:** Given functions below, determine each composite function.

$$f(x)=\frac{1}{x^2},\quad g(x)=\frac{2}{x+2},\quad h(x)=2x^2-5x-3,\quad i(x)=x^2-9,\quad k(x)=\sqrt{x}$$

a) $(h\circ g)(-3)$

b) $(k\circ g\circ f)(-2)$

**Example 1 continued…….**

$f(x)=\frac{1}{x^2}$, $g(x)=\frac{2}{x+2}$, $h(x)=2x^2-5x-3$, $i(x)=x^2-9$, $k(x)=\sqrt{x}$

c) $(h\circ i)(x)$

d) $i\left(h(x)\right)$

## Domain of a Composite Function

The domain of $(f\circ g)(x)$ has both the original restriction(s) of $g(x)$ as well as the domain restrictions of the final composite function $(f\circ g)(x)$ .

**Example 2:** Given the functions $f(x)=\frac{4}{x}$, $g(x)=\frac{2}{x-3}$ determine each composite function and its domain.

a) $(f\circ g)(x)$

b) $(g\circ f)(x)$

c) $(g\circ g)(x)$

**Example 3: Calculus Application**: Compute $\dfrac{f(x+h)-f(x)}{h}$, for $f(x)=3x^2-2x$

## Decomposing a Function:

To decompose a function, look for a simple function that could be the input of another simple function.

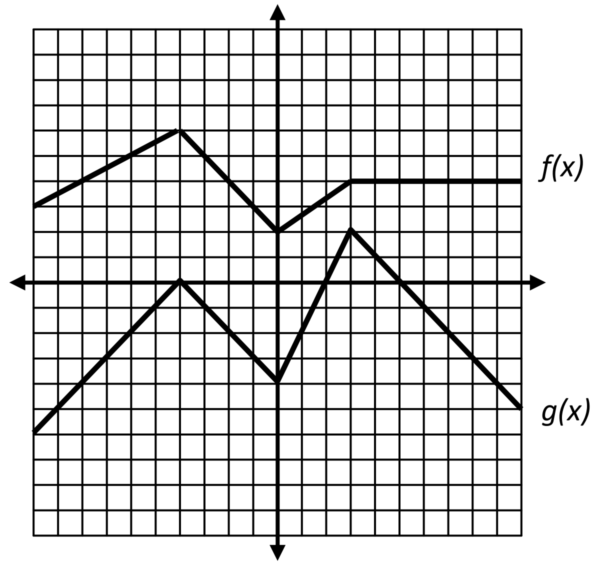
**Example 4:** Find two functions $f(x)$ and $g(x)$ such that $h(x)=(f\circ g)(x)$

a) $h(x)=\sqrt{2x^2+1}-5$

b) $h(x)=\dfrac{2}{3x+4}$

The output of a composite function can also be determined without having the actual functions as long as you have the corresponding graphs of the functions.

**Example 5:** Given the functions $f(x)$ and $g(x)$, Determine:




a) $(f \circ g)(-4)$

*b)* $(g \circ f)(3)$

*c)* $f\left((g)(-8)\right)$

*d)* $(g \circ g)(5)$

Practice: Pg 22 –26 #1 (a, b, c), 2(a, b, g), 3(a, f), 4(b, h), 5a, 6d, 7(a, m), 10b, 11

## Pre-Calculus Mathematics 12 - 1.4 – Transformations of Graphs Part 1

**Goals:**
**1.** Identify and apply a vertical translation to a function with or without a graph
**2.** Identify and apply a horizontal translation to a function with or without a graph

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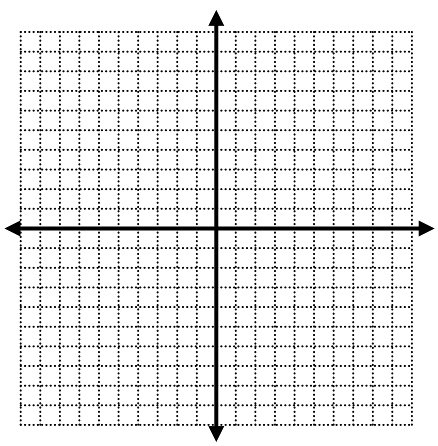
**Vertical translations** **(shifting the graph up or down)** form: $y = f(x-h) + \boldsymbol{k}$

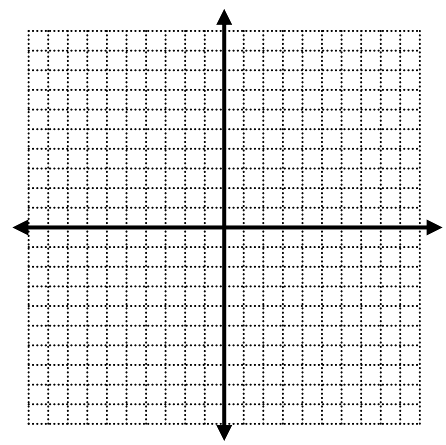
Square Root

$y = \sqrt{x} - 2$



Absolute Value

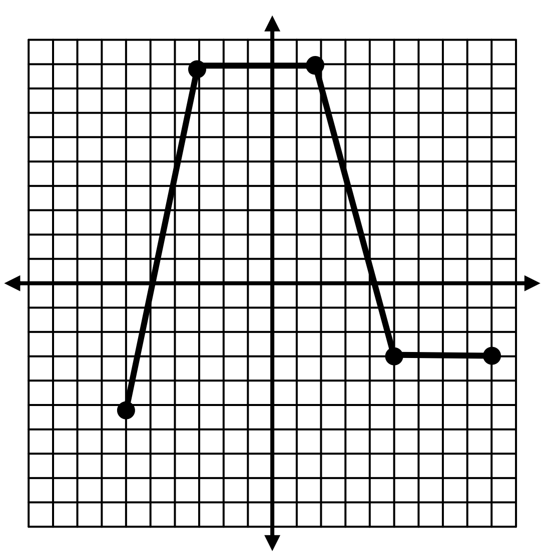
$y = |x| + 3$



So for $y = f(x-h) + \boldsymbol{k}$

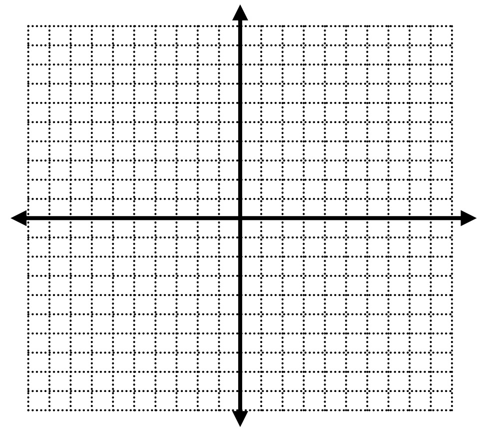
If $\boldsymbol{k}$ is positive → ______________________ If $\boldsymbol{k}$ is negative → ___________________

**Example 1:** Given the graph of $y = f(x)$ below, describe the transformation applied graph $y = f(x) - 2$, and map the coordinates of the image points.



**Horizontal Translations** **(shifting the graph left or right)** form: $y = f(x - \boldsymbol{h})$

Graph: $y = x^2$ ; $y = (x-3)^2$ ; $y = (x+2)^2$



For $y = f(x - \boldsymbol{h})$

If $\boldsymbol{h}$ is positive → ________________ If $\boldsymbol{h}$ is negative ← ____________________

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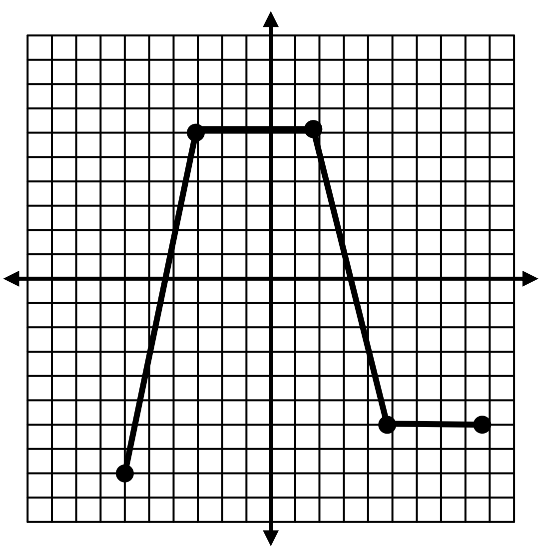
**Example 2:** Given the graph of $y = f(x)$ below, describe the transformation applied graph $y = f(x-2)$, and map the coordinates of the image points.



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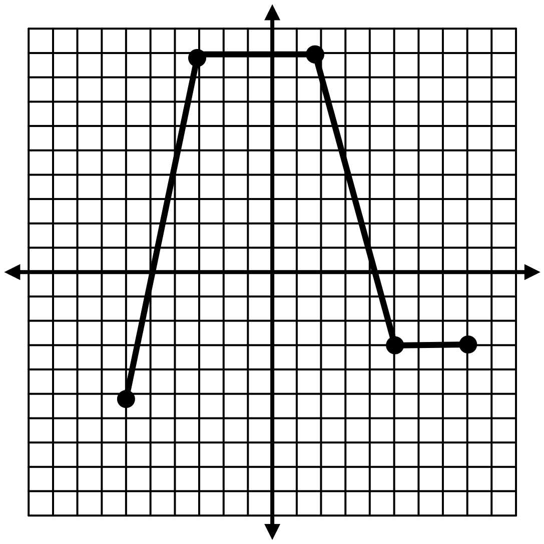
**Example 3:** Given the graph of $y = f(x)$ below, describe the transformation applied graph $y = f(x-3) - 4$, and generalize the coordinate transformation.



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**Example 4:** Given the function $y = f(x)$ below, describe the transformation applied to each of the functions below.

a) $y = f(x+3) - 2$

b) $y = f(x-5) + 6$

c) $y + 1 = f(x+5)$

d) $y - 4 = f(x-7) - 7$

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Practice: Complete the questions on the next page, the solutions are provided

## Practise

1. For each function, state the values of $h$ and $k$, the parameters that represent the horizontal and vertical translations applied to $y = f(x)$.

   a) $y - 5 = f(x)$

   b) $y = f(x) - 4$

   c) $y = f(x + 1)$

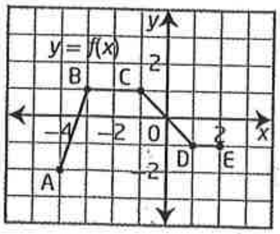
   d) $y + 3 = f(x - 7)$

   e) $y = f(x + 2) + 4$

2. Given the graph of $y = f(x)$ and each of the following transformations,
   - state the coordinates of the image points A′, B′, C′, D′ and E′
   - sketch the graph of the transformed function

   a) $g(x) = f(x) + 3$  b) $h(x) = f(x - 2)$

   c) $s(x) = f(x + 4)$  d) $t(x) = f(x) - 2$




3. Describe, using mapping notation, how the graphs of the following functions can be obtained from the graph of $y = f(x)$.

   a) $y = f(x + 10)$

   b) $y + 6 = f(x)$

   c) $y = f(x - 7) + 4$

   d) $y - 3 = f(x - 1)$

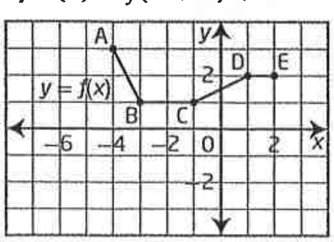
4. Given the graph of $y = f(x)$, sketch the graph of the transformed function. Describe the transformation that can be applied to the graph of $f(x)$ to obtain the graph of the transformed function. Then, write the transformation using mapping notation.

   a) $r(x) = f(x + 4) - 3$

   b) $s(x) = f(x - 2) - 4$

   c) $t(x) = f(x - 2) + 5$

   d) $v(x) = f(x + 3) + 2$




5. For each transformation, identify the values of $h$ and $k$. Then, write the equation of the transformed function in the form $y - k = f(x - h)$.

   a) $f(x) = \frac{1}{x}$, translated 5 units to the left and 4 units up

   b) $f(x) = x^2$, translated 8 units to the right and 6 units up

   c) $f(x) = |x|$, translated 10 units to the right and 8 units down

   d) $y = f(x)$, translated 7 units to the left and 12 units down

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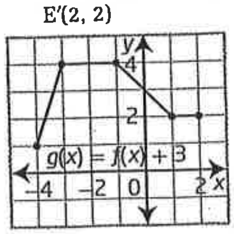
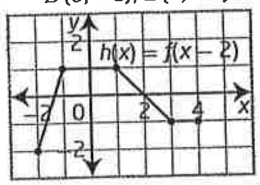
1. a) $h = 0, k = 5$ b) $h = 0, k = -4$ c) $h = -1, k = 0$ d) $h = 7, k = -3$ e) $h = -2, k = 4$

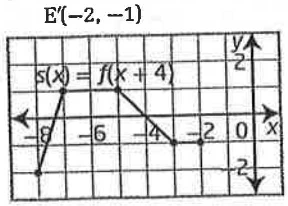
2. a) A′(−4, 1), B′(−3, 4), C′(−1, 4), D′(1, 2), E′(2, 2)



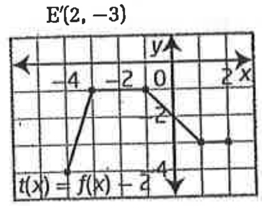

   b) A′(−2, −2), B′(−1, 1), C′(1, 1), D′(3, −1), E′(4, −1)




   c) A′(−8, −2), B′(−7, 1), C′(−5, 1), D′(−3, −1), E′(−2, −1)

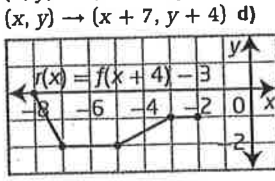



   d) A′(−4, −4), B′(−3, −1), C′(−1, −1), D′(1, −3), E′(2, −3)



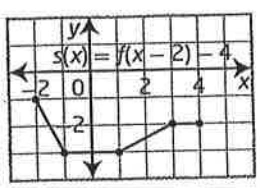

3. a) $(x, y) \rightarrow (x - 10, y)$ b) $(x, y) \rightarrow (x, y - 6)$ c) $(x, y) \rightarrow (x + 7, y + 4)$ d) $(x, y) \rightarrow (x + 1, y + 3)$

4. a)



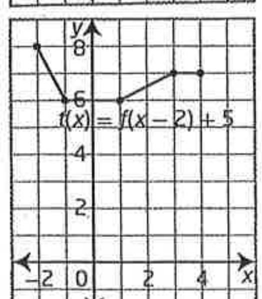

a vertical translation of 3 units down and a horizontal translation of 4 units left; $(x, y) \rightarrow (x - 4, y - 3)$

   b)



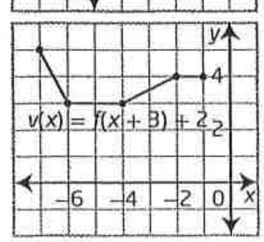

a vertical translation of 4 units down and a horizontal translation of 2 units right; $(x, y) \rightarrow (x + 2, y - 4)$

   c)




a vertical translation of 5 units up and a horizontal translation of 2 units right; $(x, y) \rightarrow (x + 2, y + 5)$

   d)




a vertical translation of 2 units up and a horizontal translation of 3 units left; $(x, y) \rightarrow (x - 3, y + 2)$

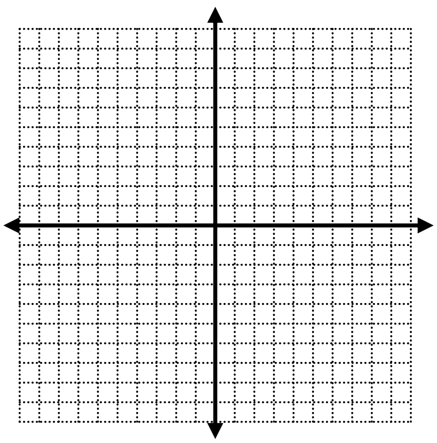
5. a) $h = -5, k = 4; y - 4 = f(x + 5)$ b) $h = 8, k = 6; y - 6 = f(x - 8)$ c) $h = 10, k = -8; y + 8 = f(x - 10)$ d) $h = -7, k = -12; y + 12 = f(x + 7)$

**Pre-Calculus Mathematics 12 - 1.4 – Transformations of Graphs Part 2**

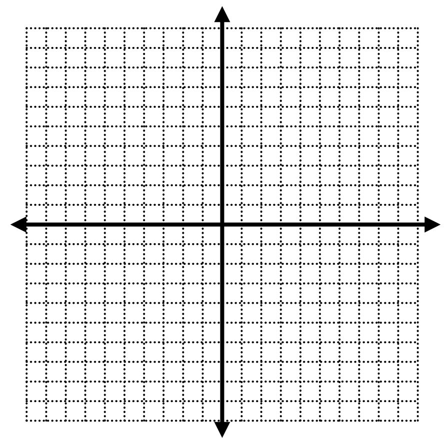
**Goals:**
**1.** Identify and apply a reflection to a function with or without a graph
**2.** Identify and apply a vertical compression/expansion to a function
**3.** Identify and apply a horizontal compression/expansion to a function

## Reflections in the Coordinate Axis (flipping the graph over the x or y axis)

$y = -\sqrt{x}$



$y = \sqrt{-x}$

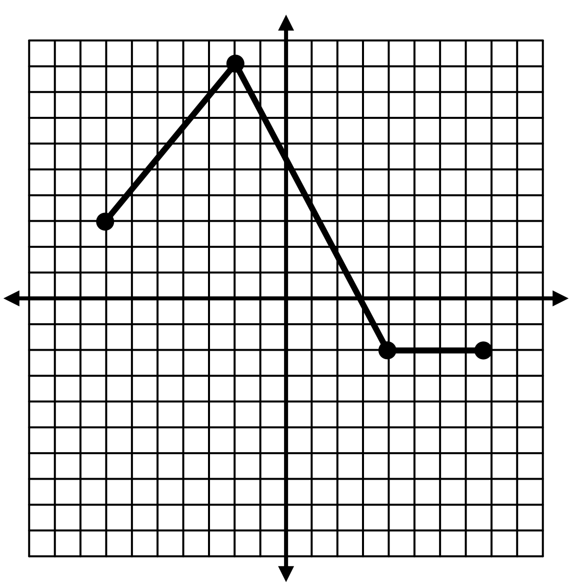


So for $y = f(x)$

If $y = -f(x)$ → ____________________

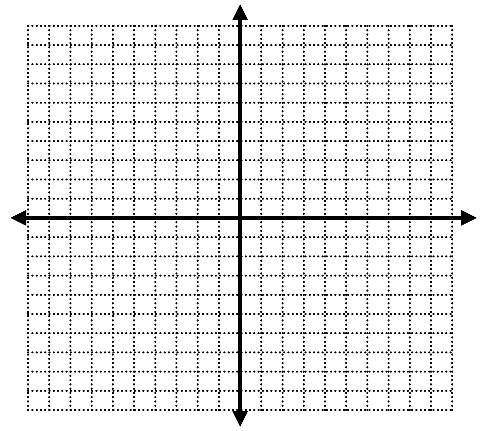
If $y = f(-x)$ → ____________________

**Example 1:** Given the graph of $y = f(x)$ below, describe the transformation applied to the graph $y = f(-x)$, and map the general coordinates of the image points.



**Vertical Expansions/Compressions** **(stretching the graph in y direction)**

Graph: $y=\sqrt{x}$ $y=2\sqrt{x}$ $y=\frac{1}{2}\sqrt{x}$

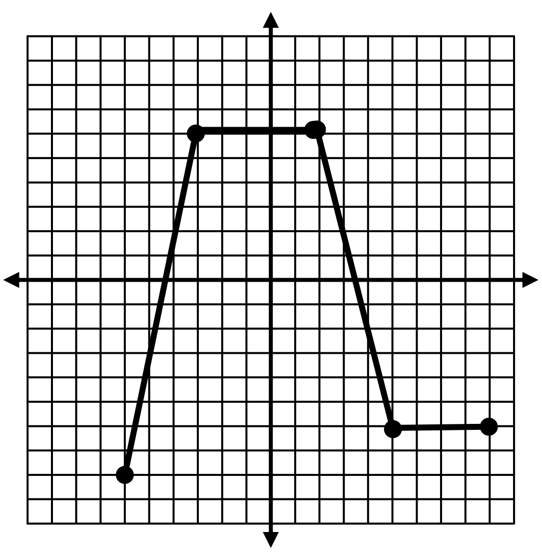


For $y=af(x)$

If **|a|** > 1 → ____________________ If **|a|** < 1 → ____________________

______________________________________________________________

**Example 2:** Given the graph of $y=f(x)$ below, describe the transformation applied to the graph $y=\frac{1}{3}f(x)$, and map the general coordinates of the image points.



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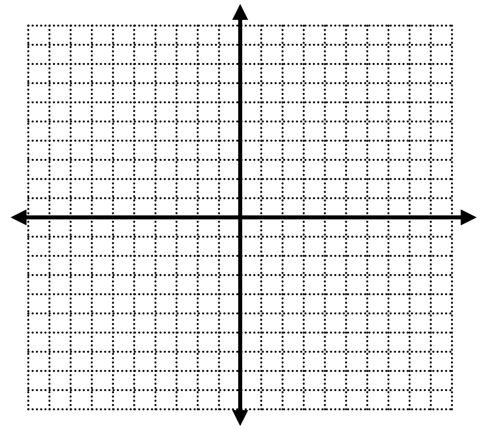
## Horizontal Compressions/Expansions **(stretching the graph in x direction)**

Graph: $y = x^2$ $y = (2x)^2$ $y = \left(\frac{1}{2}x\right)^2$

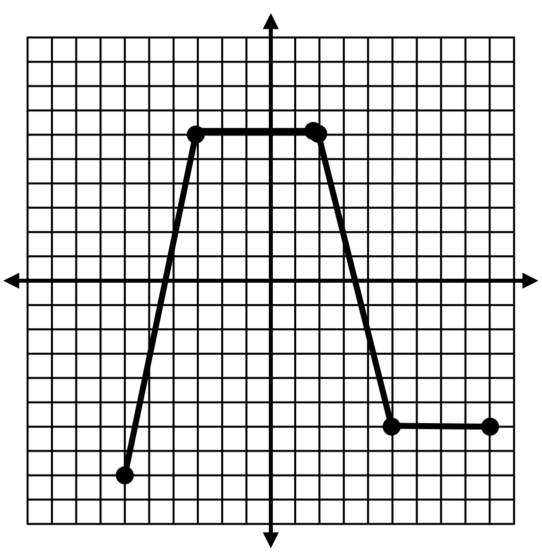


For $y = f(bx)$

If $|b| > 1$ → ____________________ If $|b| < 1$ → ____________________

___________________________________________________________

**Example 3:** Given the graph of $y = f(x)$ below, describe the transformation applied to the graph $y = f(2x)$, and map the general coordinates of the image points.



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**Example 4:** Given the function $y = f(x)$ below, describe the transformation applied to each of the functions below.

a) $y = -2f(x-4)+6$

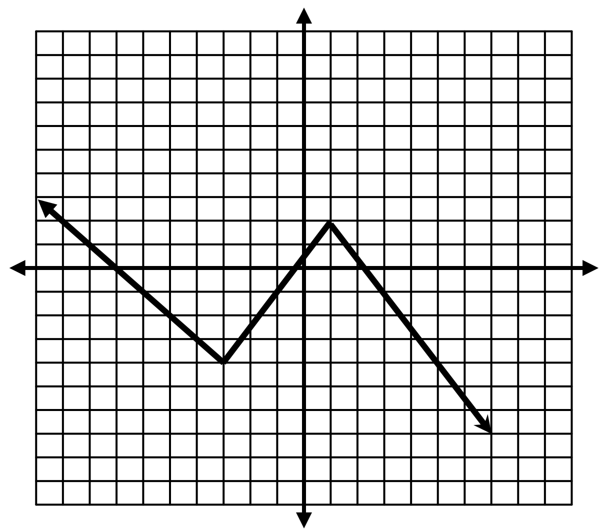
b) $y = f(-3x-12)-4$

c) $2y+10 = f(5-x)$

---

**Example 5:** Given the graph of $y = f(x)$,

sketch the graph of $y = -\frac{1}{2}f\left(\frac{1}{3}x\right)$



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**Example 6:** If (-3, 4) is a point on the graph $y = f(x)$ what must be the point on the graph $y+7 = -5f(2x-4)+3$.

---

Practice: Pg 34 # 1 (a, c, e), 2 (a, c, e, f), 3, 4, 6, 9c, 10 (b, d, f), 11d, 12(c, g)

**Goals:**
**1.** Determine the inverse of a given function
**2.** Determine whether two functions are inverses
**3.** Graph the inverse of a function

Two functions are inverse functions if one function "undoes" what the other function "does".

Inverse functions are just a reflection across the line *y* = *x*. In order for both a function $f(x)$ and its inverse $f^{-1}(x)$ to qualify as functions, $f(x)$ must be a one-to-one function.

**Determining the inverse of a function:**

1. Verify that $f(x)$ is a one-to-one function. (If not, its inverse is not a function.)
2. Replace $f(x)$ with *y* and exchange all *x's* and *y's.*
3. Solve for *y*.
4. Replace the new y with $f^{-1}(x)$. (Only if $f^{-1}(x)$ is actually a function.)

**Example 1:** If $f(x) = 3x - 2$, determine the inverse of the function.

**Example 2:** If $g(x) = \dfrac{x}{3x-1}$, determine the inverse of the function.

**Example 3:** If $h(x) = x^2 - 3$, determine the inverse of the function.

**Determining if two functions are inverses:** ('undo' each other)

Two functions $f(x)$ *and* $g(x)$ are inverses of each other **if and only if**

$(f \circ g)(x) = x$ for every value of *x* in the domain of *g* and

$(g \circ f)(x) = x$ for every value of *x* in the domain of *f.*

**Example 4:** Determine whether $f(x) = \dfrac{x}{2x-3}$ and $g(x) = \dfrac{3x}{2x-1}$ are inverses of each other.

**Graphing the inverse of a function**

The graph of a function and its inverse are symmetric about the line $y = x$.

$f(x)$ is the reflection of $f^{-1}(x)$ on the line $y = x$ and vice versa.

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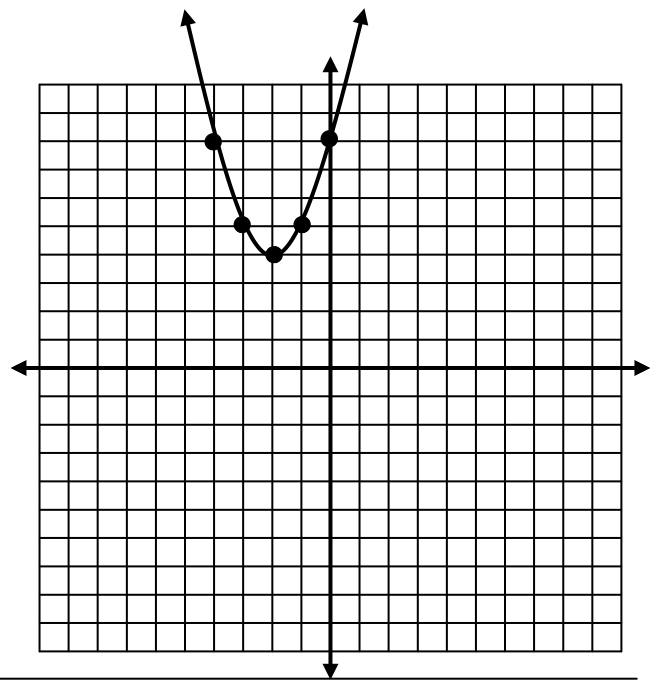
**Example 5:** Given the function $f(x) = (x+2)^2 + 4$, graph the inverse of the function. Determine whether the inverse of the function is a function.



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**Example 6:** If (-3, 4) is a point on the graph $y = f(x)$ what must be the point on the graph $y = 1 - f^{-1}(-x)$.

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Practice: Page 44 # 2(a,b,c), 4, 5(a,b,f), 6 (a, c, f), 7(a, b, c)

**Goals:**
**1.** Perform reflections, expansions, compressions and translations of a function both with and without a graph.

All of the transformations we have performed can be summarized as follows:

$$y = f(x) \text{ transforms to } y = af\left[b(x-h)\right]+k$$

**Reflections:** $a < 0$, reflection in the $x$-axis
$b < 0$, reflection in the $y$-axis
$f^{-1}$ reflection across the line y = x

**Expansion:** $|a| > 1$, vertical expansion by a factor of $|a|$
$|b| < 1$, horizontal expansion by a factor of $\frac{1}{|b|}$

**Compression:** $|a| < 1$, vertical compression by a factor of $|a|$
$|b| > 1$, horizontal compression by a factor of $\frac{1}{|b|}$

**Translation:** $k > 0$, vertical translation $k$ units up
$k < 0$, vertical translation $k$ units down
$h > 0$, horizontal translation $h$ units right
$h < 0$, horizontal translation $h$ units left

When combining transformations, the reflections/expansions/compressions must occur before the translations.

**Example 1:** Given point P (– 4, 2) on $y = f(x)$ find the new location for P on:

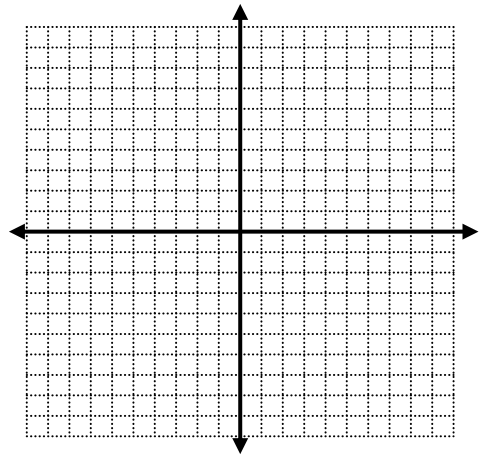
$$y = -f\left(\frac{x}{3}\right)+2$$

**Example 2:** Given point P (a, b) on $y = f(x)$ find the new location for P on:

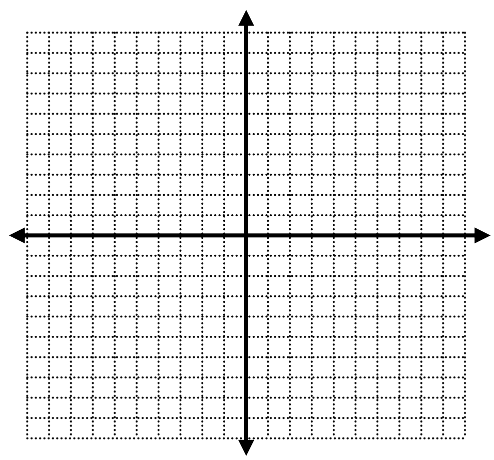
$$y = -3f\left(6-2x\right)-5$$

**Example 3:** Graph the following functions:

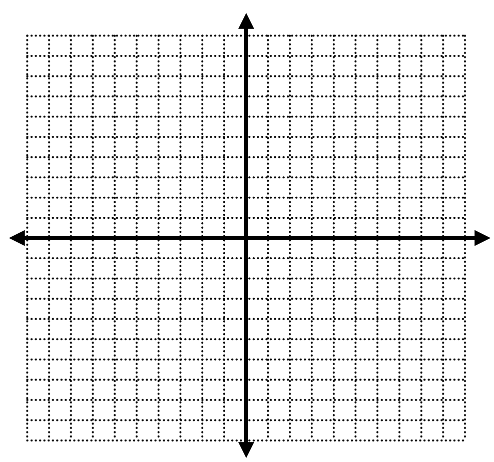
a) $y = -\frac{1}{2}\left|x-3\right|+4$



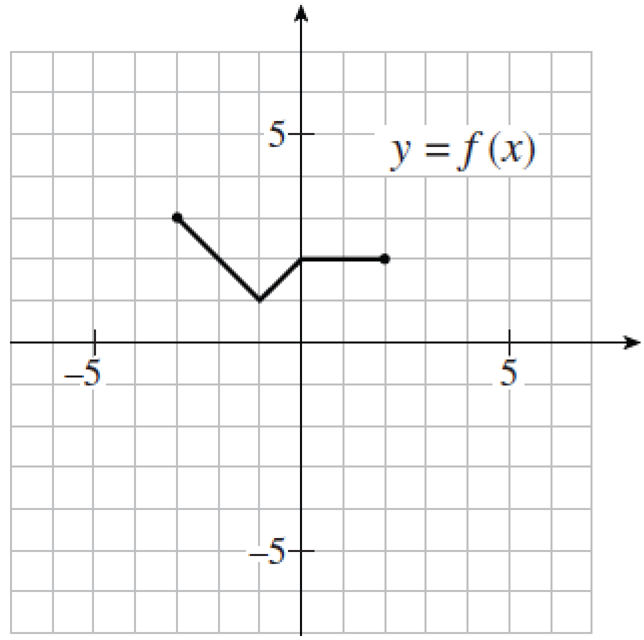
b) $h(x) = -3\sqrt{-2\left(x+1\right)}+4$



c) If $f(x) = x^2+1$ $\quad y = -\frac{1}{2}f\left(2x+4\right)-3$



**Example 4:** Given the graph of $y = f(x)$, sketch the graph $y = -2f(x+3)+1$




Practice: Page 51 #2 (a, b, c, e, g, j), 3(a, b, c, d), 6 (a, b, c), 7(a, c, e)