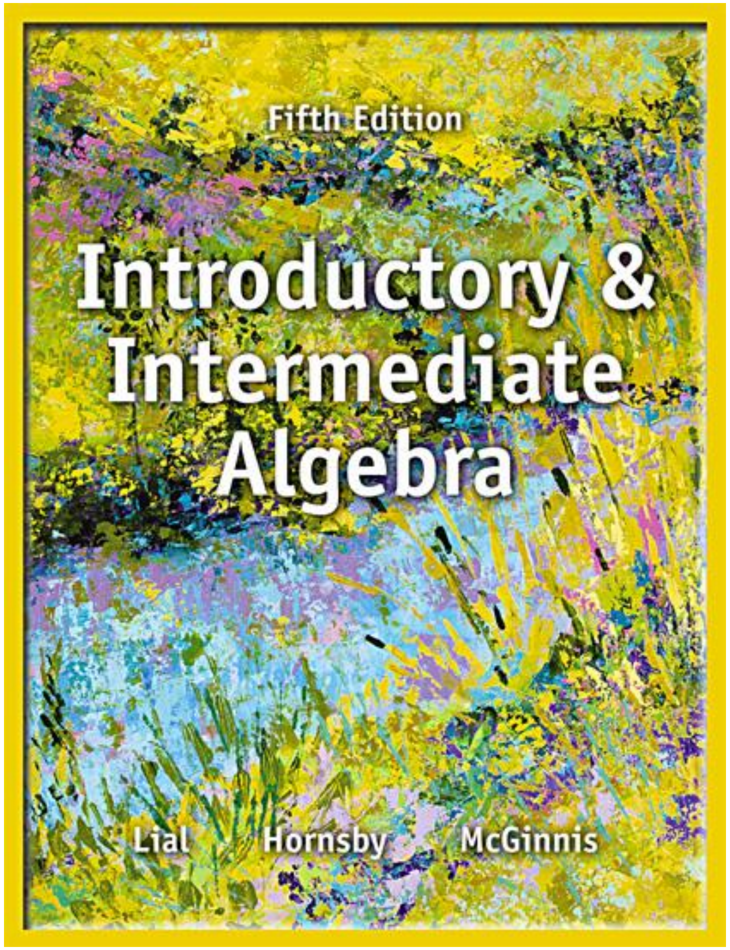
3 Graphs of Linear Equations, and Inequalities, in Two Variables



3.1 Linear Equations in Two Variables; The Rectangular Coordinate System

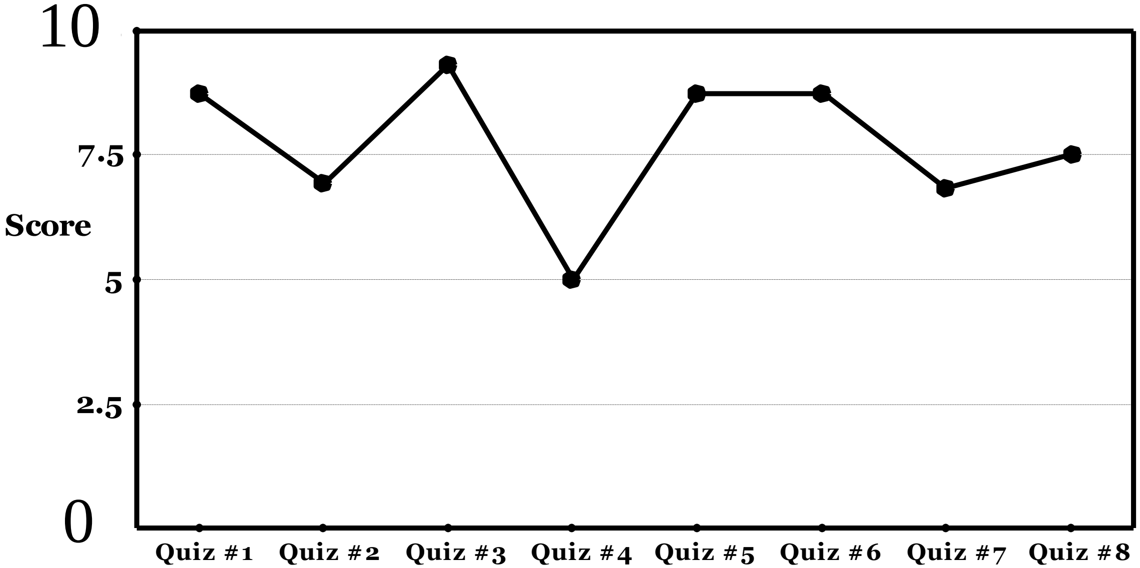
Objectives

1. Interpret graphs.
2. Write a solution as an ordered pair.
3. Decide whether a given ordered pair is a solution of a given equation.
4. Complete ordered pairs for a given equation.
5. Complete a table of values.
6. Plot ordered pairs.

Interpret Graphs

Example 2

The line graph below shows the class averages on the first eight quizzes in a college math course.



(a) Which quiz had the highest class average? Quiz #3

(b) Which quiz experienced the biggest jump in class average from the previous quiz? Quiz #5

(c) Estimate the difference between the class average on Quiz #2 and Quiz #3.

Difference $\approx 9 - 7$

Difference ≈ 2

Linear Equation in Two Variables

A **linear equation in two variables** is an equation that can be written in the form

$$Ax + By = C,$$

where A, B, and C are real numbers and A and B are not both 0.

Note

Other linear equations in two variables, such as

$$y = 4x + 5 \quad \text{and} \quad 3x = 7 - 2y,$$

are not written in standard form but could be. We discuss the forms of linear equations in **Section 3.4.**

Write a Solution as an Ordered Pair

A solution of a linear equation in two variables requires two numbers, one for each variable. For example, a true statement results when we replace x with 2 and y with 13 in the equation $y = 4x + 5$ since

$$13 = 4(2) + 5. \qquad \text{Let } x = 2,\ y = 13.$$

The pair of numbers $x = 2$ and $y = 13$ gives one solution of the equation $y = 4x + 5$. The phrase "$x = 2$ and $y = 13$" is abbreviated

x-value $\qquad$ y-value

$$(2,13)$$

Ordered Pair

The x-value is always given first. A pair of numbers such as (2,13) is called an **ordered pair**.

Decide Whether an Ordered Pair is a Solution

Example 3

Decide whether each ordered pair is a solution to $5x - 2y = 4$.

(a) (2,3)

To see whether (2,3) is a solution, substitute 2 for x and 3 for y.

$$5(2) - 2(3) \stackrel{?}{=} 4$$

$$10 - 6 \stackrel{?}{=} 4$$

$$4 = 4 \quad \text{True}$$

Thus, (2,3) is a solution.

(b) (–2,–3)

To see whether (–2,–3) is a solution, substitute –2 for x and –3 for y.

$$5(-2) - 2(-3) \stackrel{?}{=} 4$$

$$-10 + 6 \stackrel{?}{=} 4$$

$$-4 = 4 \quad \text{False}$$

Thus, (–2,–3) is *not* a solution.

Complete Ordered Pairs

Example 4

Complete each ordered pair for the equation $-3x + y = 4$.

(a) (3,)

Substitute 3 for x and solve for y.

$$-3(3) + y = 4$$

$$-9 + y = 4$$

$$\underline{+9 \qquad +9}$$

$$y = 13$$

The ordered pair is (3,13).

(b) (,1)

Substitute 1 for y and solve for x.

$$-3x + 1 = 4$$

$$\underline{\quad -1 \quad -1}$$

$$\frac{-3x}{-3} = \frac{3}{-3}$$

$$x = -1$$

The ordered pair is (–1,1).

Complete a Table of Values

Example 5

Complete the table of values for each equation.

(a) $x - 3y = 6$

x	y
	-1
12	

$$x - 3(-1) = 6$$
$$x + 3 = 6$$
$$\underline{\quad -3 \quad -3}$$
$$x = 3$$

$$12 - 3y = 6$$
$$\underline{-12 \qquad -12}$$
$$\frac{-3y}{-3} = \frac{-6}{-3}$$
$$y = 2$$

For the first ordered pair, let $y = -1$.

For the second ordered pair, let $x = 12$.

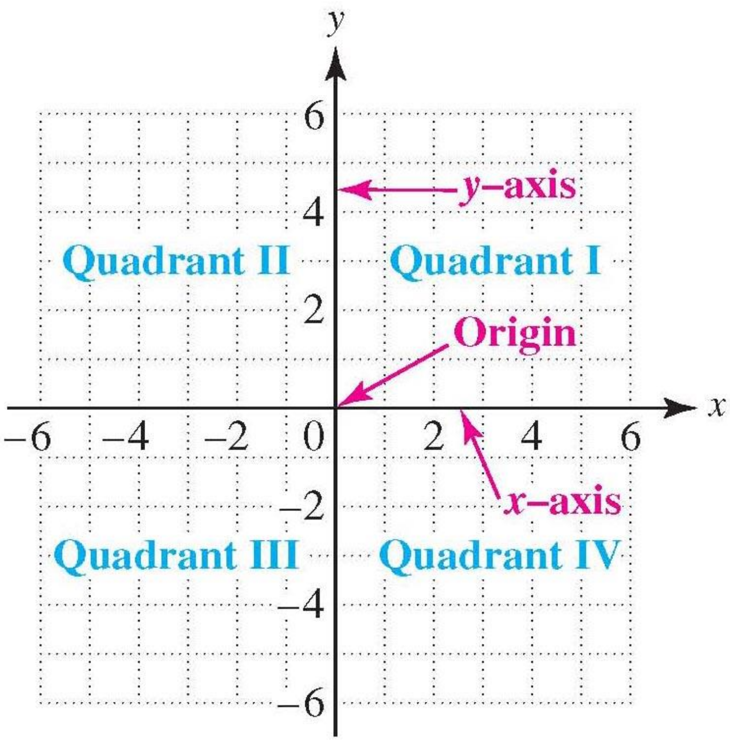
Thus, the completed table is:

x	y
3	-1
12	2

Plot Ordered Pairs

To graph solutions, represented as the ordered pairs (x, y), we need *two* number lines, one for each variable, as drawn below. The horizontal number line is called the ***x*-axis**, and the vertical line is called the ***y*-axis**.

Together, the *x*-axis and the *y*-axis form a **rectangular coordinate system**, also called the **Cartesian coordinate system**, in honor of René Descartes, the French mathematician who is credited with its invention.



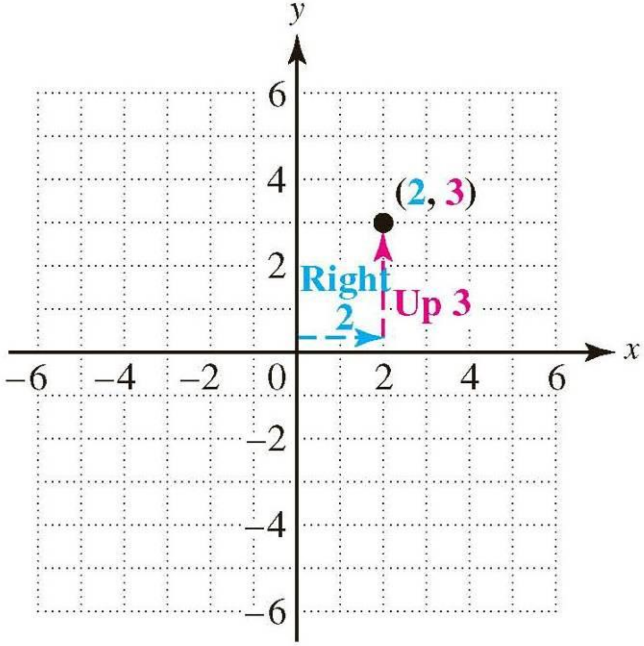
Plot Ordered Pairs

The coordinate system is divided into four regions, called **quadrants**. These quadrants are numbered counter-clockwise, as shown on the previous slide. ***Points on the axes themselves are not in any quadrant***. The point at which the x-axis and the y-axis meet is called the **origin**. The origin, which is labeled 0 in the previous figure, is the point corresponding to (0,0).

The x-axis and y-axis determine a **plane**. By referring to the two axes, every point in the plane can be associated with an ordered pair. The numbers in the ordered pair are called the **coordinates** of the point.

Plot Ordered Pairs

For example, we locate the point associated with the ordered pair (2,3) by starting at the origin. Since the x-coordinate is 2, we go 2 units to the right along the x-axis. Then, since the y-coordinate is 3, we turn and go up 3 units on a line parallel to the y-axis. Thus, the point (2,3) is **plotted**.



Note

When we graph on a number line, one number corresponds to each point. On a plane, however, *both* numbers in an ordered pair are needed to locate a point. The ordered pair is a name for the point.

Plot Ordered Pairs

Example 6

Plot each ordered pair on a coordinate system.

(a) (1, 5)

(b) (–2, 3)

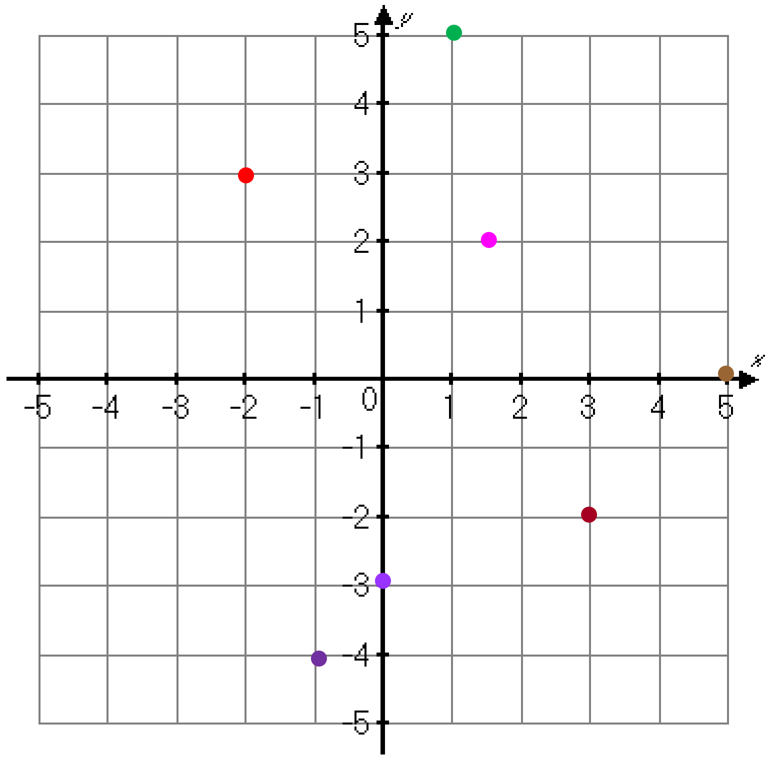
(c) (–1, –4)

(d) (3, –2)

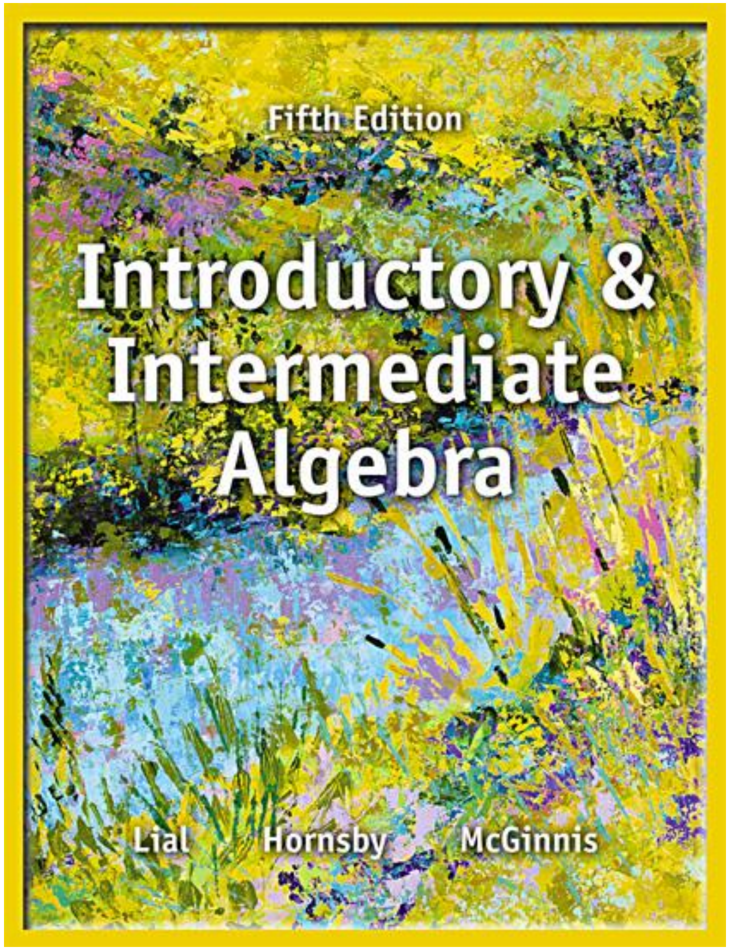
(e) $\left(\frac{3}{2}, 2\right)$

(f) (5, 0)

(g) (0, –3)



3 Graphs of Linear Equations, and Inequalities, in Two Variables



3.2 Graphing Linear Equations in Two Variables

Objectives

1. Graph linear equations by plotting ordered pairs.
2. Find intercepts.
3. Graph linear equations of the form $Ax + By = 0$.
4. Graph linear equations of the form $y = k$ or $x = a$.
5. Use a linear equation to model data.

Graph by Plotting Ordered Pairs

The graph of any linear equation in two variables is a straight line.

Example 1 Graph the linear equation $y = 2x - 1$.

Note that although this equation is not of the form $Ax + By = C$, it could be. Therefore, it is linear. To graph it, we will first find two points by letting $x = 0$ and then $y = 0$.

If $x = 0$, then

$y = 2(0) - 1$

$y = -1$

So, we have the ordered pair $(0,-1)$.

If $y = 0$, then

$$
\begin{aligned}
0 &= 2x - 1 \\
+1 & \quad\;\; +1 \\
\hline
1 &= 2x \\
\frac{1}{2} &= \frac{2x}{2} \\
\tfrac{1}{2} &= x
\end{aligned}
$$

So, we have the ordered pair $(\tfrac{1}{2},0)$.

Graph by Plotting Ordered Pairs

Example 1 (concluded) Graph the linear equation $y = 2x – 1$.

Now we will find a third point (just as a check) by letting $x = 1$.

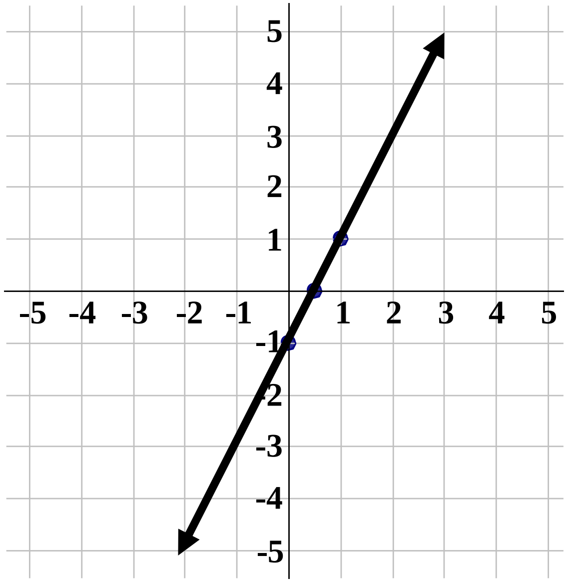
If $x = 1$, then

$y = 2(1) – 1$

$y = 1$

So, we have the ordered pair (1,1).

When we graph, all three points, (0,–1), (½,0), and (1,1), should lie on the same straight line.



Find Intercepts

Finding Intercepts

To find the x-intercept, let $y = 0$ in the given equation and solve for x. Then $(x, 0)$ is the x-intercept.

To find the y-intercept, let $x = 0$ in the given equation and solve for y. Then $(0, y)$ is the y-intercept.

Find Intercepts

Example 2

Find the intercepts for the graph of $x + y = 2$. Then draw the graph.

To find the y-intercept, let $x = 0$; to find the x-intercept, let $y = 0$.

$0 + y = 2$

$y = 2$

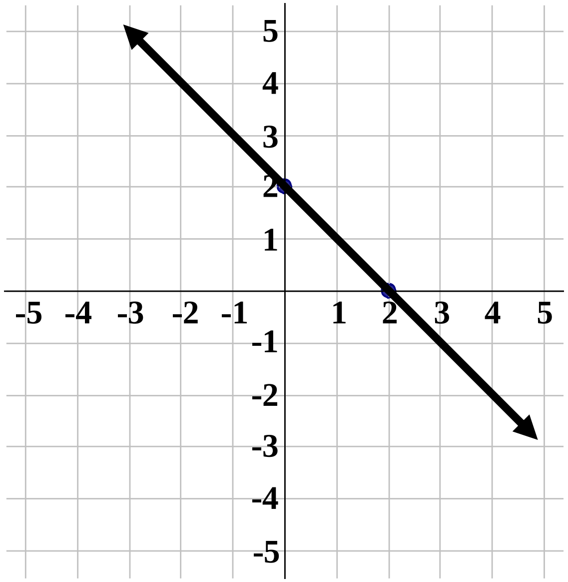
The y-intercept is (0, 2).

$x + 0 = 2$

$x = 2$

The x-intercept is (2, 0).

Plotting the intercepts gives the graph.



Graph Linear Equations of the Form *Ax* + *By* = *0*

Example 4

Graph the linear equation $-6x + 2y = 0$.

First, find the intercepts.

$$-6(0) + 2y = 0$$

$$\frac{2y}{2} = \frac{0}{2}$$

$$y = 0$$

The *y*-intercept is (0, 0).

$$-6x + 2(0) = 0$$

$$\frac{-6x}{-6} = \frac{0}{-6}$$

$$x = 0$$

The *x*-intercept is (0, 0).

Since the *x* and *y* intercepts are the same (the origin), choose a different value for *x* or *y*.

Graph Linear Equations of the Form *Ax* + *By* = *0*

Example 4 (concluded)

Graph the linear equation $-6x + 2y = 0$.

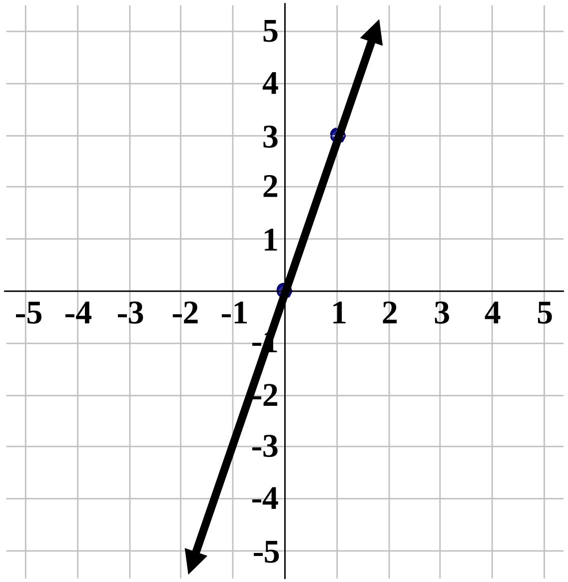
Let $x = 1$.

$$-6(1) + 2y = 0$$

$$\begin{array}{r} -6 + 2y = 0 \\ +6 \qquad\quad +6 \\ \hline 2y = 6 \end{array}$$

$$\frac{2y}{2} = \frac{6}{2}$$

$$y = 3$$



A second point is (1, 3).

Graph Linear Equations of the Form *Ax* + *By* = *0*

Line through the Origin

The graph of a linear equation of the form

$$Ax + By = 0$$

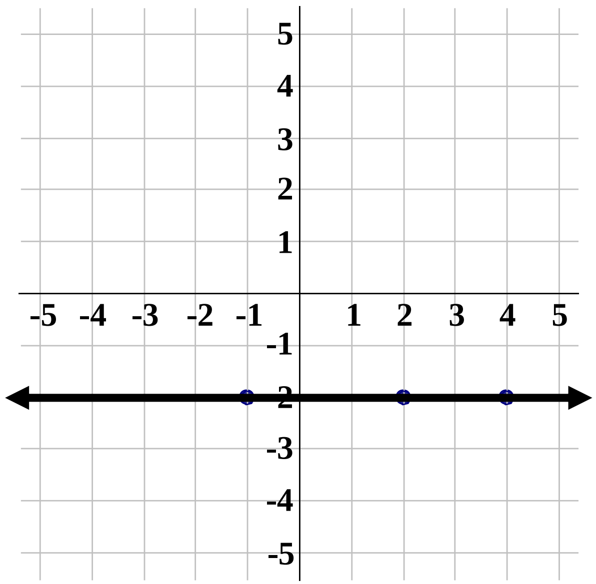
where *A* and *B* are nonzero real numbers, passes through the origin (0,0).

Graph Linear Equations of the Form *y* = *k* or *x* = *a*

Example 5

Graph $y = –2$.

The expanded version of this linear equation would be $0 \cdot x + y = –2$. Here, the *y*-coordinate is unaffected by the value of the *x*-coordinate. Whatever *x*-value we choose, the *y*-value will be –2. Thus, we could plot the points (–1, –2), (2,–2), (4,–2), etc.



Note that this is the graph of a **horizontal line** with *y*-intercept (0,–2).

Graph Linear Equations of the Form *y* = *k* or *x* = *a*

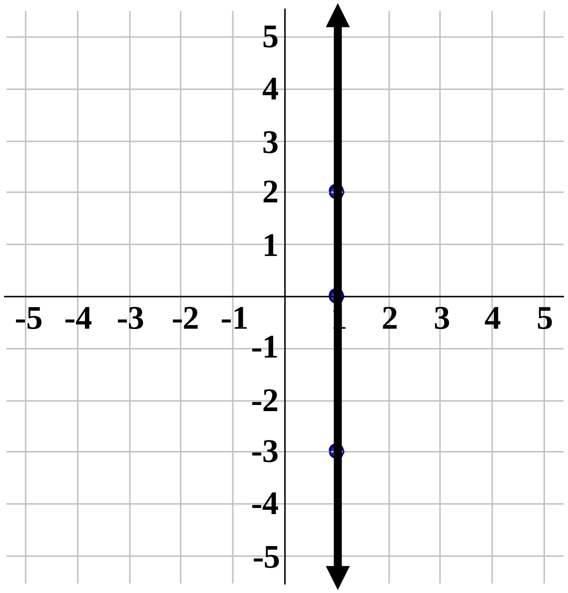
Example 6

Graph $x - 1 = 0$.

Add 1 to each side of the equation. $x = 1$.

The *x*-coordinate is unaffected by the value of the *y*-coordinate.

Thus, we could plot the points (1, –3), (1, 0), (1, 2), etc.



Note that this is the graph of a **vertical line** with no *y*-intercept.

Graph Linear Equations of the Form ***y* = *k* or *x* = *a***

Horizontal and Vertical Lines

The graph of the linear equation $\boldsymbol{y = k}$, where k is a real number, is the horizontal line with y-intercept $(0, k)$ and no x-intercept.

The graph of the linear equation $\boldsymbol{x = a}$, where a is a real number, is the vertical line with x-intercept $(a, 0)$ and no y-intercept.

Use a Linear Equation to Model Data

Example 7

Bob has owned and managed Bob's Bagels for the past 5 years and has kept track of his costs over that time. Based on his figures, Bob has determined that his total monthly costs can be modeled by $C = 0.75x + 2500$, where x is the number of bagels that Bob sells that month.

(a) Use Bob's cost equation to determine his costs if he sells 1000 bagels next month, 4000 bagels next month.

$C = 0.75(1000) + 2500$ $\qquad$ $C = 0.75(4000) + 2500$

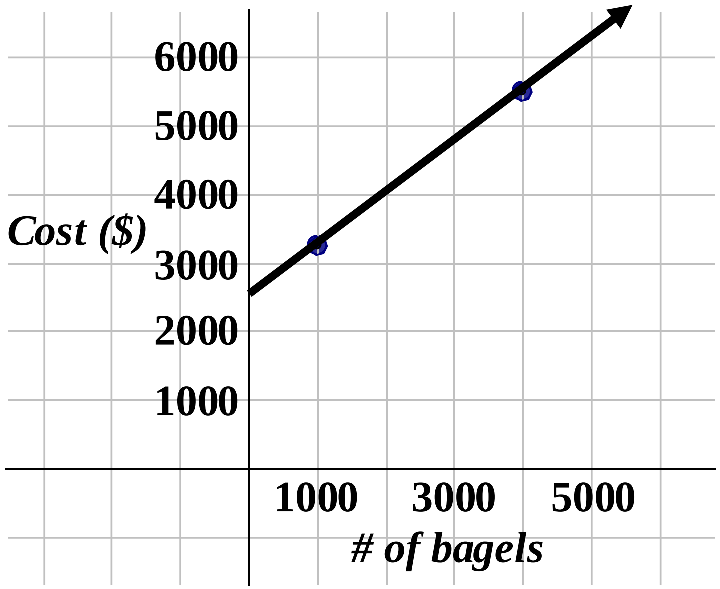
$C = \$3250$ $\qquad$ $C = \$5500$

Use a Linear Equation to Model Data

Example 7 (concluded)

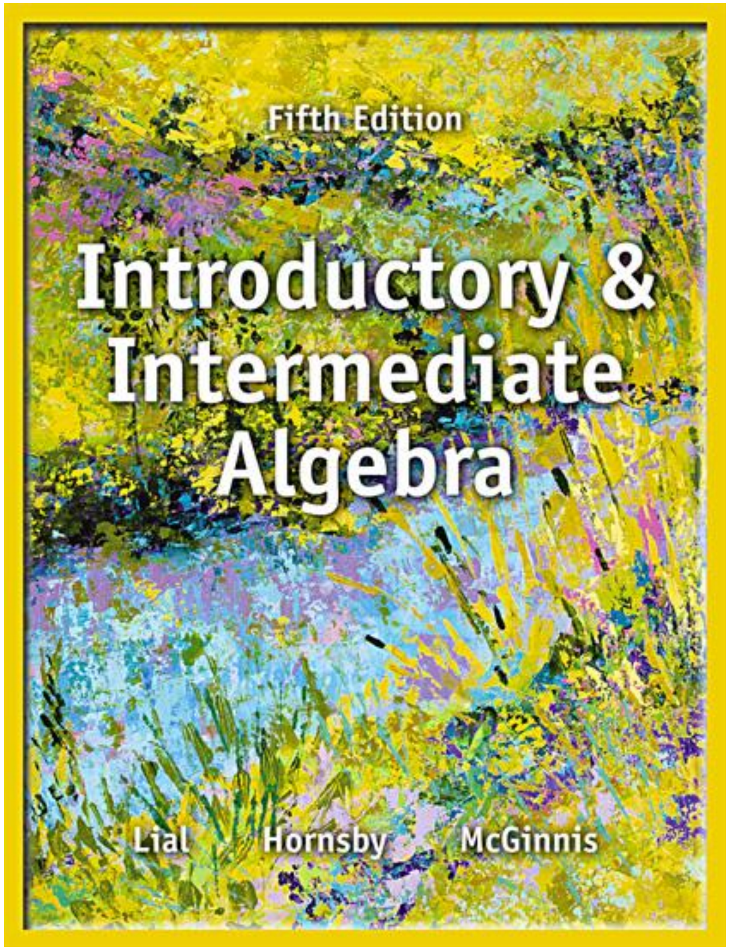
(b) Write the information from part (a) as two ordered pairs and use them to graph Bob's cost equation.

From part (a) we have (1000, 3250) and (4000, 5500).



Note that we did not extend the graph to the left beyond the vertical axis. That area would correspond to a negative number of bagels, which does not make sense.

3 Graphs of Linear Equations, and Inequalities, in Two Variables

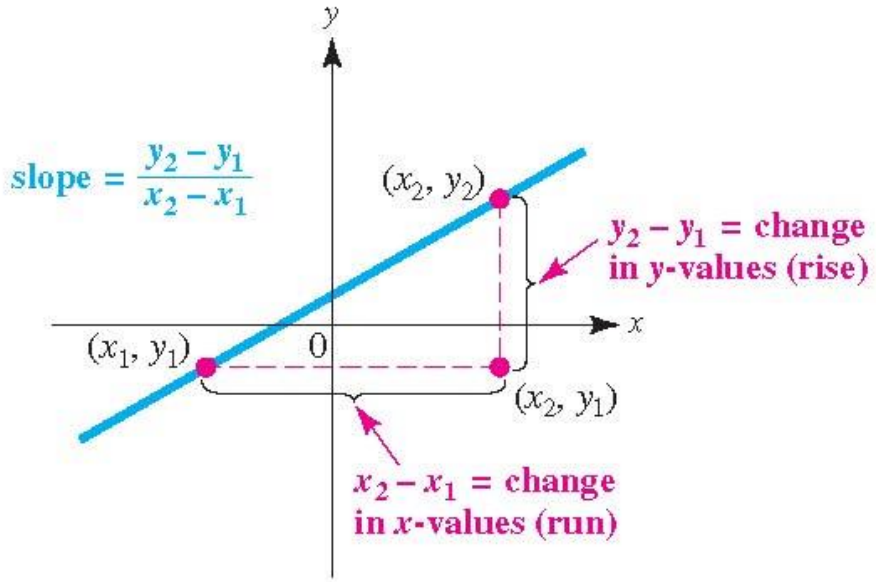


Objectives

1. Find the slope of a line given two points.
2. Find the slope from the equation of a line.
3. Use slope to determine whether two lines are parallel, perpendicular, or neither.
4. Solve problems involving average rate of change.

Find the Slope of a Line Given Two Points

Find the slope of the line through two nonspecific points (x_1, y_1) and (x_2, y_2).



Moving along the line from the point (x_1, y_1) to the point (x_2, y_2), we see that y changes by $y_2 - y_1$ units. This is the vertical change (**rise**). Similarly, x changes by $x_2 - x_1$ units, which is the horizontal change (**run**). The **slope** of the line is the ratio of $y_2 - y_1$ to $x_2 - x_1$.

Find the Slope of a Line Given Two Points

Slope Formula

The **slope *m*** of the line through the points (x_1, y_1) and (x_2, y_2) is

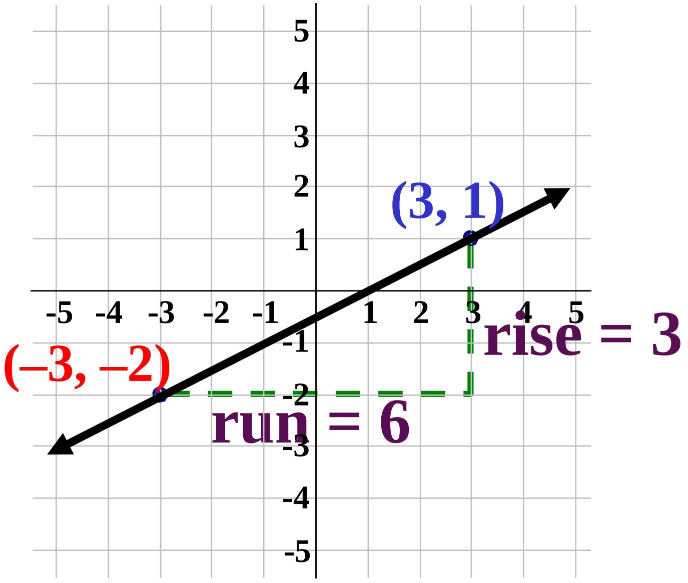
$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_1 \neq x_2).$$

Find the Slope of a Line Given Two Points

Example 2

Find the slope of each line.

(a) The line through (–3, –2) and (3, 1).



Use the slope formula. Let (–3, –2) be (x_1, y_1), and let (3, 1) be (x_2, y_2).

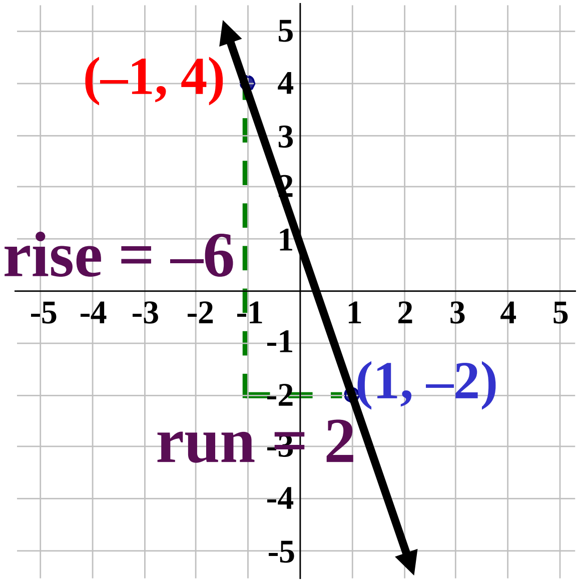
$$m = \frac{1-(-2)}{3-(-3)} = \frac{1+2}{3+3} = \frac{3}{6} = \frac{1}{2}$$

Find the Slope of a Line Given Two Points

Example 2 (concluded)

Find the slope of each line.

(b) The line through (–1, 4) and (1, –2).



Use the slope formula. Let (–1, 4) be (x_1, y_1), and let (1, –2) be (x_2, y_2).

$$m = \frac{-2-4}{1-(-1)} = \frac{-6}{1+1} = \frac{-6}{2} = -3$$

Find the Slope of a Line Given Two Points

CAUTION

It makes no difference which point is (x_1, y_1) or (x_2, y_2); however, be consistent. Start with the x- and y-values of one point (either one), and subtract the corresponding values of the other point.

Positive and Negative Slopes

A line with a positive slope rises (slants up) from left to right.

A line with a negative slope falls (slants down) from left to right.

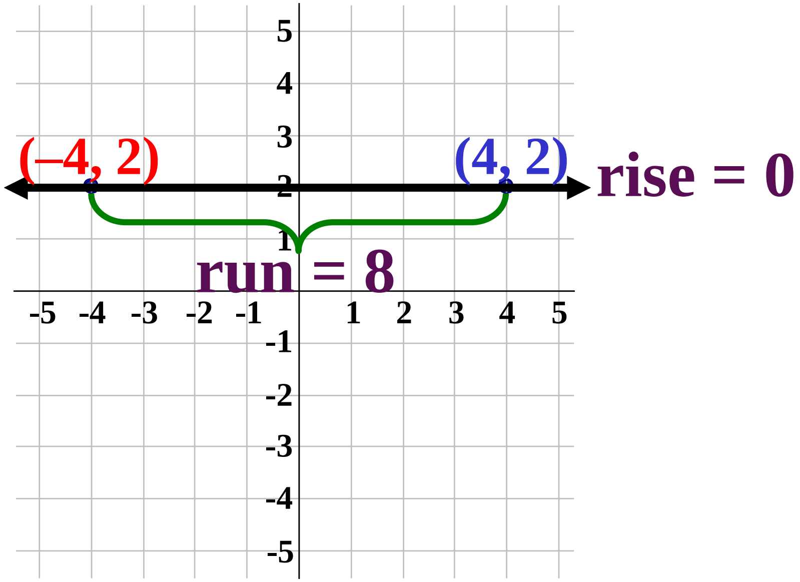
Find the Slope of a Line Given Two Points

Example 3

Find the slope of each line.

The line through (–4, 2) and (4, 2).

$$m = \frac{2-2}{4-(-4)} = \frac{0}{4+4} = \frac{0}{8} = \boxed{0}$$



Note that this is a horizontal line.

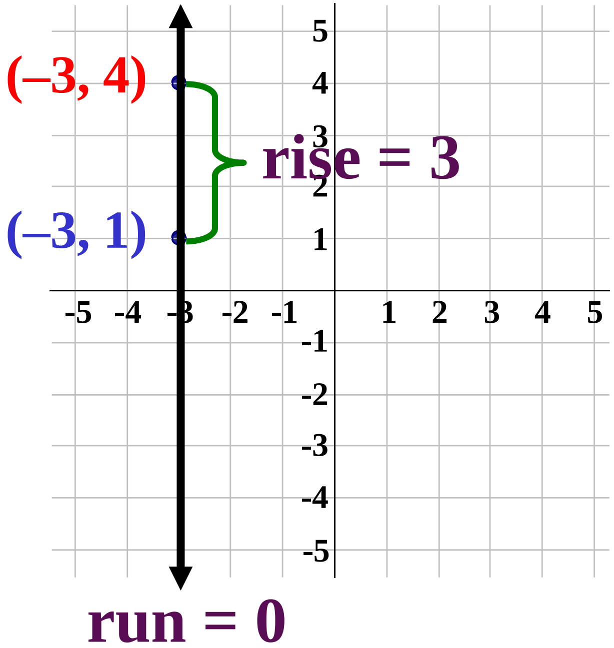
Find the Slope of a Line Given Two Points

Example 4

Find the slope of each line.

The line through (–3, 4) and (–3, 1).

$$m = \frac{4-1}{-3-(-3)} = \frac{3}{-3+3} = \frac{3}{0} \quad \text{undef}$$



Note that this is a vertical line.

Find the Slope of a Line Given Two Points

Slopes of Horizontal and Vertical Lines

Horizontal lines, which have equations of the form $y = k$, have **slope 0**.

Vertical lines, which have equations of the form $x = k$, have **undefined slope**.

Find Slope From the Equation of a Line

Finding the Slope of a Line from Its Equation

Step 1 Solve the equation for y.

Step 2 The slope is given by the coefficient of x.

Finding the Slope from an Equation

Example 5

Find the slope of the graph $5x - 6y = 18$.

Solve the equation for y.

The graph of $x = 2$ is a vertical line.

$$5x - 6y = 18$$

$$-6y = -5x + 18$$

$$\frac{-6y}{-6} = \frac{-5x}{-6} + \frac{18}{-6}$$

$$y = \frac{5}{6}x - 3$$

The slope is the coefficient of x, so the slope is 5/6.

Use Slope to Determine if Two Lines Parallel or Perpendicular

Slopes of Parallel and Perpendicular Lines

Two lines with the same slope are parallel.

Two lines whose slopes have a product of –1 are perpendicular.

Use Slope to Determine if Two Lines Parallel or Perpendicular

Example 6

Decide whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

(a) $x - 2y = 4$ and $6x + 3y = 9$

Find the slope of each line by first solving each equation for y.

$$\begin{array}{rcl} x - 2y &=& 4 \\ -x & & -x \\ \hline -2y &=& -x + 4 \end{array}$$

$$\frac{-2y}{-2} = \frac{-x}{-2} + \frac{4}{-2}$$

$$y = \tfrac{1}{2}x - 2$$

Slope is ½.

$$\begin{array}{rcl} 6x + 3y &=& 9 \\ -6x & & -6x \\ \hline 3y &=& -6x + 9 \end{array}$$

$$\frac{3y}{3} = \frac{-6x}{3} + \frac{9}{3}$$

$$y = -2x + 3$$

Slope is –2.

Use Slope to Determine if Two Lines Parallel or Perpendicular

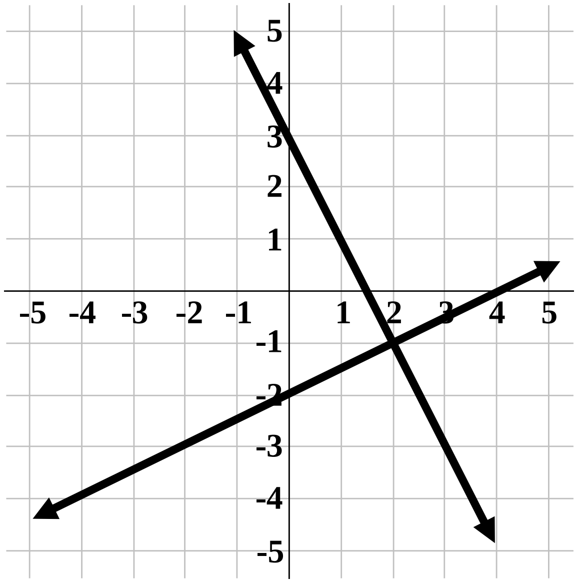
Example 6 (continued)

Decide whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

Because the slopes are not equal, the lines are not parallel. To see if the lines are perpendicular, find the product of the slopes.

$$\tfrac{1}{2} \cdot -2 = -1$$

The two lines are perpendicular because the product of their slopes is –1.



Use Slope to Determine if Two Lines Parallel or Perpendicular

Example 6 (continued)

Decide whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

(b) $8x - 2y = 4$ and $5x + y = -3$

Find the slope of each line by first solving each equation for *y*.

$$\begin{aligned} 8x - 2y &= 4 \\ -8x \qquad &\quad -8x \\ \hline -2y &= -8x + 4 \\ \frac{-2y}{-2} &= \frac{-8x}{-2} + \frac{4}{-2} \\ y &= 4x - 2 \end{aligned}$$

Slope is 4.

$$\begin{aligned} 5x + y &= -3 \\ -5x \qquad &\quad -5x \\ \hline y &= -5x - 3 \end{aligned}$$

Slope is –5.

Use Slope to Determine if Two Lines Parallel or Perpendicular

Example 6 (continued)

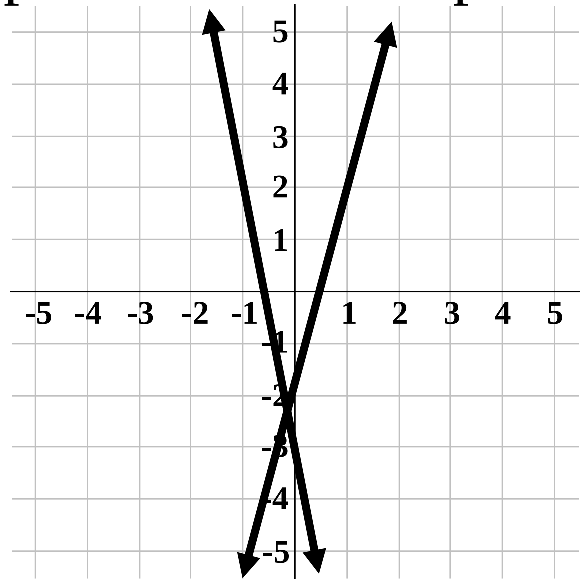
Decide whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

Because the slopes are not equal, the lines are not parallel. To see if the lines are perpendicular, find the product of the slopes.

$$4 \cdot (-5) = -20$$

The lines are not perpendicular because the product of their slopes is not –1.

The lines are neither parallel nor perpendicular.



Use Slope to Determine if Two Lines Parallel or Perpendicular

Example 6 (continued)

Decide whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

(c) $y - x = -3$ and $5x - 5y = -10$

Find the slope of each line by first solving each equation for y.

$$\begin{array}{r} y - x = -3 \\ +x \quad +x \\ \hline y = x - 3 \end{array}$$

Slope is 1.

$$\begin{array}{r} 5x - 5y = -10 \\ -5x \qquad\qquad -5x \\ \hline \dfrac{-5y}{-5} = \dfrac{-5x}{-5} - \dfrac{10}{-5} \end{array}$$

$$y = x + 2$$

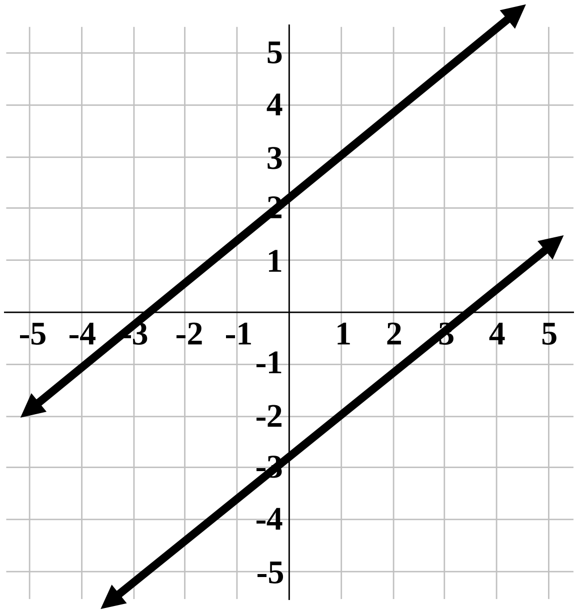
Slope is 1.

Use Slope to Determine if Two Lines Parallel or Perpendicular

Example 6 (concluded)

Decide whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

The slopes are equal, so the lines are parallel.



Interpreting Slope as Average Rate of Change

Example 8

Cindy purchased a new car in 2006 for $18,000. In 2011, the car had a value of $7500. At what rate is the car's value changing with respect to time?

To determine the average rate of change, we need two pairs of data. If x = 2006, then y = 18,000 and if x = 2011, then y = 7500.

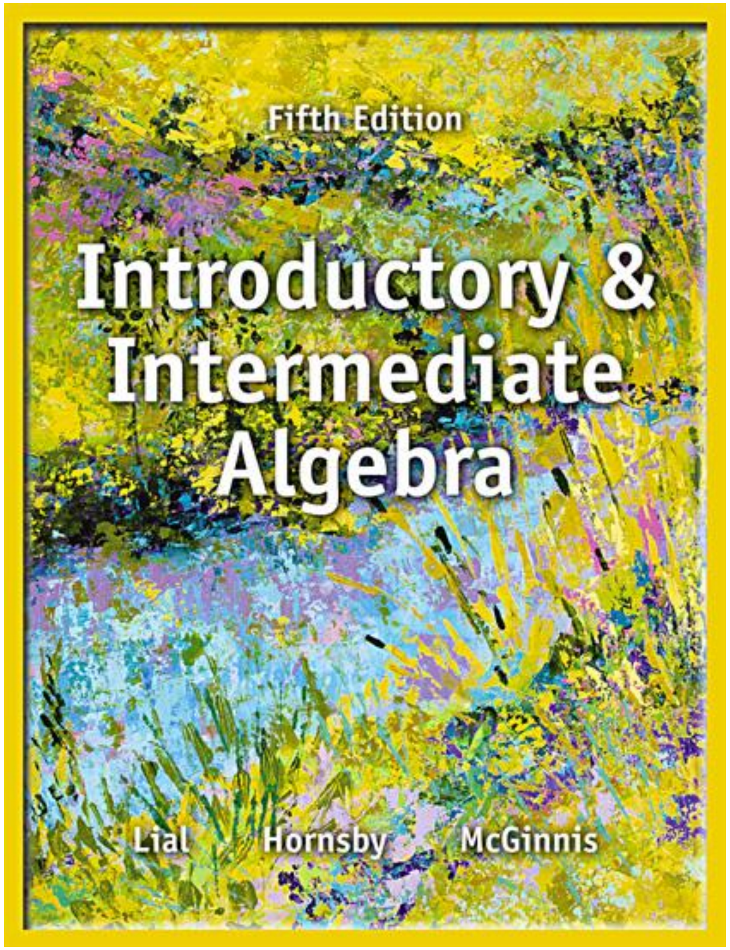
$$\text{average rate of change} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{7500 - 18{,}000}{2011 - 2006} = -\frac{10{,}500}{5}$$

$$= -2100$$

This means the car decreased in value by $2100 each year from 2006 to 2011.

3 Graphs of Linear Equations, and Inequalities, in Two Variables



3.4 Writing and Graphing Equations of Lines

Objectives

1. Use the slope-intercept form of the equation of a line.
2. Graph a line given its slope and a point on the line.
3. Write an equation of a line given its slope and any point on the line.
4. Write an equation of a line given two points on the line.
5. Write equations of horizontal and vertical lines.
6. Write an equation of a line parallel or perpendicular to a given line.
7. Write an equation of a line that models real data.

Use the Slope-Intercept Form

Slope-Intercept Form

The slope-intercept form of the equation of a line with slope m and y-intercept $(0, b)$ is

$$y = \underbrace{m}_{\text{Slope}}x + \underbrace{b}_{(0,\,b)\text{ is the }y\text{-intercept.}}.$$

Use the Slope-Intercept Form

Example 2

Write an equation of the line with slope –4 and y-intercept (0, 9).

Here $m = -4$ and $b = 9$, so an equation is

$$y = mx + b$$

$$y = -4x + 9.$$

Graph a Line Given Its Slope and a Point on the Line

Graphing a Line by Using the Slope and y-intercept

Step 1 Write the equation in slope-intercept form, if necessary, by solving for *y.*

Step 2 Identify the y-intercept. Graph the point $(0, b)$.

Step 3 Identify slope *m* of the line. Use the geometric interpretation of slope (“rise over run”) to find another point on the graph by counting from the y-intercept.

Step 4 Join the two points with a line to obtain the graph.

Graph a Line Given Its Slope and a Point on the Line

Example 3

Graph $y = \frac{2}{3}x - 1$

Step 1 The equation is in slope-intercept form.

$m = 2/3$

y-intercept = (0, –1)

Step 2 The y-intercept is (0, –1) . Graph this point.

Graph a Line Given Its Slope and a Point on the Line

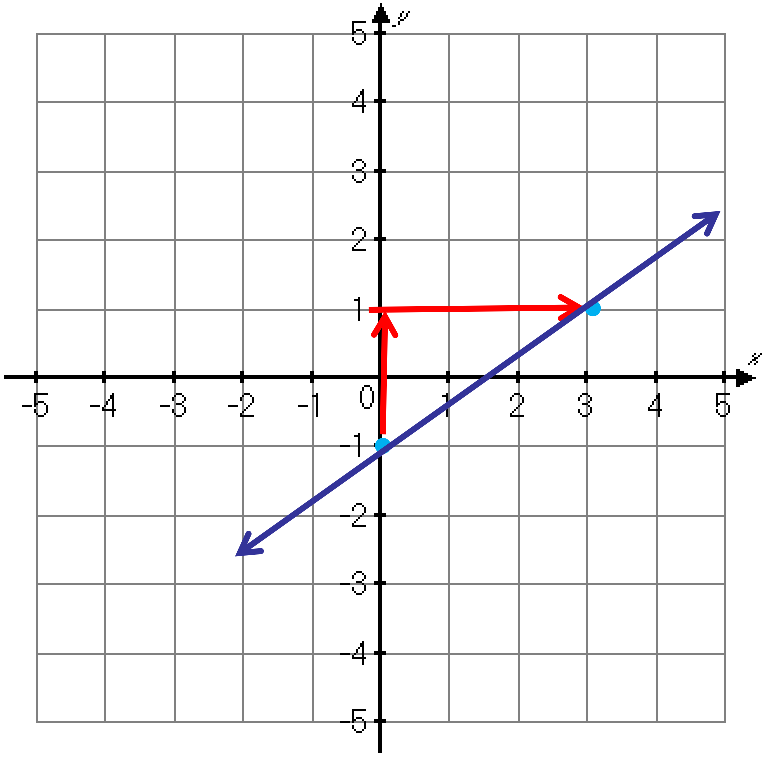
Example 3 (concluded)

Graph $y = \frac{2}{3}x - 1$

Step 3 The slope is 2/3.

$$\frac{\text{rise}}{\text{run}} = \frac{2}{3}$$

Count up 2 units and to the right 3 units.



Step 4 Connect the two points.

Write a Linear Equation Given Slope and a Point

Point-Slope Form

The **point-slope form** of the equation of a line with slope m passing through the point (x_1, y_1) is

$$\boldsymbol{y - y_1 = m(x - x_1).}$$

Write a Linear Equation Given Slope and a Point

Example 6

Write an equation for the line through (3, 2), with slope ⅓. Give the final answer in slope-intercept form.

Use point-slope form. Here $m = \frac{1}{3}$, $x_1 = 3$, and $y_1 = 2$.

$$(y - 2) = \tfrac{1}{3}(x - 3)$$

$$\begin{array}{r} y - 2 = \tfrac{1}{3}x - 1 \\ +2 \qquad\quad +2 \\ \hline \boxed{y = \tfrac{1}{3}x + 1} \end{array}$$

Write a Linear Equation Given Two Points

Example 7 Write an equation of the line through the points (–2, 2) and (4, –1). Give the final answer in slope-intercept form.

First, find the slope of the line, using the slope formula.

$$m = \frac{-1-2}{4-(-2)} = \frac{-3}{4+2} = \frac{-3}{6} = -\frac{1}{2}$$

Now use either (–2, 2) or (4, –1) and the point-slope form. Using (4, –1) gives

$$y - (-1) = -\tfrac{1}{2}(x - 4)$$

$$\begin{aligned} y + 1 &= -\tfrac{1}{2}x + 2 \\ -1 &\qquad\quad -1 \\ \hline y &= -\tfrac{1}{2}x + 1 \end{aligned}$$

Equations of Horizontal and Vertical Lines

Equations of Horizontal and Vertical Lines

The horizontal line through the point (a, b) has equation $\boldsymbol{y = b}$.

The vertical line through the point (a, b) has equation $\boldsymbol{x = a}$.

Writing Equations of Horizontal and Vertical Lines

Example 8

Write an equation of the line passing through the point (–3, 3) that satisfies the given condition.

a. The line has slope 0.

Since the slope is 0, this is a horizontal line.

The equation is $y = 3$.

b. The line has undefined slope.

This is a vertical line.

The equation is $x = –3$.

Writing Equations of Parallel or Perpendicular Lines

Example 9

Write an equation in slope-intercept form of the line passing through the point (4, –7) that is parallel to the graph of $x + 2y = 6$.

Find the slope of the given line by solving for *y*.

$$x + 2y = 6$$
$$2y = -x + 6$$
$$y = -\frac{1}{2}x + 3$$

A line parallel will have the same slope.

$$y - y_1 = m(x - x_1)$$
$$y - (-7) = -\frac{1}{2}(x - 4)$$

Writing Equations of Parallel or Perpendicular Lines

Continued.

Write an equation in slope-intercept form of the line passing through the point (4, –7) that satisfies the given condition.

a. The line is parallel to the graph of $x + 2y = 6$.

$$y-(-7)=-\frac{1}{2}(x-4)$$

$$-2(y+7)=x-4$$

$$2y-14=x-4$$

$$2y=x+10$$

$$y=\frac{1}{2}x+5$$

Writing Equations of Parallel or Perpendicular Lines

Example 9b

Write an equation in slope-intercept form of the line passing through the point (0, 0) that is perpendicular to the graph of $2x + 3y = 7$.

Find the slope of the given line by solving for *y*.

$$x + 3y = 7$$

$$3y = -2x + 7$$

$$y = -\frac{2}{3}x + \frac{7}{3}$$

A line perpendicular will have a slope of 3/2.

Writing Equations of Parallel or Perpendicular Lines

Continued.

Write an equation in slope-intercept form of the line passing through the point (0, 0) that is perpendicular to the graph of $2x + 3y = 7$.

Use the point (0, 0) and the point-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{3}{2}(x - 0)$$

$$y = \frac{3}{2}x$$

Write a Linear Equation Given Two Points

Forms of Linear Equations

$x = a$	**Vertical line** Slope is undefined; x-intercept is $(a, 0)$.
$y = k$	**Horizontal line** Slope is 0; y-intercept is $(0, k)$.
$y = mx + b$	**Slope-Intercept Form** Slope is m; y-intercept is $(0, b)$.

Write a Linear Equation Given Two Points

Forms of Linear Equations (cont.)

$y - y_1 = m(x - x_1)$ **Point-slope form**

Slope is m; line passes through (x_1, y_1).

$Ax + By = C$ **Standard form**

Slope is $-\dfrac{A}{B}$; x-intercept is $\left(\dfrac{C}{A}, 0\right)$;

y-intercept is $\left(0, \dfrac{C}{B}\right)$.

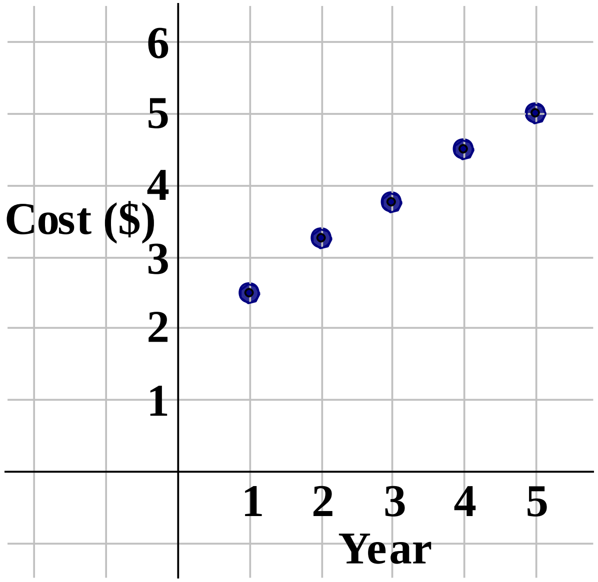
Write a Linear Equation that Fits a Data Set

Example 10

The table below gives average annual cost (in dollars) for a medium sized cup of regular coffee at Muckity-Muck's over a 5-year period. Plot the data and find an equation that approximates it.

Year	Cost ($)
1	2.50
2	3.25
3	3.75
4	4.50
5	5.00

Letting y = Cost in year x, we get



Write a Linear Equation that Fits a Data Set

Example 10 (concluded)

The points appear to lie approximately in a straight line. We can use two of the data pairs and the point-slope form of a line to find an equation. Say we choose (1, 2.50) and (5, 5.00).

$$m = \frac{5-2.5}{5-1} = \frac{2.5}{4} = 0.625$$

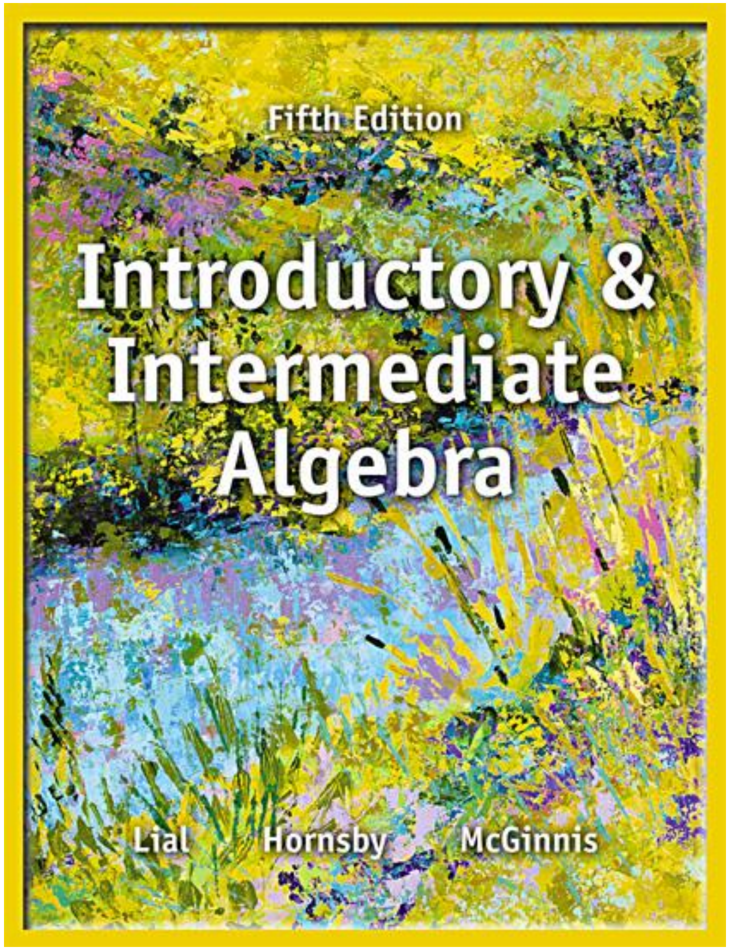
Then we will use (5,5) with $m = 0.625$ in the point-slope form.

$$y - 5 = 0.625(x - 5)$$

$$\begin{array}{r} y - 5 = 0.625x - 3.125 \\ +5 \qquad\qquad\qquad\quad +5 \\ \hline y = 0.625x + 1.875 \end{array}$$

Note: In Example 8, if we had chosen two different data points, we would have gotten a slightly different equation.

3 Graphs of Linear Equations, and Inequalities, in Two Variables



3.5 Graphing Linear Inequalities in Two Variables

Objectives

1. Graph linear inequalities in two variables.
2. Graph an inequality with a boundary line through the origin.

Graph Linear Inequalities in Two Variables

In this section we learn to graph linear inequalities in two variables on a rectangular coordinate system.

Linear Inequality in Two Variables

An inequality that can be written as

$$Ax + By < C,\ Ax + By \le C,\ Ax + By > C, \text{ or } Ax + By \ge C,$$

where A, B, and C are real numbers and A and B are not both 0, is a **linear inequality in two variables.**

Graph Linear Inequalities

Example 1

Graph the inequality $2x + 3y \le 6$.

The inequality $2x + 3y \le 6$ means that

$$2x + 3y < 6 \quad \text{or} \quad 2x + 3y = 6.$$

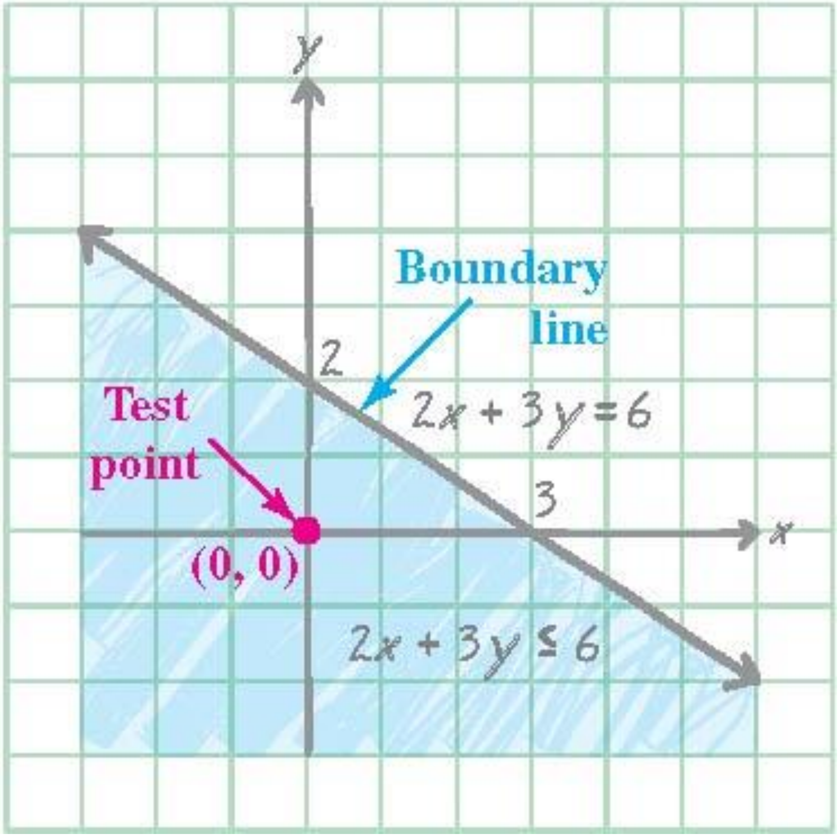
The graph of $2x + 3y = 6$ is a line. This **boundary line** divides the plane into two regions. The graph of the solutions of the inequality $2x + 3y < 6$ will include only *one* of these regions. We find the required region by checking a test point.

We choose any point *not* on the boundary line. Because $(0, 0)$ is easy to substitute, we often use it.

Graph Linear Inequalities

Example 1 (continued)

Graph the inequality $2x + 3y \le 6$.



Check (0, 0)

$$2x + 3y \le 6$$

$$2(0) + 3(0) \le 6$$

$$0 + 0 \le 6$$

$$0 \le 6 \qquad \text{True.}$$

Since the last statement is true, we shade the region that includes the test point (0, 0).

Graphing Linear Inequalities

Graphing a Linear Inequality

Step 1 **Graph the boundary**. Graph the line that is the boundary of the region. Draw a solid line if the inequality has ≤ or ≥; draw a dashed line if the inequality has < or >.

Step 2 **Shade the appropriate side**. Use any point not on the line as a test point. Substitute for x and y in the *inequality.* If a true statement results, shade the side containing the test point. If a false statement results, shade the other region.

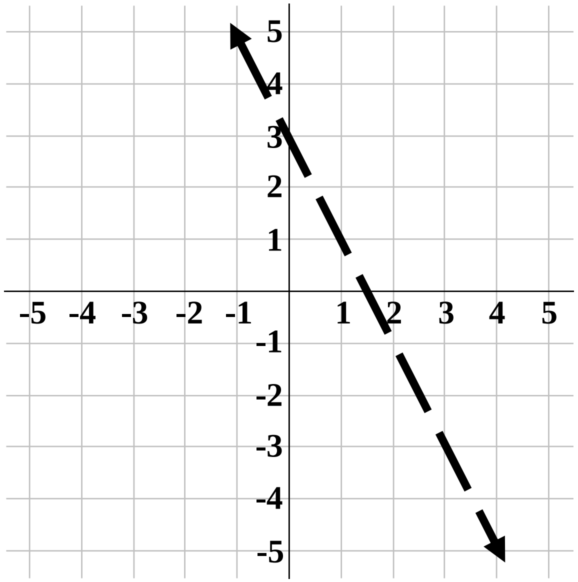
Graph Linear Inequalities

Example 2

Graph the inequality $2x + y > 3$.

Start by graphing the equation $2x + y = 3$.

$$\begin{array}{rcl} 2x + y &=& 3 \\ -2x & & -2x \\ \hline y &=& -2x + 3 \end{array}$$



Use a dashed line to show that the points on the line are <u>not</u> solutions of the inequality.

Graph Linear Inequalities

Example 2 (concluded)

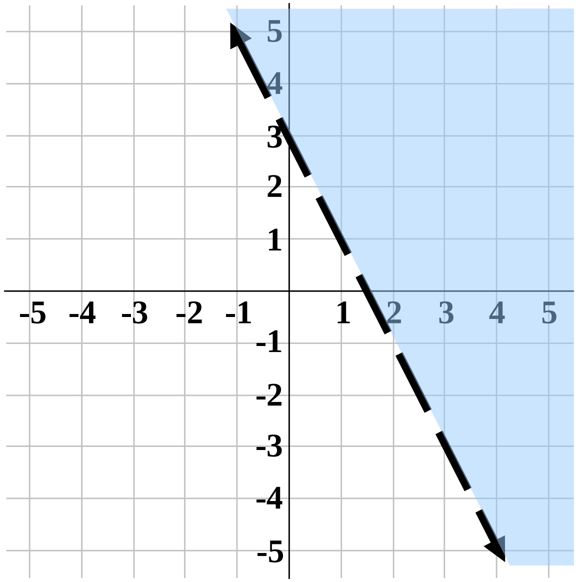
Graph the inequality $2x + y > 3$.

Choose any test point not on the line. Here, we choose (0,0)

$$2(0) + (0) > 3$$

$$0 > 3 \quad \text{False}$$

Because $0 > 3$ is false, shade the region not containing (0, 0).



Graph an Inequality with Boundary Through the Origin

Example 4

Graph the inequality $y < 3x$.

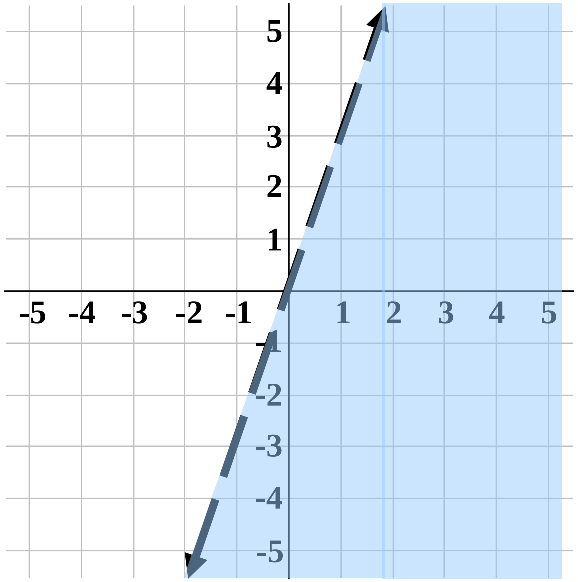
Begin by graphing $y = 3x$, using a dashed line.

Since (0, 0) is on the boundary line, choose a different test point. Here, we choose (1,1).

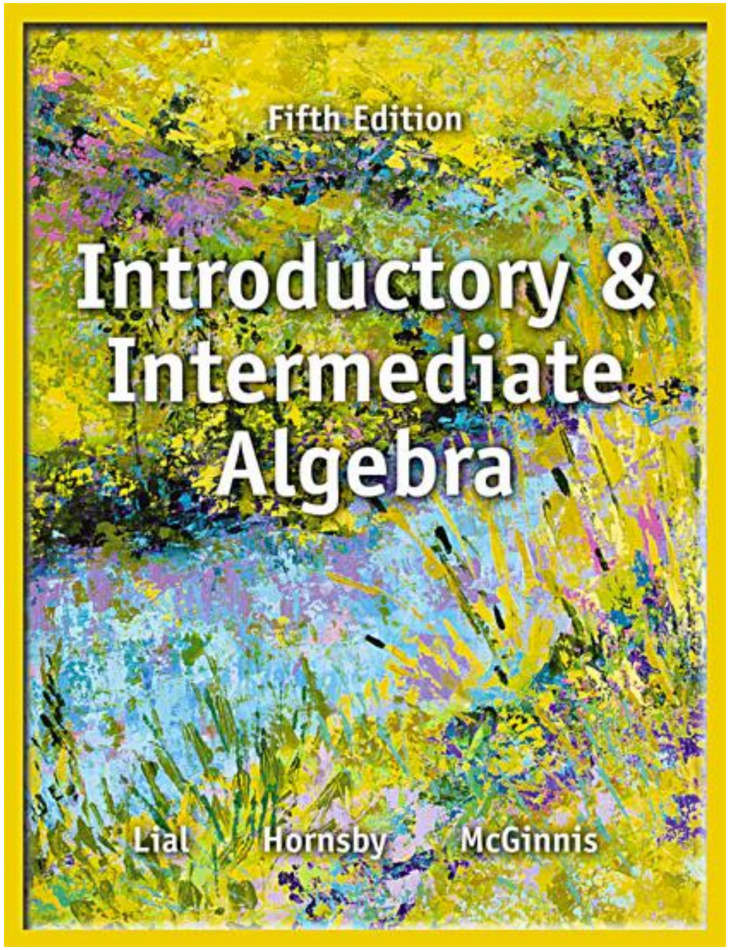
$1 < 3(1)$

$1 < 3$ True

Thus, we shade the region containing (1,1).



3 Graphs of Linear Equations, and Inequalities, in Two Variables



3.7 Function Notation and Linear Functions

Objectives

1. Use function notation.
2. Graph linear and constant functions.

Function Notation

When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say "y is a function of x" to emphasize that y *depends on* x. We use the notation

$$y = f(x),$$

called **function notation**, to express this and read $f(x)$, as "f of x".
The letter f stands for *function*. For example, if $y = 5x - 2$, we can name this function f and write

$$f(x) = 5x - 2.$$

Note that $f(x)$ ***is just another name for the dependent variable*** y**.**

Function Notation

CAUTION

The symbol $f(x)$ *does not* indicate "f times x," but represents the y-va
for the indicated x-value. As shown below, $f(3)$ is the y-value that corresponds to the x-value 3.

$$y = f(x) = 5x - 2$$

$$y = f(3) = 5(3) - 2 = 13$$

Example 2 Using Function Notation

Let $f(x) = x^2 + 2x - 1$. Find the following.

(a) $f(4)$

$f(x) = x^2 + 2x - 1$

$f(4) = 4^2 + 2 \cdot 4 - 1$ Replace x with 4.

$f(4) = 16 + 8 - 1$

$f(4) = 23$

Since $f(4) = 23$, the ordered pair (4, 23) belongs to f.

Continued. **Using Function Notation**

Let $f(x) = x^2 + 2x - 1$. Find the following.

(b) $f(w)$

$$f(x) = x^2 + 2x - 1$$

$$f(w) = w^2 + 2w - 1$$ Replace x with w.

The replacement of one variable with another is important in later courses.

Example 3 Using Function Notation

Let $g(x) = 5x + 6$. Find and simplify $g(n + 2)$.

$$g(x) = 5x + 6$$

$$g(n + 2) = 5(n + 2) + 6$$ Replace x with $n + 2$.

$$= 5n + 10 + 6$$

$$= 5n + 16$$

Example 4 **Evaluating Functions**

For each function, find $f(7)$.

(a) $f(x) = -x + 2$

$$f(x) = -x + 2$$

$$f(7) = -7 + 2$$ Replace x with 7.

$$= -5$$

Continued. **Evaluating Functions**

For each function, find $f(7)$.

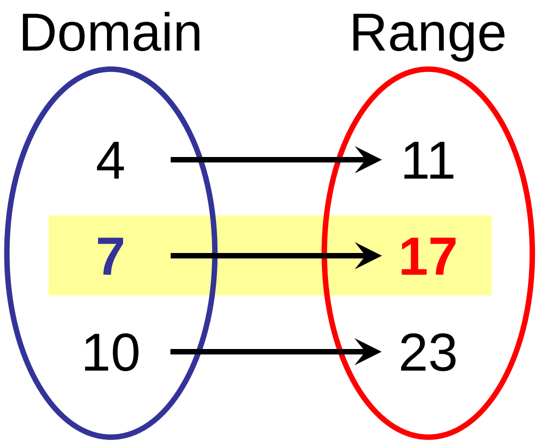
(b) $f = \{(–5, –9), (–1, –1), (3, 7), (7, 15), (11, 23)\}$

We want $f(7)$, the y-value of the ordered pair where $x = 7$. As indicated by the ordered pair **7**, **15**), when $x = 7$, $y = 15$, so $f(\mathbf{7}) = \mathbf{15}$.

Continued. **Evaluating Functions**

For each function, find f(7).

(c) f



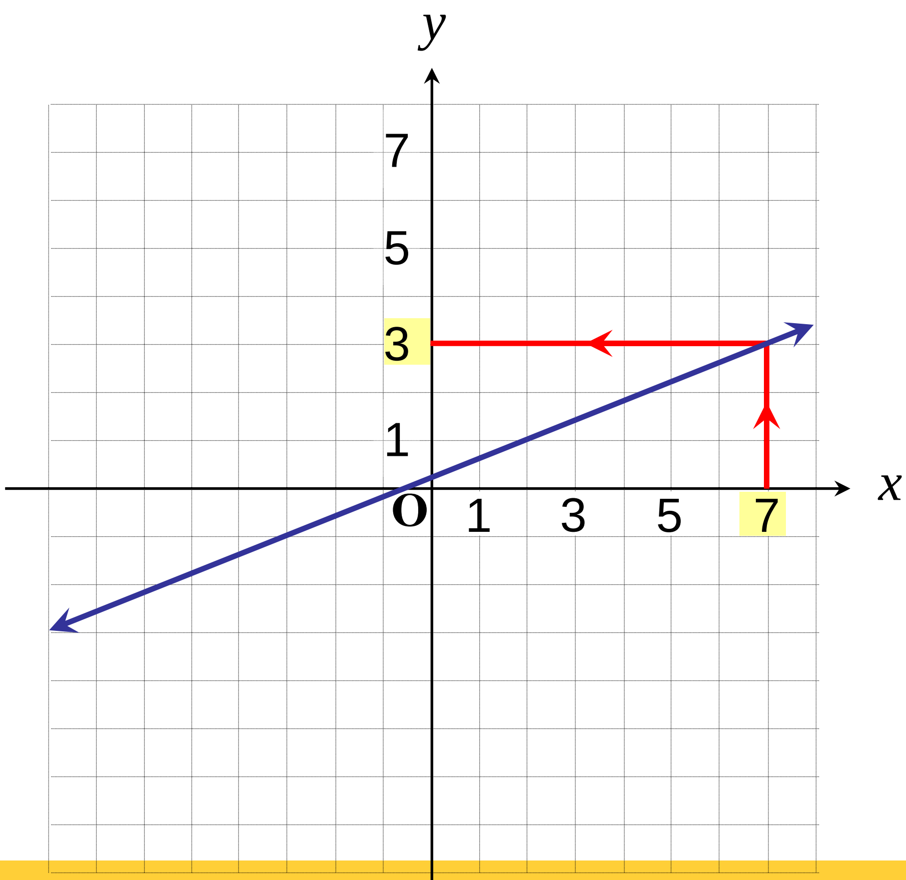
The domain element 7 is paired with 17 in the range, so $f(7) = 17$.

Finding Function Values from a Graph

Continued.

For each function, find $f(7)$.

(d)



To evaluate $f(7)$, find 7 on the x-axis. Then move up until the graph of f is reached. Moving horizontally to the y-axis gives 3 for the corresponding y-value. Thus, $f(\mathbf{7}) = \mathbf{3}$.

Writing an Equation Using Function Notation

Step 1 Solve the equation for y if it is not given in that form.

Step 2 Replace y with $f(x)$.

Writing Equations Using Function Notation

Example 6

Rewrite each equation using function notation. Then find $f(-3)$ and $f(n)$.

(a) $y = x^2 - 1$

This equation is already solved for y. Since $y = f(x)$,

$$f(x) = x^2 - 1.$$

To find $f(-3)$, let $x = -3$.

$$f(-3) = (-3)^2 - 1$$
$$= 9 - 1$$
$$= 8$$

To find $f(n)$, let $x = n$.

$$f(n) = n^2 - 1$$

Writing Equations Using Function Notation

Continued.

Rewrite each equation using function notation. Then find $f(-3)$ and $f(n)$.

(b) $x - 5y = 3$

First solve $x - 5y = 3$ for y. Then replace y with $f(x)$.

$$x - 5y = 3$$

$$x - 3 = 5y \quad \text{Add } 5y\text{; subtract 3.}$$

$$y = \frac{x-3}{5} \quad \text{so} \quad f(x) = \frac{1}{5}x - \frac{3}{5}$$

Now find $f(-3)$ and $f(n)$.

$$f(-3) = \frac{1}{5}(-3) - \frac{3}{5} = -\frac{6}{5} \qquad \text{Let } x = -3$$

$$f(n) = \frac{1}{5}(n) - \frac{3}{5} \qquad \text{Let } x = n$$

Linear Function

A function that can be defined by

$$f(x) = ax + b,$$

for real numbers a and b is a **linear function.** The value *of a* is the slope of *m* of the graph of the function. The domain of a linear function, unless specified otherwise, is $(-\infty, \infty)$.

Linear Function

A linear function defined by $\boldsymbol{f(x) = b}$ (whose graph is a horizontal line) is sometimes called a **constant function.** The domain of any linear function is $(-\infty, \infty)$. The range of a nonconstant linear function is $(-\infty, \infty)$, while the range of the constant function defined by $f(x) = b$ is $\{b\}$.

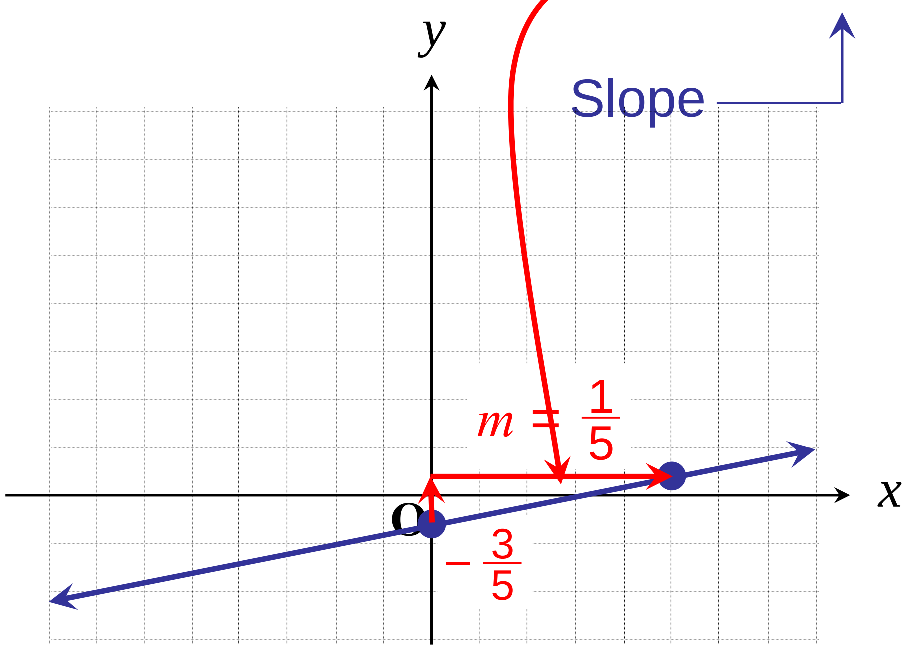
Linear Function

Recall that m is the slope of the line and $(0, b)$ is the y-*intercept*. In a previous example, we wrote $x - 5y = 3$ as the linear function defined by

$$f(x) = \frac{1}{5}x - \frac{3}{5}$$

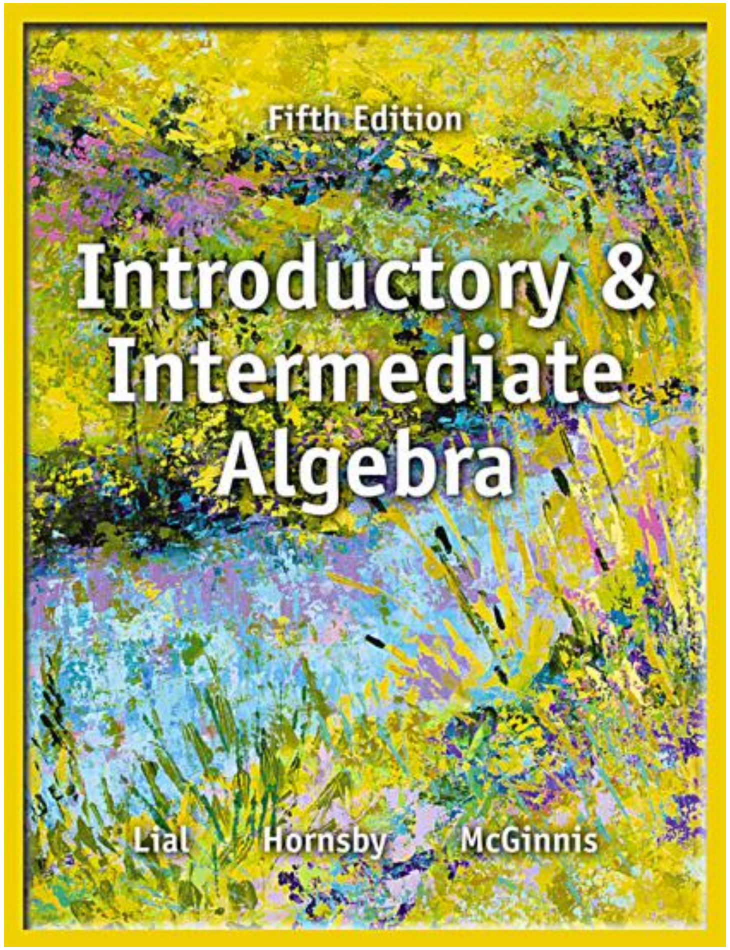
Slope

y-intercept is $(0, -\frac{3}{5})$.



To graph this function, plot the y-*intercept* and use the definition of slope as $\frac{\text{rise}}{\text{run}}$ to find a second point on the line. Draw a straight line through these points to obtain the graph.

3 Graphs of Linear Equations, and Inequalities, in Two Variables



3.6 Introduction to Relations and Functions

Objectives

1. Distinguish between independent and dependent variables.
2. Define and identify relations and functions.
3. Find domain and range.
4. Identify functions defined by graphs and equations.

Independent and Dependent Variables

We often describe one quantity in terms of another. We can indicate the relationship between these quantities by writing ordered pairs in which the first number is used to arrive at the second number. Here are some examples.

(5, $11)

5 gallons of gasoline

will cost $11. The total cost depends on the number of gallons purchased.

(8, $17.60)

8 gallons of gasoline

will cost $17.60. Again, the total cost depends on the number of gallons purchased.

Independent and Dependent Variables

We often describe one quantity in terms of another. We can indicate the relationship between these quantities by writing ordered pairs in which the first number is used to arrive at the second number. Here are some examples.

(the number of gallons, the total cost)

depends on

Independent and Dependent Variables

We often describe one quantity in terms of another. We can indicate the relationship between these quantities by writing ordered pairs in which the first number is used to arrive at the second number. Here are some examples.

(10, $150)

Working for 10 hours, you will earn $150. The total gross pay depends on the number of hours worked.

(15, $225)

Working for 15 hours, you will earn $225. The total gross pay depends on the number of hours worked.

Independent and Dependent Variables

We often describe one quantity in terms of another. We can indicate the relationship between these quantities by writing ordered pairs in which the first number is used to arrive at the second number. Here are some examples.

(the number of hours worked, the total gross pay)

depends on

Independent and Dependent Variables

We often describe one quantity in terms of another. We can indicate the relationship between these quantities by writing ordered pairs in which the first number is used to arrive at the second number. Here are some examples.

Generalizing, if the value of the variable y depends on the value of the variable x, then y is called the **dependent variable** and x is the **independent variable.**

Independent variable

$$(x, y)$$

Dependent variable

Define and identify relations and functions.

Relation

A **relation** is any set of ordered pairs.

A special kind of relation, called a *function*, is very important in mathematics and its applications.

Function

A **function** is a relation in which, for each value of the first component of the ordered pairs, there is *exactly one value* of the second component.

Determining Whether Relations Are Functions

Example 1

Tell whether each relation defines a function.

L = { (2, 3), (–5, 8), (4, 10) }

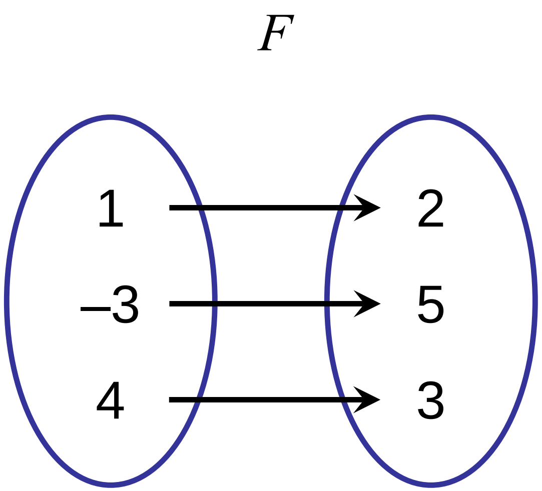
M = { (–3, 0), (–1, 4), (1, 7), (3, 7) }

N = { (6, 2), (–4, 4), (6, 5) }

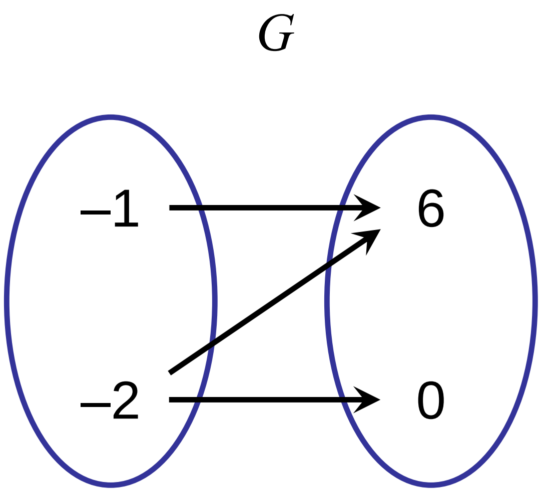
Relations L and M are functions, because for each different x-value there is exactly one y-value.

In relation N, the first and third ordered pairs have the *same* x-value paired with *two different* y-values (6 is paired with both 2 and 5), so N is a relation but not a function. ***In a function, no two ordered pairs can have the same first component and different second components.***

Mapping Relations



F is a function.

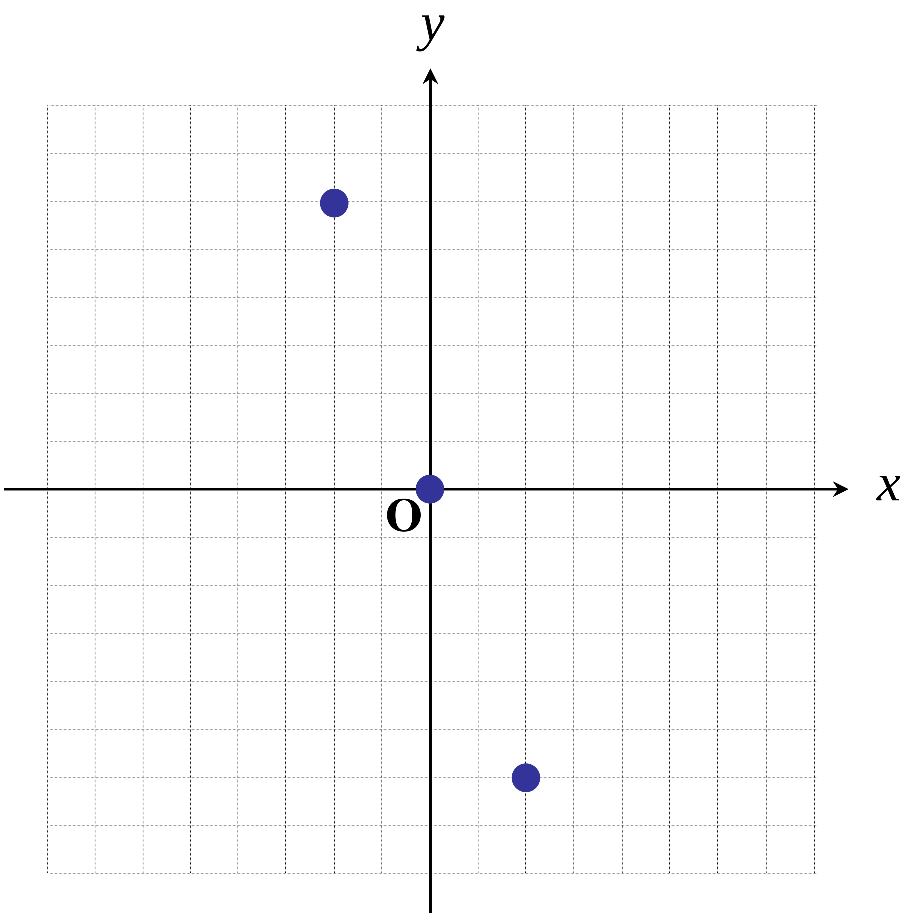


G is not a function.

Tables and Graphs

x	y
–2	6
0	0
2	–6

Table of the function, F



Graph of the function, F

Using an Equation to Define a Relation or Function

Relations and functions can also be described using rules. Usually, the rule is given as an equation. For example, from the previous slide, the chart and graph could be described using the following equation.

$$y = -3x$$

Dependent variable Independent variable

An equation is the most efficient way to define a relation or function.

Functions

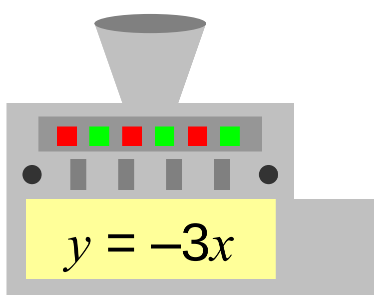
NOTE

Another way to think of a function relationship is to think of the independent variable as an input and the dependent variable as an output. This is illustrated by the input-output (function) machine (below) for the function defined by $y = -3x$.

(Input x)	(Output y)
2	**–6**
–5	**15**
4	**–12**

–25

(Input x)



(Output y)

Domain and Range

In a relation, the set of all values of the independent variable (x) is the **domain**. The set of all values of the dependent variable (y) is the **range.**

Finding Domains and Ranges of Relations

Example 2

Give the domain and range of each relation. Tell whether the relation defines a function.

(a) { (**3**, **–8**), (**5**, **9**), (**5**, **11**), (**8**, **15**) }

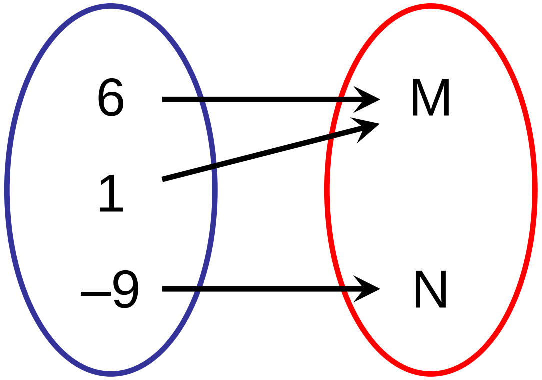
The domain, the set of x-values, is {**3**, **5**, **8**}; the range, the set of y-values, is {**–8**, **9**, **11**, **15**}. This relation is not a function because the same x-value 5 is paired with two different y-values, 9 and 11.

Finding Domains and Ranges of Relations

Continued.

Give the domain and range of each relation. Tell whether the relation defines a function.

(b)



The domain of this relation is {6, 1, –9}. The range is {M, N}.

This mapping defines a function – each x-value corresponds to exactly one y-value.

Finding Domains and Ranges of Relations

Continued.

Give the domain and range of each relation. Tell whether the relation defines a function.

(c)

x	y
–2	3
1	3
2	3

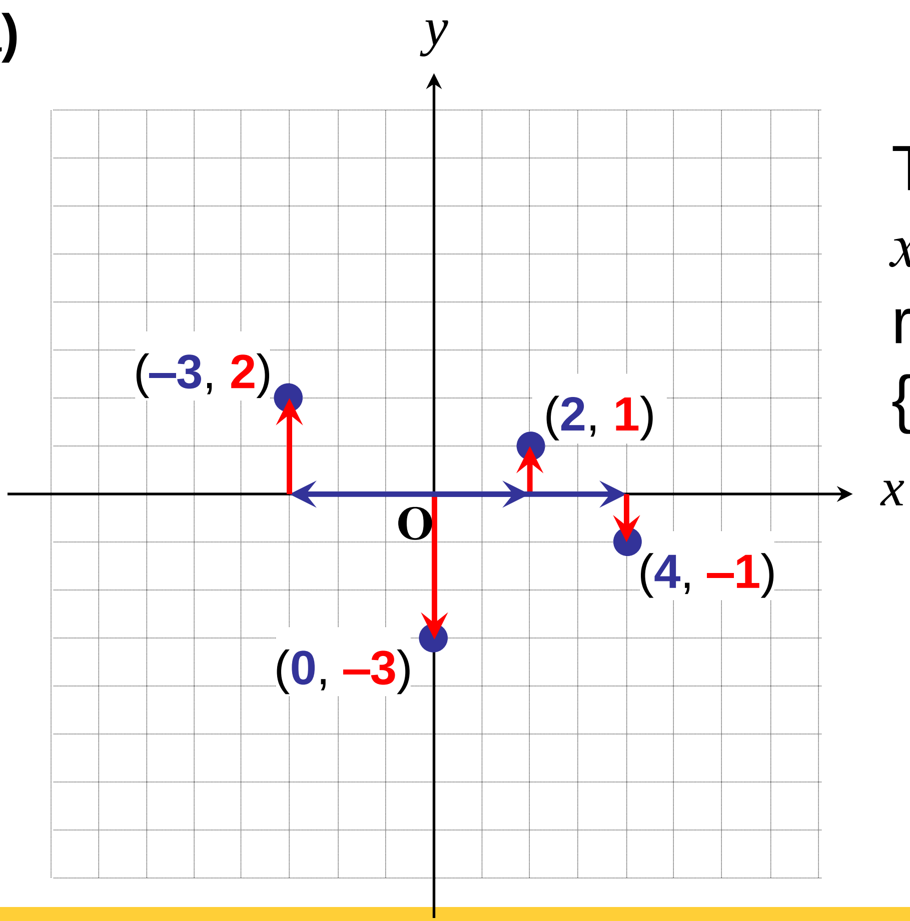
This is a table of ordered pairs, so the domain is the set of x-values, {–2, 1, 2}, and the range is the set of y-values, {3}. The table defines a function because each different x-value corresponds to exactly one y-value (even though it is the same y-value).

Finding Domains and Ranges from Graphs

Example 3

Give the domain and range of each relation.

(a)



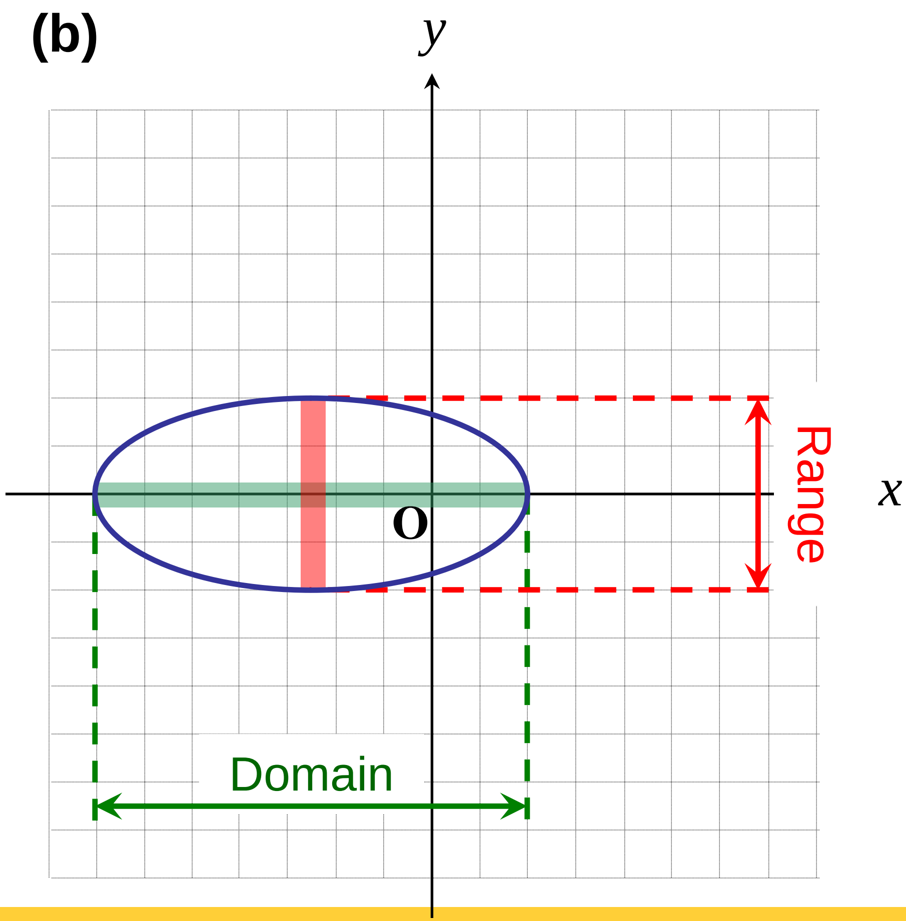
The domain is the set of x-values, {**–3**, **0**, **2** , **4**}. The range, the set of y-values, is {**–3**, **–1**, **1**, **2**}.

Finding Domains and Ranges from Graphs

Continued.

Give the domain and range of each relation.

(b)



The x-values of the points on the graph include all numbers between –7 and 2, inclusive. The y-values include all numbers between –2 and 2, inclusive. Using interval notation,

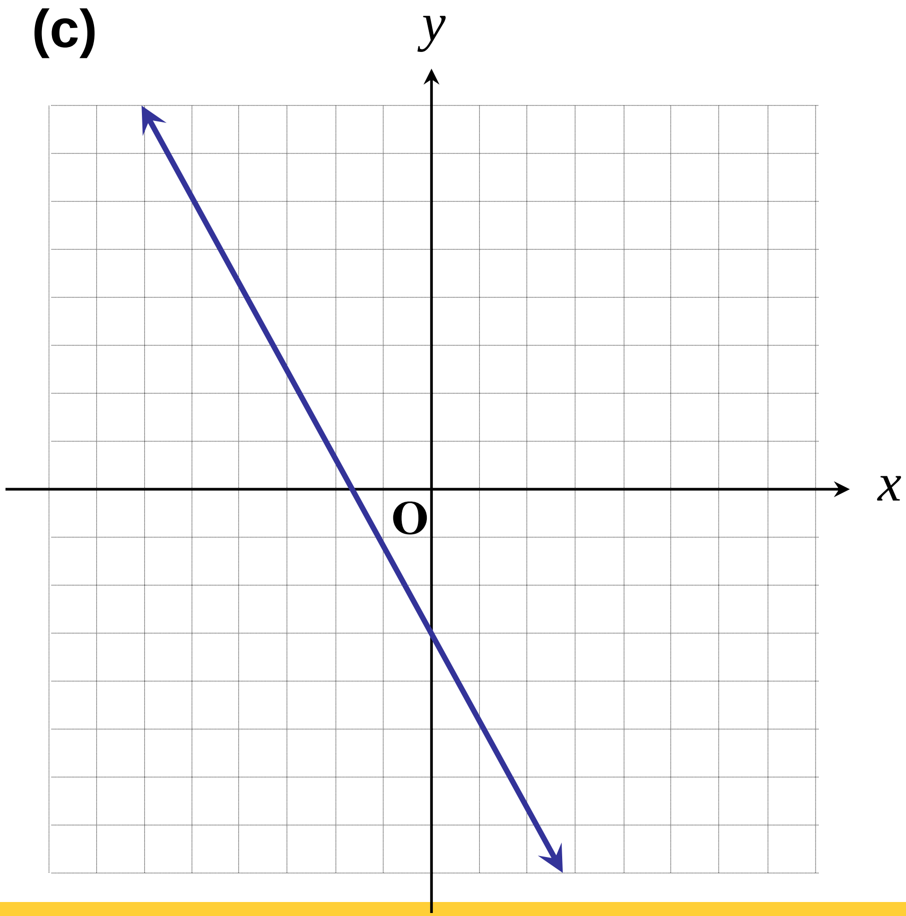
the domain is [**–7**, **2**];
the range is [**–2**, **2**].

Finding Domains and Ranges from Graphs

Continued.

Give the domain and range of each relation.

(c)



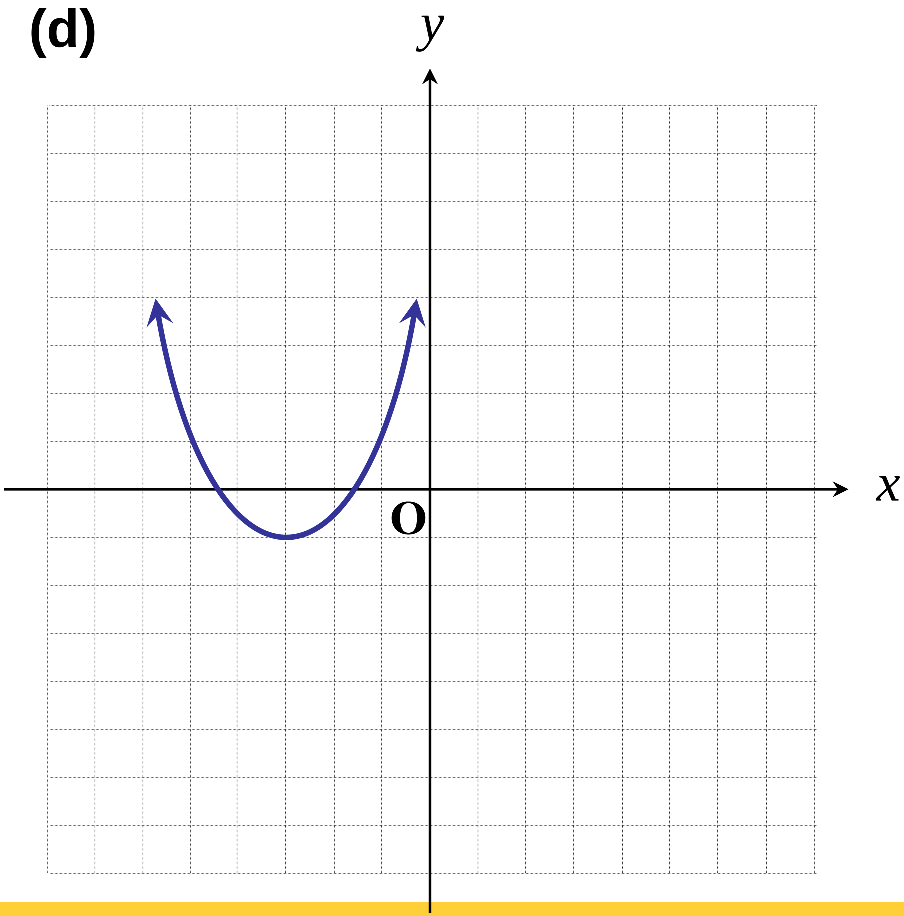
The arrowheads indicate that the line extends indefinitely left and right, as well as up and down. Therefore, both the domain and range include all real numbers, written $(-\infty, \infty)$.

Finding Domains and Ranges from Graphs

Continued.

Give the domain and range of each relation.

(d)



The arrowheads indicate that the graph extends indefinitely left and right, as well as upward. The domain is (-∞, ∞). Because there is a least *y*-value, –1, the range includes all numbers greater than or equal to –1, written [–1, ∞).

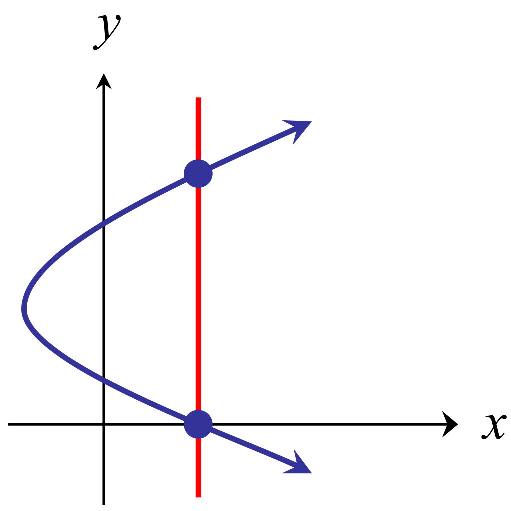
Agreement on Domain

The domain of a relation is assumed to be all real numbers that produce real numbers when substituted for the independent variable.

Vertical Line Test

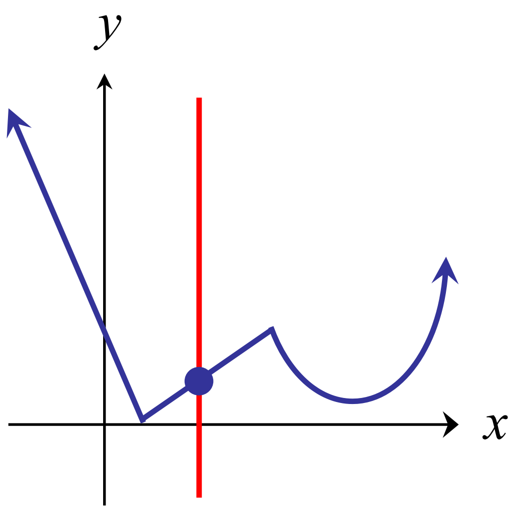
If every vertical line intersects the graph of a relation in no more than one point, then the relation represents a function.

(a)



Not a function – the same *x*-value corresponds to two different *y*-values.

(b)

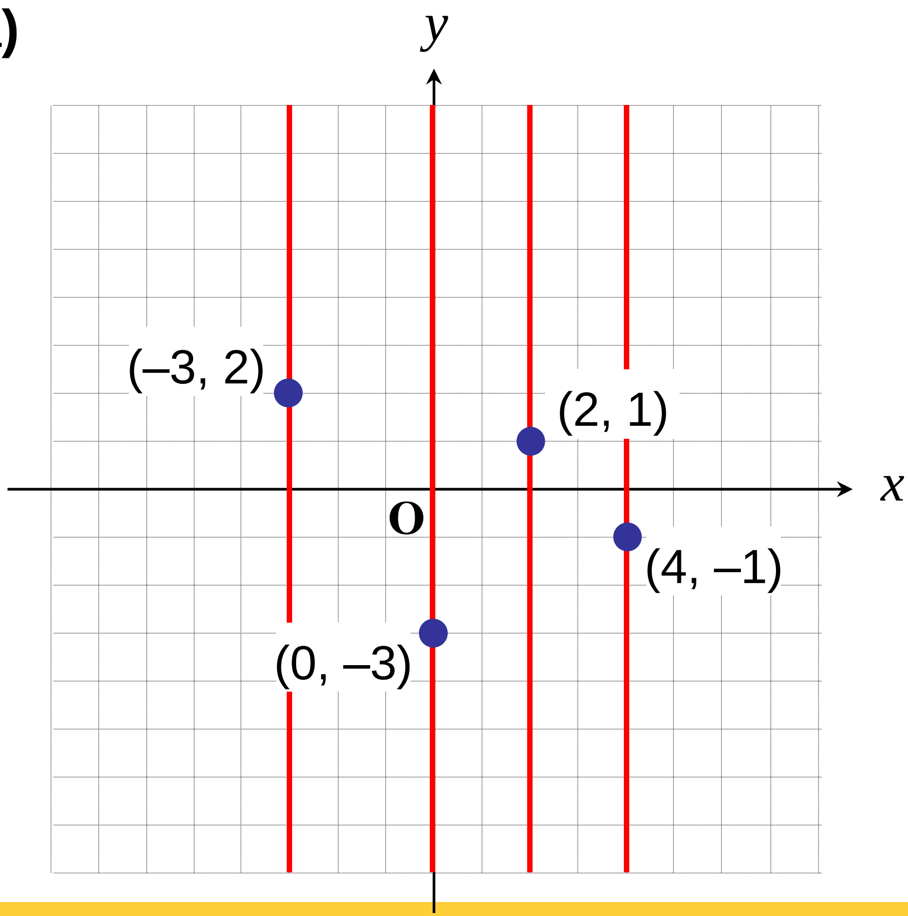


Function – each *x*-value corresponds to only one *y*-value.

Example 4 Using the Vertical Line Test

Use the vertical line test to determine whether each relation is a function.

(a)

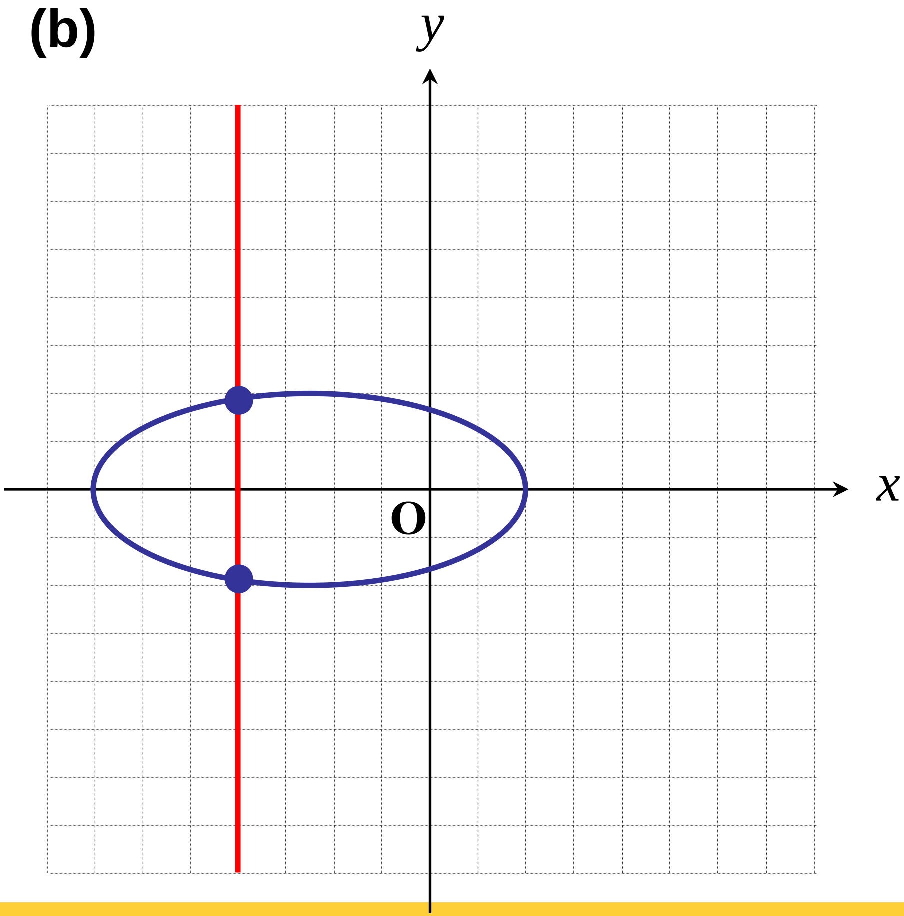


This relation is a function.

Continued. **Using the Vertical Line Test**

Use the vertical line test to determine whether each relation is a function.

(b)

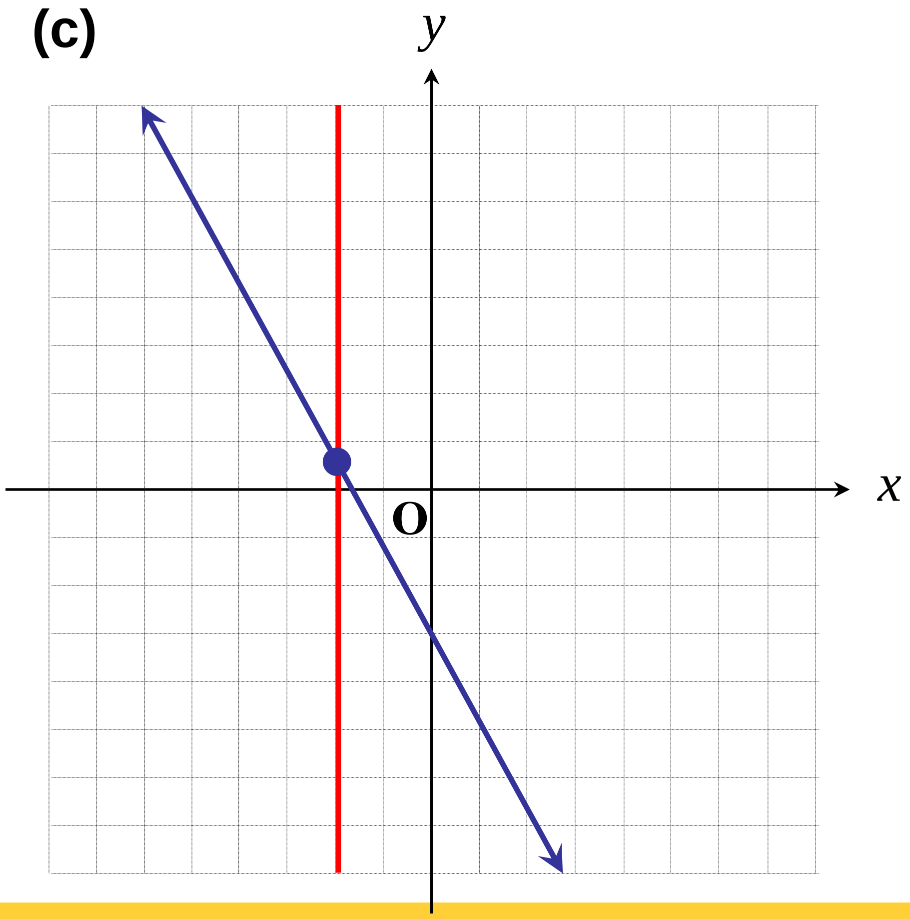


This graph fails the vertical line test since the same x-value corresponds to two different y-values; therefore, it is not the graph of a function.

Continued. **Using the Vertical Line Test**

Use the vertical line test to determine whether each relation is a function.

(c)

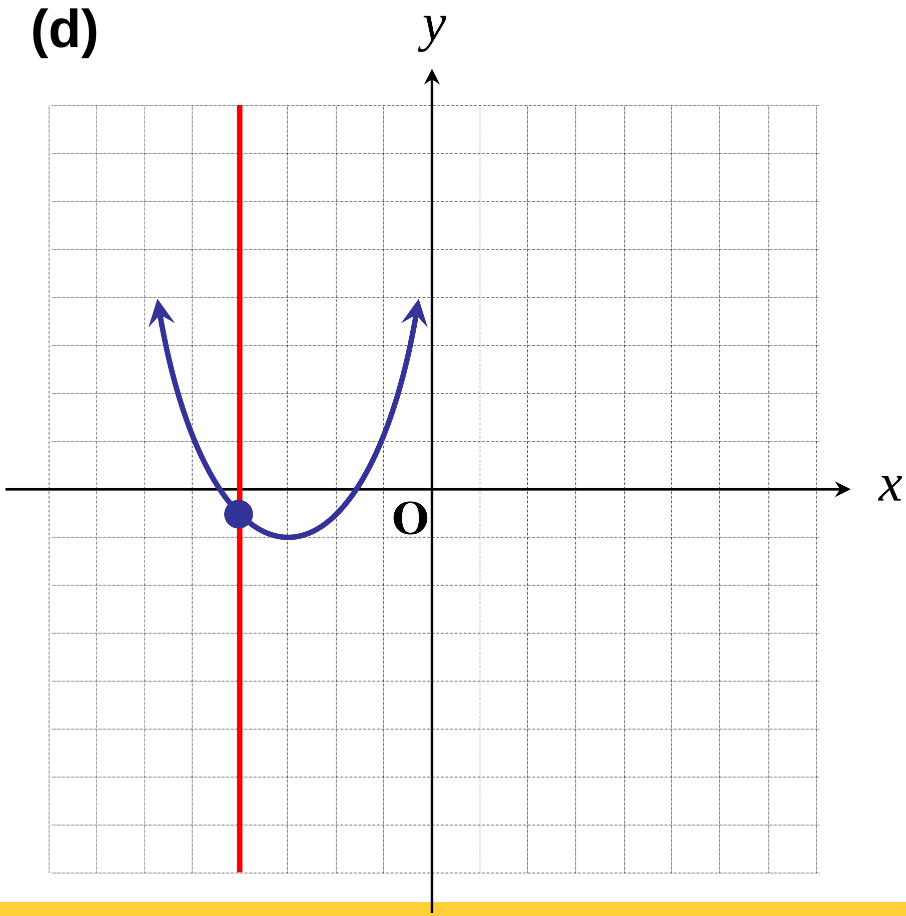


This relation is a function.

Continued. **Using the Vertical Line Test**

Use the vertical line test to determine whether each relation is a function.

(d)



This relation is a function.

Identifying Functions from Their Equations

Example 5

Decide whether each relation defines a function and give the domain.

(a) $y = x - 5$

In the defining equation, $y = x - 5$, y is always found by subtracting 5 from x. Thus, each value of x corresponds to just one value of y and the relation defines a function; x can be any real number, so the domain is $(-\infty, \infty)$.

Identifying Functions from Their Equations

Continued.

Decide whether each relation defines a function and give the domain.

(b) $y = \sqrt{3x - 1}$

For any choice of x in the domain, there is exactly one corresponding value for y (the radical is a nonnegative number), so this equation defines a function. Since the equation involves a square root, the quantity under the radical sign cannot be negative. Thus,

$$3x - 1 \geq 0$$

$$3x \geq 1$$

$$x \geq \frac{1}{3},$$

and the domain of the function is $[\frac{1}{3}, \infty)$.

Identifying Functions from Their Equations

Continued.

Decide whether each relation defines a function and give the domain.

(c) $y^2 = x$

The ordered pair (9, 3) and (9, –3) both satisfy this equation. Since one value of x, 9, corresponds to two values of y, 3 and –3, this equation does not define a function. Because x is equal to the square of y, the values of x must always be nonnegative. The domain of the relation is [0, ∞).

Identifying Functions from Their Equations

Continued.

Decide whether each relation defines a function and give the domain.

(d) $y \geq x - 3$

By definition, y is a function of x if every value of x leads to exactly one value of y. Here a particular value of x, say 4, corresponds to many values of y. The ordered pairs (4, 7), (4, 6), (4, 5), and so on, all satisfy the inequality. Thus, ***an inequality never defines a function*.** Any number can be used for x so the domain is the set of real numbers $(-\infty, \infty)$.

Identifying Functions from Their Equations

Continued.

Decide whether each relation defines a function and give the domain.

(e) $y = \dfrac{3}{x + 4}$

Given any value of x in the domain, we find y by adding 4, then dividing the result into 3. This process produces exactly one value of y for each value in the domain, so this equation defines a function. The domain includes all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving for x.

$$x + 4 = 0$$

$$x = -4$$

The domain includes all real numbers *except* –4, written (–∞, –4) U (–4, ∞).

Variations of the Definition of Function

1. A **function** is a relation in which, for each value of the first component of the ordered pairs, there is exactly one value of the second component.
2. A **function** is a set of ordered pairs in which no first component is repeated.
3. A **function** is an equation (rule) or correspondence (mapping) that assigns exactly one range value to each distinct domain value.