Production and Cost functions

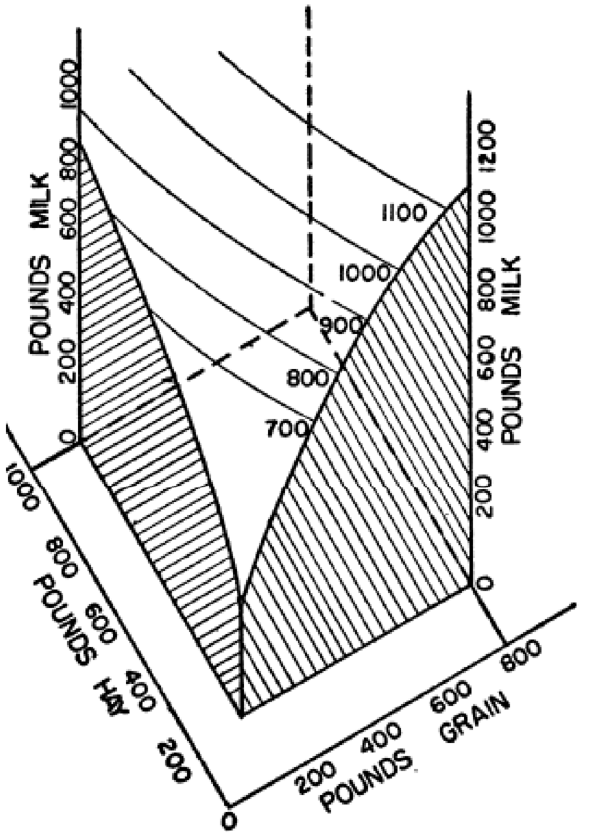
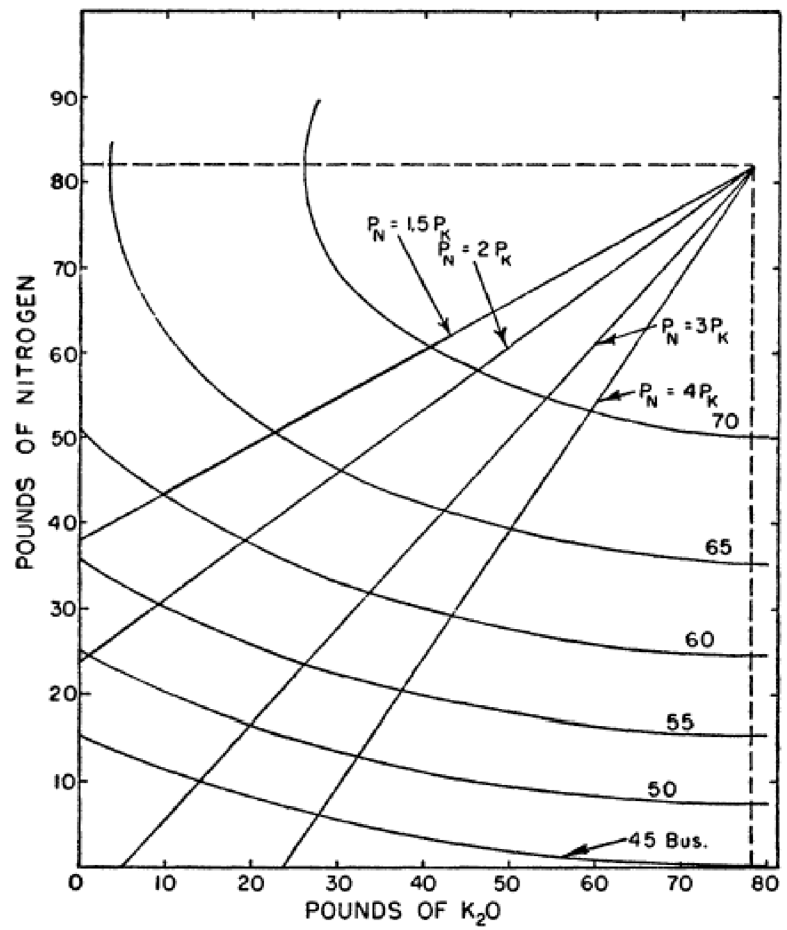


FIGURE 8.—Milk Production Surface from Equation (15). Ability and Time Set at Mean.



From: Heady 1957. An Econometric Investigation of the Technology of Agricultural Production Functions Econometrica, Vol. 25, No. 2: 249-268

Outline

- Production function
- The Firm's problem
- Parallel with Consume
- From Production function to cost functions

FIGURE 10.—Pork Costulator for Determining Least-Cost Rations

Production function-facts

- Its easy, its engineering data.
  - You go and figure out what happens when you vary the quantities of inputs.
  - You may get nothing
    - because some ingredient is necessary (strict complement)
  - You may get faster output, or slower…
  - Tabulate all that data and you can get a set of points that represent different ways of producing the same amount.
  - Run plant 24 hours a day (three shifts) or Increase number of assembly lines
  - Use your workers for their strength, give them more machines. (unload ships with cranes or with people)

Production and Cost functions

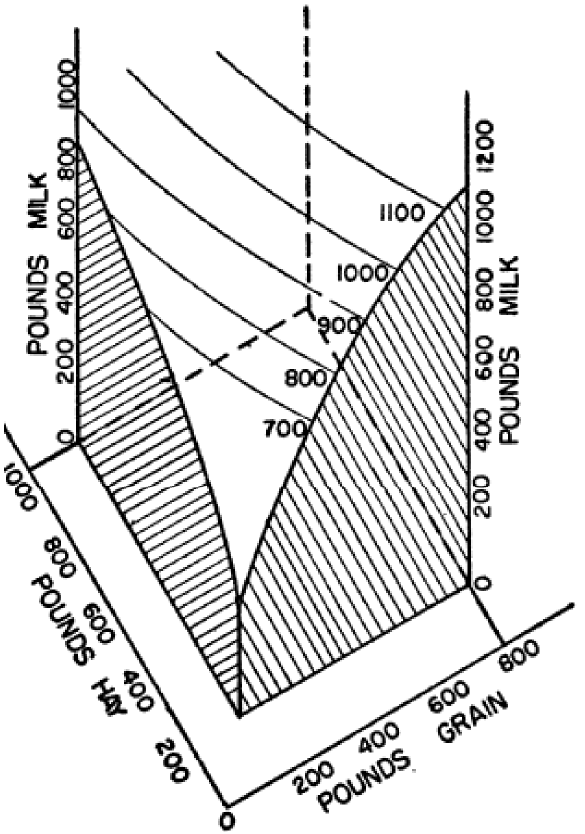
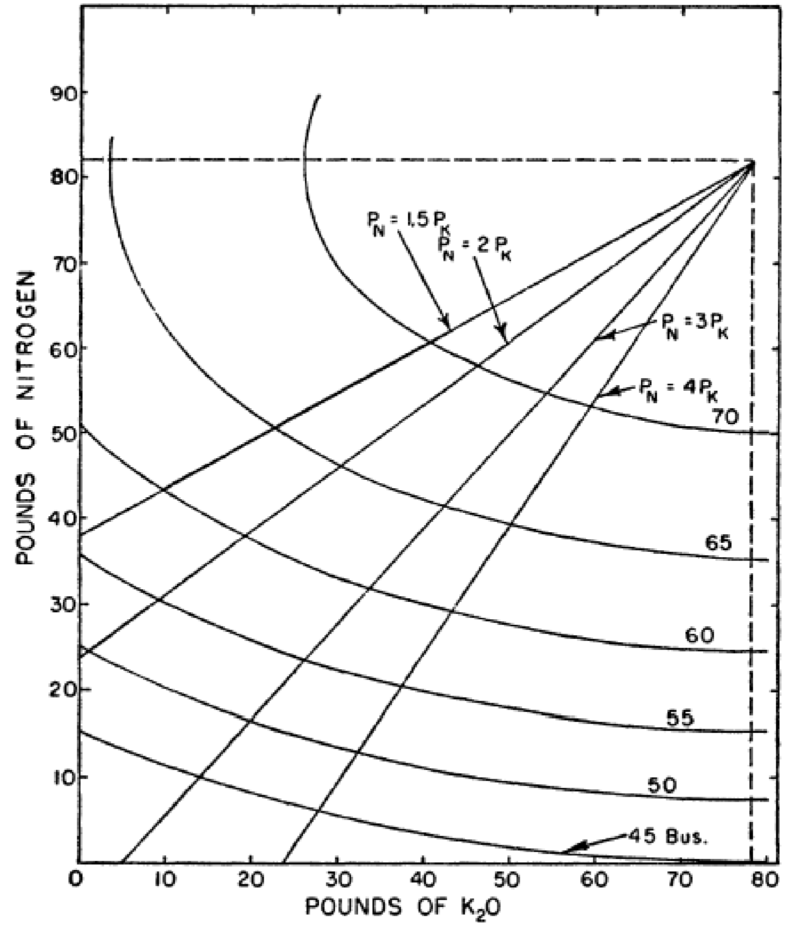


FIGURE 8.—Milk Production Surface from Equation (15). Ability and Time Set at Mean.

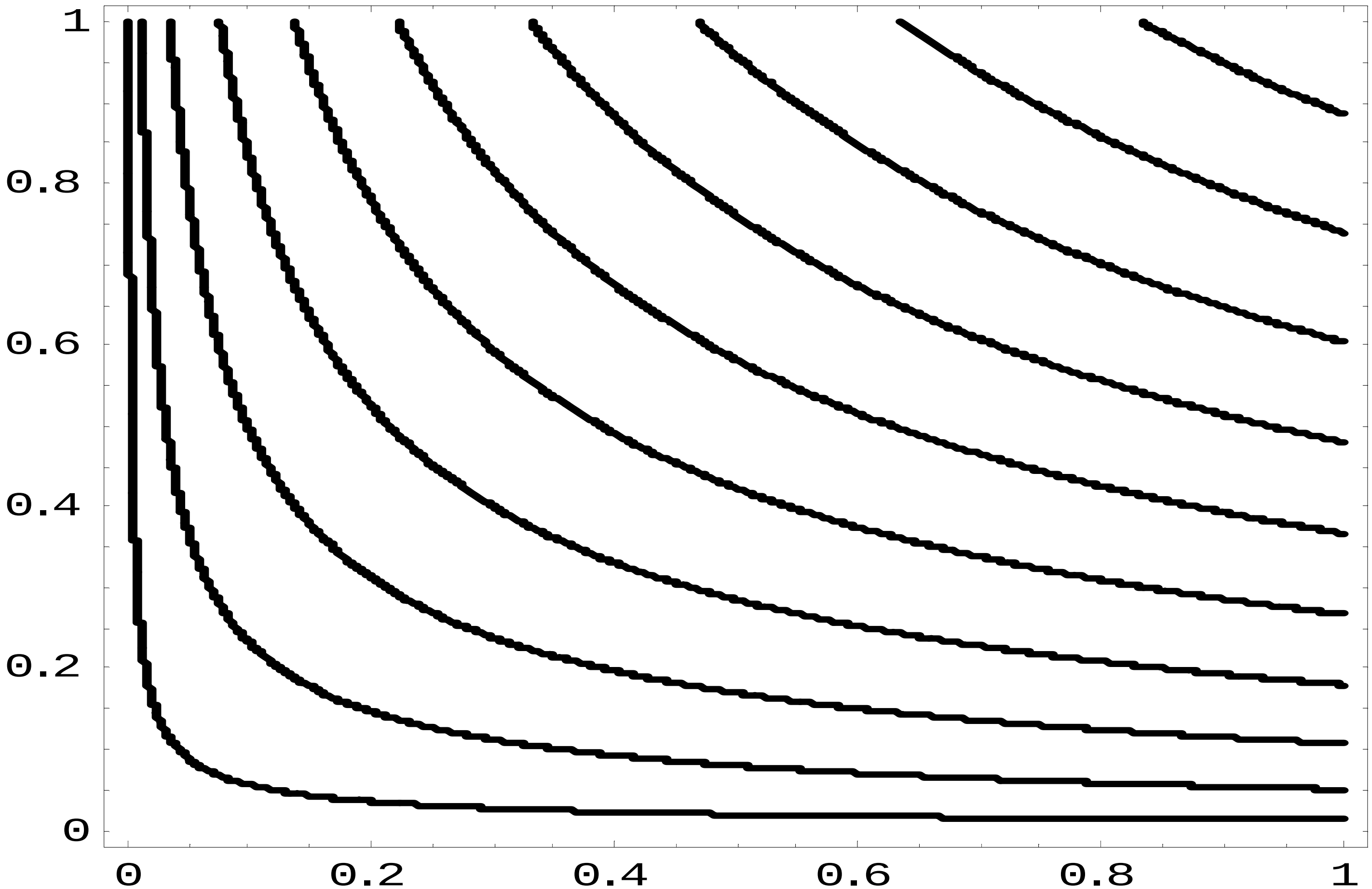


From: Heady 1957. An Econometric Investigation of the Technology of Agricultural Production Functions Econometrica, Vol. 25, No. 2: 249-268

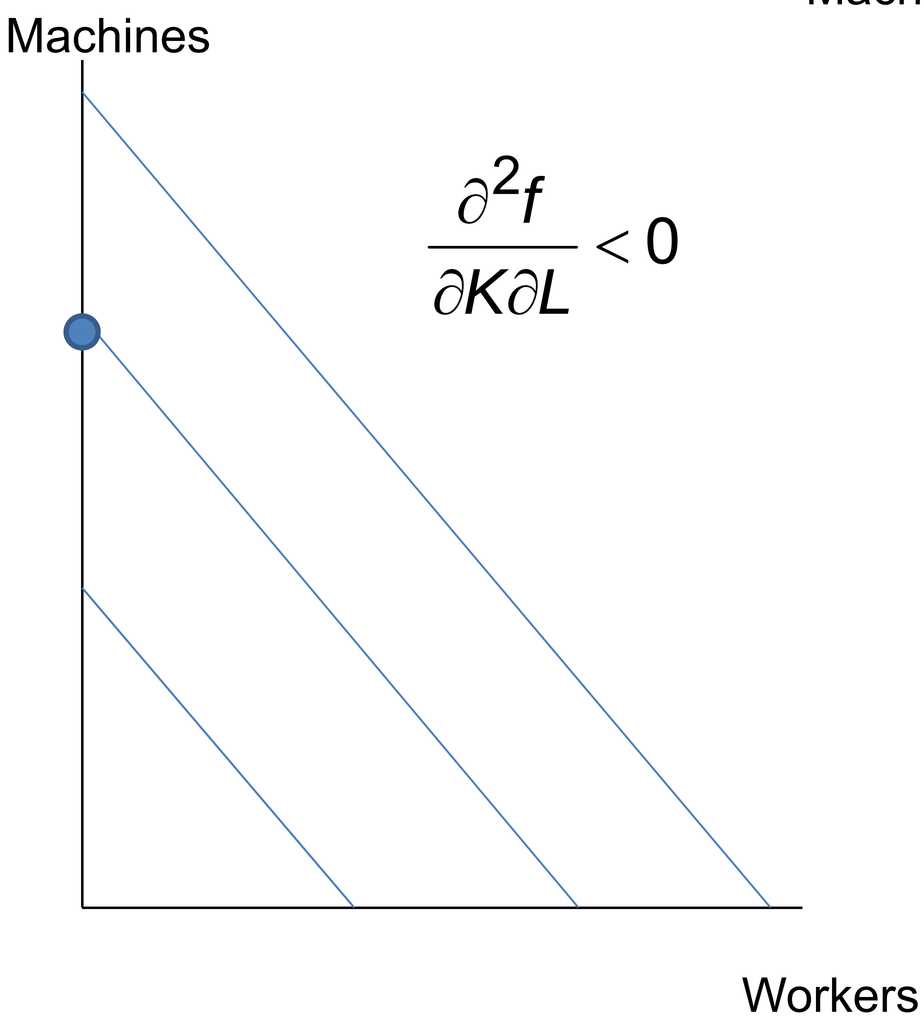
From facts to Theory

- When you measure them, you tend to find
- Isoquants are weakly convex
  - marginal diminishing returns in increasing one input.
- They can have decreasing, constant, or increasing returns.
  - Involves issues of indivisibility (example plant)
    - Returns increase as you come close to capacity
    - Then fall because some input is not adjusted.

Cobb-Douglas Isoquants

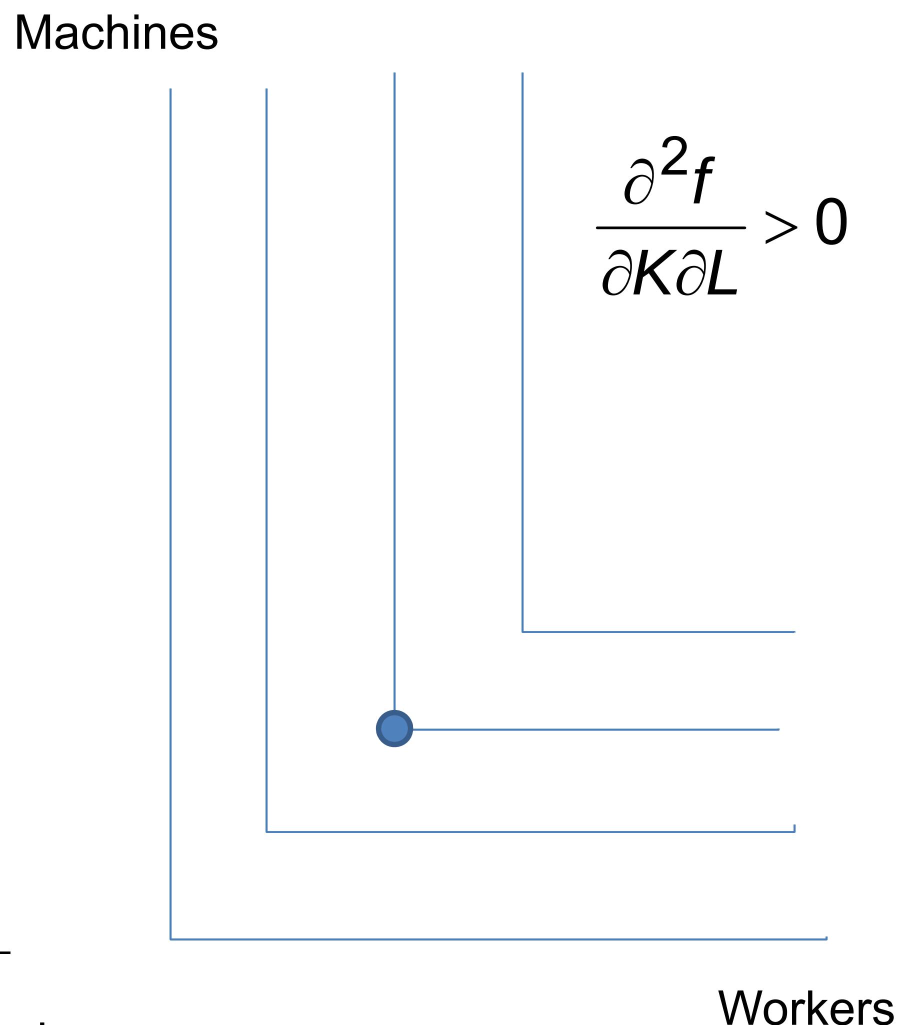


Substitutes



$$\frac{\partial^2 f}{\partial K \partial L} < 0$$

Complements



$$\frac{\partial^2 f}{\partial K \partial L} > 0$$

Firms vs consumer

- Consumer has
  - Utility function
  - Cost of goods
  - Faces budget constraint
- Consumer problem 1
  - Max U sbj to $PX \leq Y$
- Consumer problem 2
  - Min PX sbj to $U(X) \geq \underline{U}$

- Firm has
  - Production function
  - Costs of inputs
  - Faces production constraint.
- Firm problem 1
  - Max F(X) sbj to $PX \leq Y$
- Firm problem 2
  - Min PX sbj to $F(X) \geq \underline{Q}$

If you can solve 1 you can solve 2 if you can solve the consumer's problem you can solve the firm's.

More symmetry

- Consumer

$$\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{p_1}{p_2}$$

Ratio of marginal utilities equal to the price ratio

- Firm

$$\frac{\frac{\partial F}{\partial x_1}}{\frac{\partial F}{\partial x_2}} = \frac{p_1}{p_2}$$

$$\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}} = \frac{w}{r}$$

Ratio of marginal product equal to the price ratio

Short run vs long run

- Marginal product is non-negative

$$\frac{\partial f}{\partial k} > 0 \qquad \frac{\partial^2 f}{(\partial k)^2} < 0.$$

- More is better but there are marginal diminishing returns
- Some inputs (labor, raw materials) more readily changed than others (plant and equipment)…so there is a long run and a short run.
- Short run, take fixed assets and technology as given
  - choose the right mix of inputs to run that firm
- Long run chose your plant size, technology.
  - Suppose the price of gas increases for a taxi firm
    - What should it do?

Short Run Profit Maximization (take p as given)

$$\pi = pF(K,L) - rK - wL.$$

$$0 = \frac{\partial \pi}{\partial L} = p\frac{\partial F}{\partial L}(K,L^*) - w.$$

- FOC

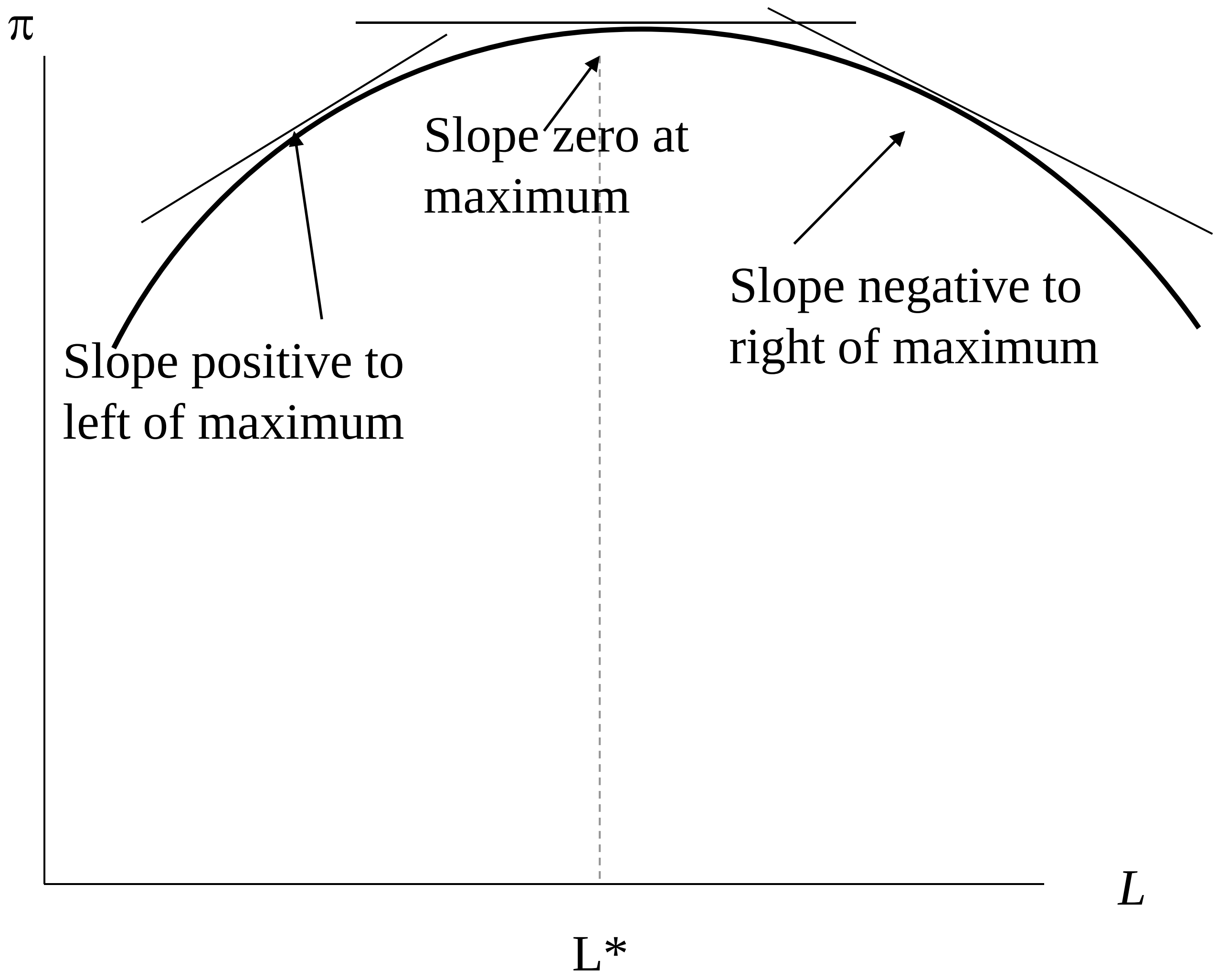
- Wage (price of input) is equal to the value of its marginal product

$$0 \geq \frac{\partial^2 \pi}{(\partial L)^2} = p\frac{\partial^2 F}{(\partial L)^2}(K,L^*).$$

- SOC

- Guaranteed if there are marginal decreasing returns

Graphical Depiction



Short-run Effect of a Wage Increase

$$\frac{\partial \pi}{\partial L} = p\frac{\partial F}{\partial L}(K, L^*) - w = 0.$$

I can differentiate the FOC with respect to w (the wage)

$$0 = p\frac{\partial^2 F}{(\partial L)^2}(K, L^*(w))L^{*\prime}(w) - 1,$$

$$L^{*\prime}(w) = \frac{1}{p\frac{\partial^2 F}{(\partial L)^2}(K, L^*(w))} \leq 0.$$

P>0, F''<0

Aside: Revealed Preference

- Revealed preference is a powerful technique to prove comparative statics
- Works without assumptions about continuity or differentiability
- Suppose $w1 < w2$ are two wage levels
- The entrepreneur chooses $L1$ when the wage is $w1$ and $L2$ when the wage is $w2$

Revealed Preference Proof

Prefer L_1 to L_2 when wage = w_1

$$pf(K,L_1)-rK-w_1L_1 \geq pf(K,L_2)-rK-w_1L_2$$

Prefer L_2 to L_1 when wage = w_2

$$pf(K,L_2)-rK-w_2L_2 \geq pf(K,L_1)-rK-w_2L_1.$$

Sum these two

$$pf(K,L_1)-rK-w_1L_1+pf(K,L_2)-rK-w_2L_2 \geq$$

$$pf(K,L_1)-rK-w_2L_1+pf(K,L_2)-rK-w_1L_2$$

$$-w_1L_1-w_2L_2 \geq -w_2L_1-w_1L_2$$

Revealed Preference, Cont'd

$$-w_1 L_1 - w_2 L_2 \geq -w_2 L_1 - w_1 L_2$$

- *But remember* $w1 < w2$

$$(w_1 - w_2)(L_2 - L_1) \geq 0.$$

- So the only way for above to be true is
- $0 \geq (L_2 - L_1) <=> L_1 \geq L_2.$
- Revealed preference shows that profit maximization implies *L* falls as *w* rises.

Cost Minimization

- Profit maximization requires minimizing cost
- Cost minimization for fixed output

c(y) = Min $wL + rK$

subject to

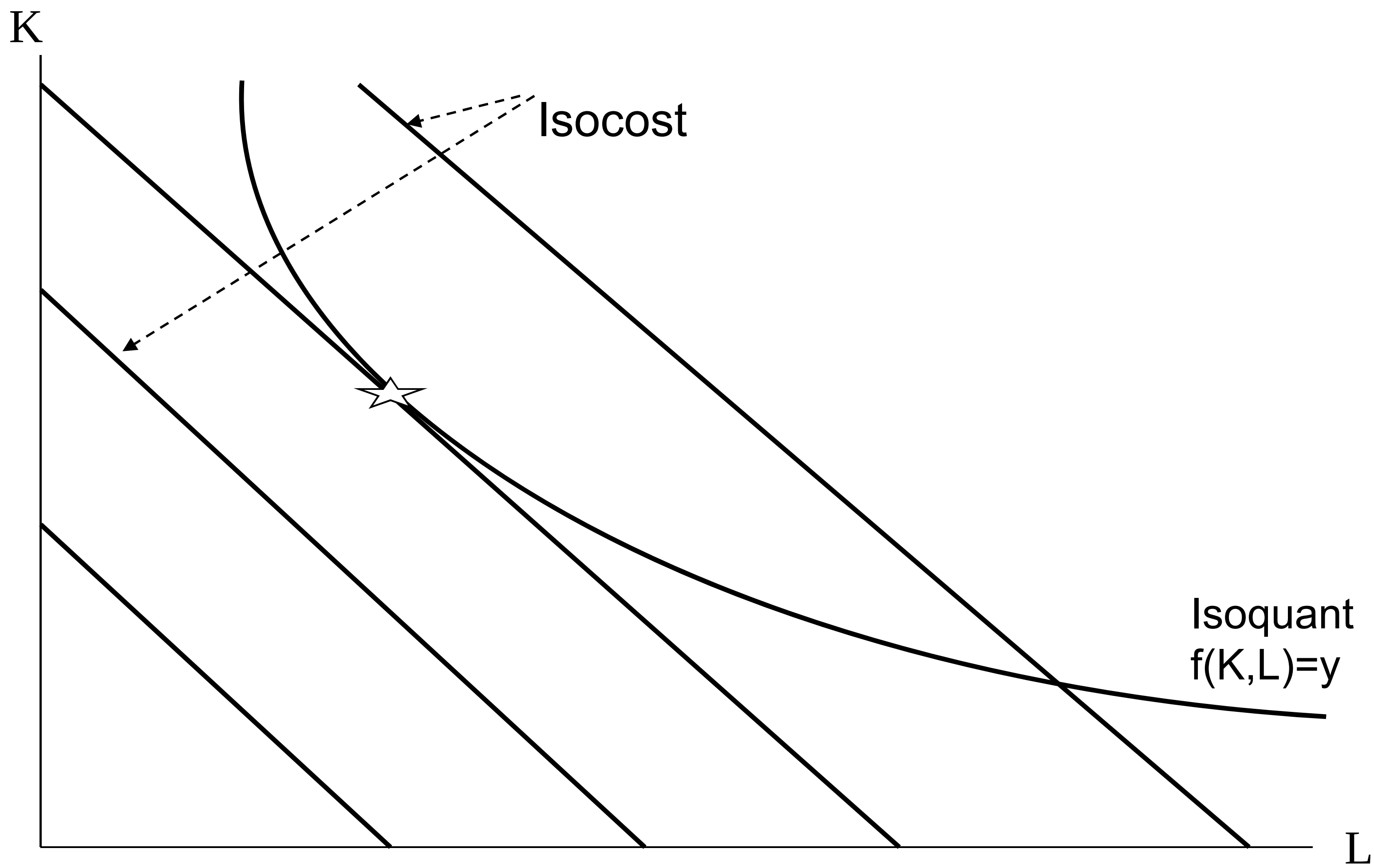
$$f(K,L) = y$$

Cost Minimization, Continued

- Profit maximization:
- max py – (wL + rK) s.t. $f(K,L) = y$
- For given y, this is equivalent to minimizing cost.
- Cost minimization equation:

$$-\frac{\partial f / \partial L}{\partial f / \partial K} = \left.\frac{dK}{dL}\right|_{f(K,L)=y} = -\frac{w}{r}$$

Cost Min Diagram



Short run vs Long run

$$F(K_1, K_2, L) = \beta K_1 (K_2^{\alpha} L^{\beta})$$

- Short run fix K_1 (Plant size)
- Long run chose K_1
- Short run Min $wL+rK_2$ sbj $F(K_2,L)>Q$
- $G(L, K_2, \lambda)= wL+rK_2 + \lambda\,(Q-F(K_2,L))$

FOC

$$\frac{dG}{dL} = w - \lambda \frac{dF}{dL} = w - \lambda\gamma K_1 \beta K_2^{\alpha} L^{\beta - 1} = 0$$

$$\frac{dG}{dK_2} = w - \lambda \frac{dF}{dK_2} = r - \lambda\gamma K_1 \alpha K_2^{\alpha - 1} L^{\beta} = 0$$

$$\frac{dG}{d\lambda} = \mathrm{Q} - \mathrm{F}(\mathrm{K2}, \mathrm{L}) = 0$$

- Ratio of the first two gives

$$\frac{w}{r} = \frac{\lambda\gamma K_1 \beta K_2^{\alpha} L^{\beta-1}}{\lambda\gamma K_1 \alpha K_2^{\alpha-1} L^{\beta}} = \frac{\beta}{\alpha}\frac{K_2^{\alpha} L^{\beta-1}}{K_2^{\alpha-1} L^{\beta}} = \frac{(1-\alpha)}{\alpha}\frac{K_2}{L}$$

- Ratio of MP equals price ratio, or

$$L = \frac{r}{w}\frac{\beta}{\alpha}K_2$$

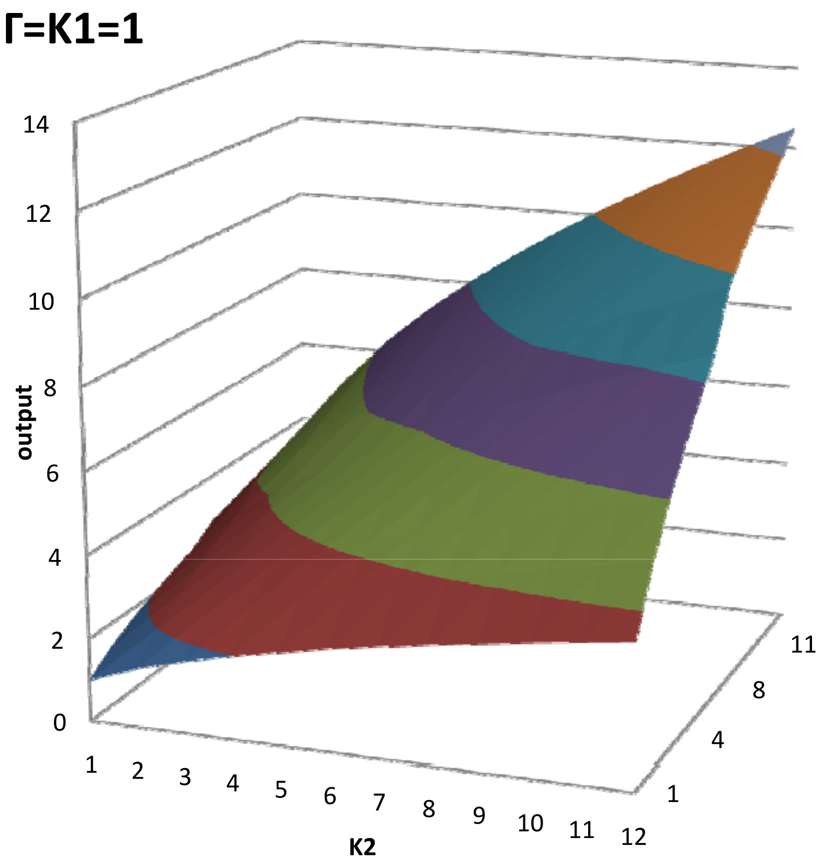
- Plug back into last FOC

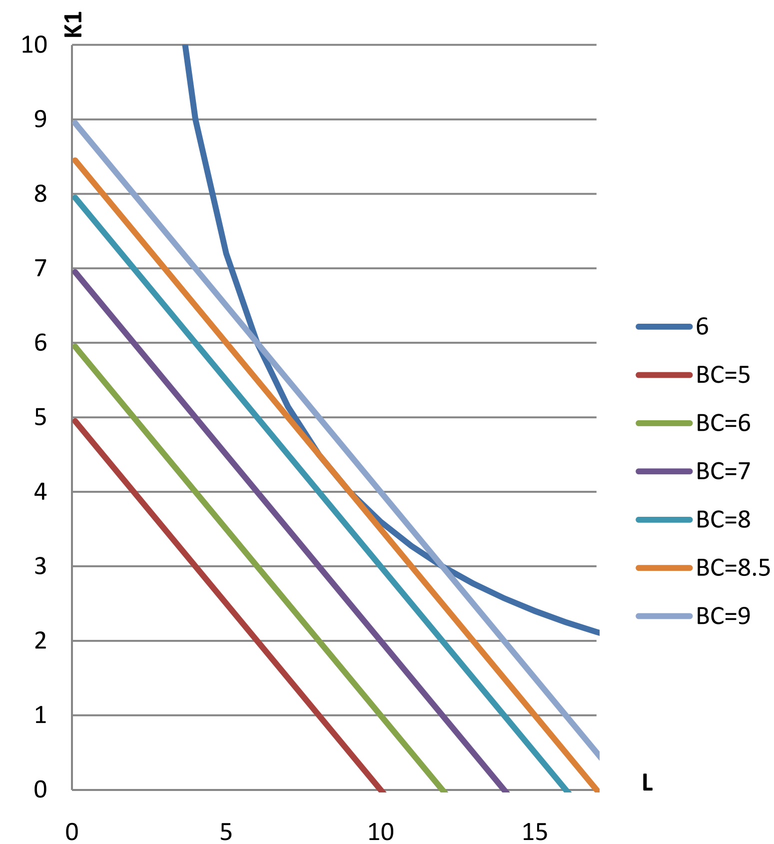
$$Q = \gamma K_1 K_2^{\alpha} L^{1-\alpha} = \gamma K_1 K_2^{\alpha}\left(\frac{r}{wr}\frac{\beta}{\alpha}\right)^{\beta} K_2^{\beta} = \gamma K_1 \left(\frac{r}{w}\frac{\beta}{\alpha}\right)^{\beta} K_2^{\alpha+\beta}$$

$$K_2 = Q^{\frac{1}{\alpha+\beta}}\left(\frac{1}{\gamma K_1}\left(\frac{w\alpha}{r\beta}\right)^{\beta}\right)^{\frac{1}{\alpha+\beta}} \qquad L = Q^{\frac{1}{\alpha+\beta}}\left(\frac{1}{\gamma K_1}\left(\frac{r\beta}{w\alpha}\right)^{\alpha}\right)^{\frac{1}{\alpha+\beta}}$$

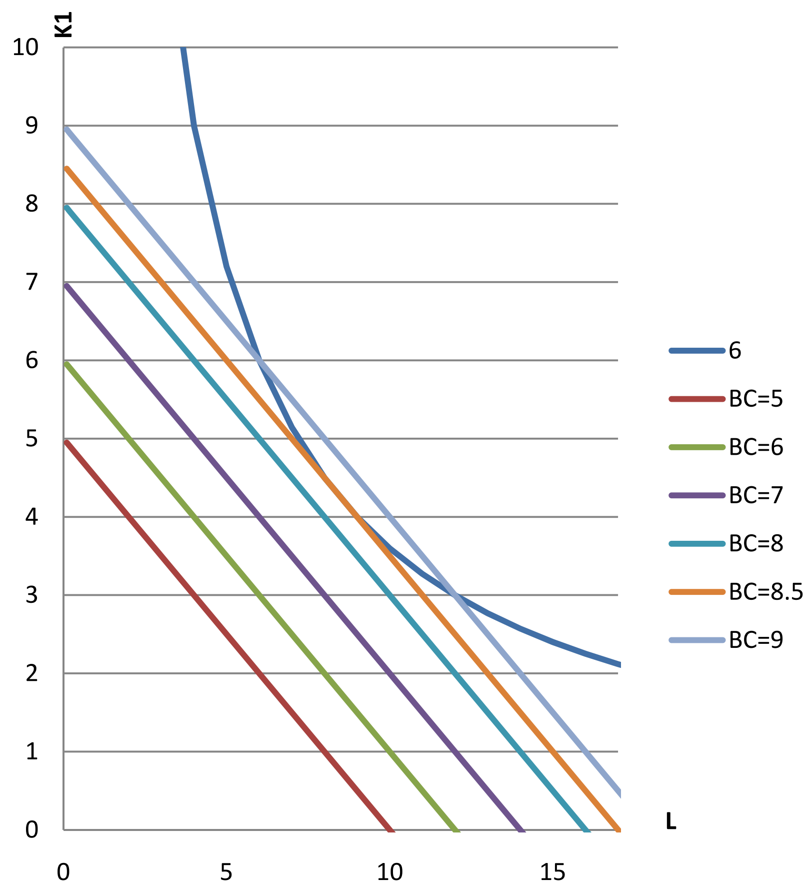
A bit of a mess but in fact its linear in Q if $\alpha+\beta=1$; its also decreasing in K_1

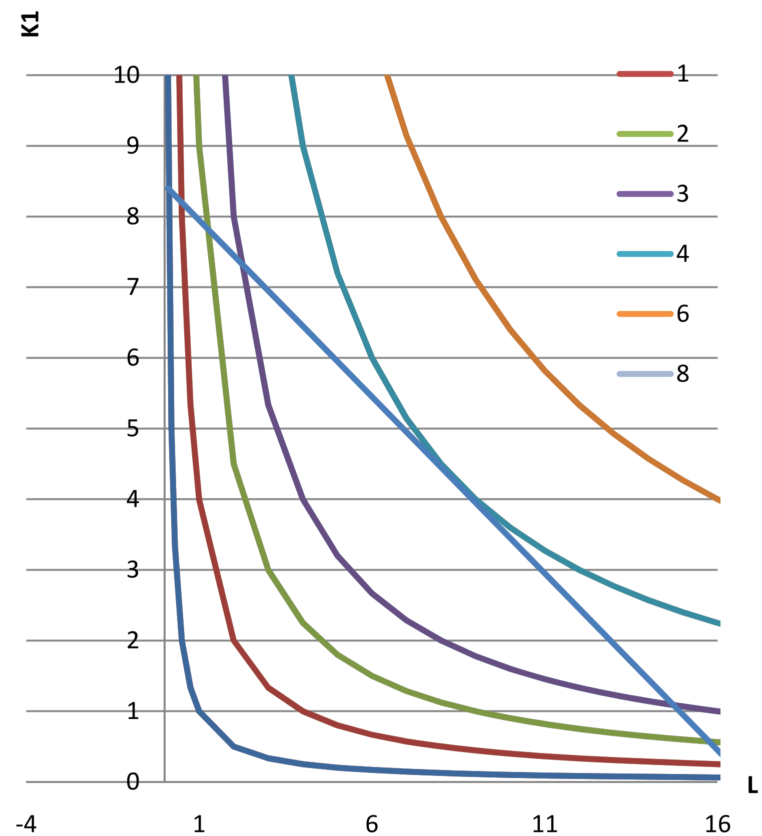
Short run


Γ=K1=1
output
14
12
10
8
6
4
2
0
1
2
3
4
5
6
7
8
9
10
11
12
K2
1
4
8
11


K1
10
9
8
7
6
5
4
3
2
1
0
0
5
10
15
L
6
BC=5
BC=6
BC=7
BC=8
BC=8.5
BC=9

On the left Cost Min; on the right Max Q





Long run

- First notice that L and K_2 are optimal solutions given any K_1 …so we could just search out the optimal K_1 given L and K_2

$$\frac{dG}{dK_1} = r - \lambda \frac{dF}{dK_1} = r - \lambda\gamma(K_2^{\alpha}L^{\beta}) = 0$$

$$r = \lambda\gamma\left\{Q^{\frac{1}{\alpha+\beta}}\left(\frac{1}{\gamma K_1}\left(\frac{w\alpha}{r\beta}\right)^{\beta}\right)^{\frac{1}{\alpha+\beta}}\right\}^{\alpha} + \lambda\gamma\left\{Q^{\frac{1}{\alpha+\beta}}\left(\frac{1}{\gamma K_1}\left(\frac{r\beta}{w\alpha}\right)^{\alpha}\right)^{\frac{1}{\alpha+\beta}}\right\}^{\beta} = 0$$

$$r = \lambda\gamma\left\{\left(\frac{Q}{\gamma K_1}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{w\alpha}{r\beta}\right)^{\frac{\alpha\beta}{\alpha+\beta}} + \left(\frac{Q}{\gamma K_1}\right)^{\frac{\beta}{\alpha+\beta}}\left(\frac{r\beta}{w\alpha}\right)^{\frac{\alpha\beta}{\alpha+\beta}}\right\}$$

What a mess! Notice that the solution (K_1^*) is going to be increasing in Q and γ and declining in r

When α+β=0.5

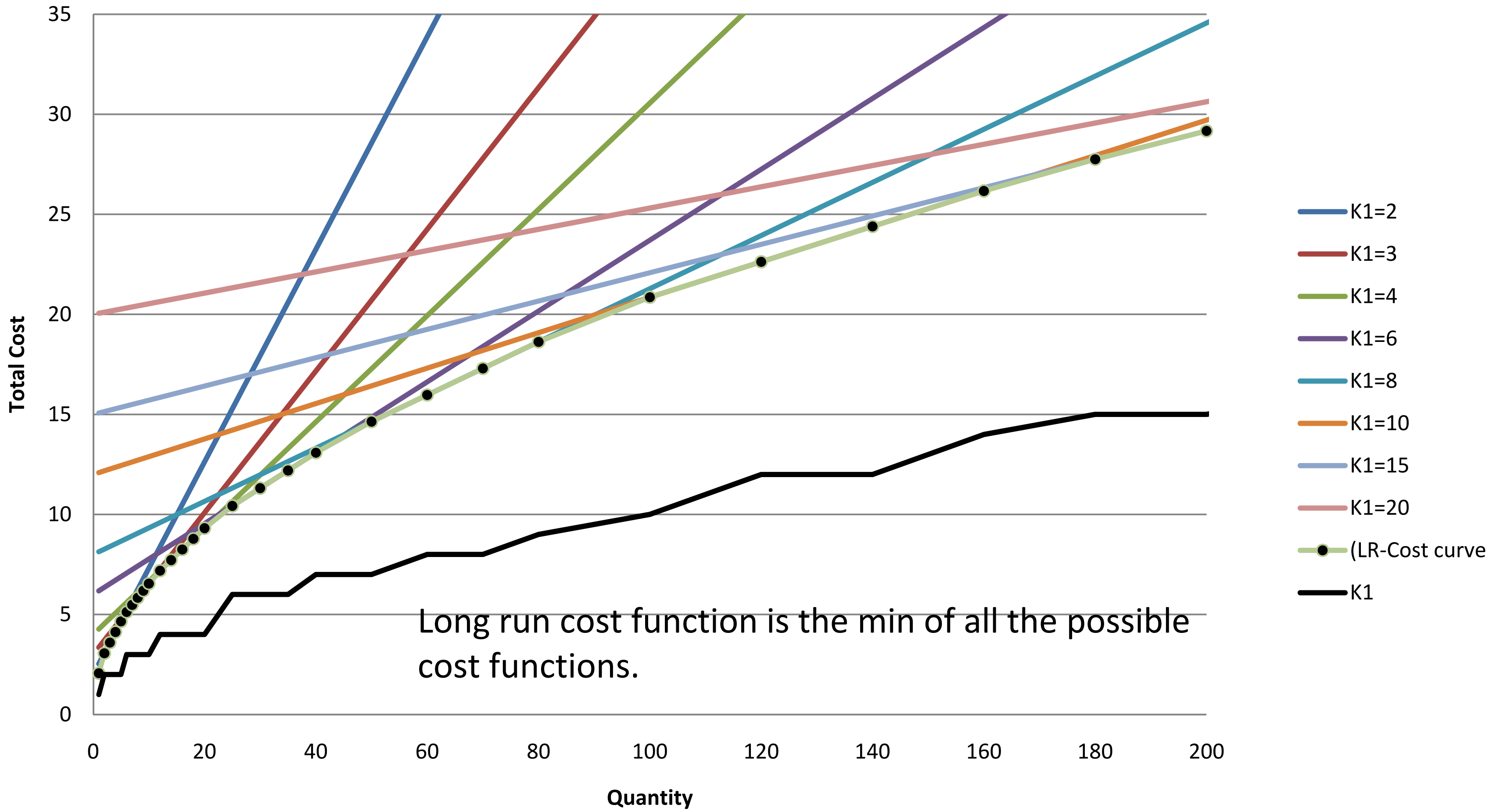
$$r = \lambda\gamma\left\{\left(\frac{Q}{\gamma K_1}\right)^{.5}\left(\frac{w}{r}\right)^{.25} + \left(\frac{Q}{\gamma K_1}\right)^{.5}\left(\frac{r}{w}\right)^{.25}\right\}$$

$$r = \lambda\gamma\left(\frac{Q}{\gamma K_1}\right)^{.5}\left\{\left(\frac{w}{r}\right)^{.25} + \left(\frac{r}{w}\right)^{.25}\right\}$$

$$\left(\frac{Q}{\gamma K_1}\right)^{.5} = \frac{r}{\lambda\gamma\left\{\left(\frac{w}{r}\right)^{.25} + \left(\frac{r}{w}\right)^{.25}\right\}}$$

$$K_1 = Q\gamma\frac{\lambda^2}{r^2}\left\{\left(\frac{w}{r}\right)^{.25} + \left(\frac{r}{w}\right)^{.25}\right\}^2$$

- As the target quantity Q, increases you invest more in plant, same if crowding is a big deal (γ) or if capital is cheap.
- Note this does not depend on α+β=0.5



Note Marginal cost is constant for each level of K1

From production function to costs

- Rather than look at a production function one can summarize the firm's decision into a simple cost function.
- Note: that implies that we are tracing out the optimal input mix given prices, and technology.
- In our last example cost were increasing in a linear way for each level of K1. So each K1 corresponds to a different short run cost function.
- Key: Cost functions assume that input prices are stable.

Reminder

- For every production function and input price set, you can find the vector X ={x_1,..,x_i,..,x_n} that minimizes the cost of producing a given level of output Q. The cost of producing Q is there fore PX. Now one can do this for every Q over the relevant range.
- The function that maps Q into cost exists if the production function is convex. C(Q)
- Marginal cost is simply the derivative of the cost function with respect to quantity.

From production to Firm

- One possibility is for the Entrepeneur to solve one big problem
  - Max $\pi = pF(x_1,..,x_i,..,x_n) - (p_1x_1+\ldots+p_ix_i..\ p_nx_n\ )$
- Another takes two steps
  - (1) The engineers to do the cost minimization
    - Min $p_1x_1+\ldots+p_ix_i..p_nx_n$ **sbj to** $F(x_1,..,x_i,..,x_n)<Q$
  - (2) Marketing finds the quantity to produce
    - Max $\pi = pQ-C(Q)$

Short-run Costs

- Short-run total cost
- L varies, K does not
- Short-run marginal cost
  - Derivative of cost with respect to output
- Short-run average cost
  - average over output
  - infinite at zero, due to fixed costs
- Short-run average variable cost
  - average over output, omits fixed costs

Comparative Statics

- What happens to *L* as *K* rises?

$$L^{*\prime}(K) = \frac{-\dfrac{\partial^2 F}{\partial K \partial L}(K, L^*(K))}{\dfrac{\partial^2 F}{(\partial L)^2}(K, L^*(K))}.$$

- Remember the lower part of the ratio has to be negative (condition for a max)
- Thus, *L* rises if *L* and *K* are complements, and falls if substitutes

Conclusion and Wrap up

- 1) remember the symmetry of our two key decision problems (consumer and producer)
- 2) short run problem is exactly symmetric
- 3) long run problem involves choosing scale