WHAT IS THE AREA OF AN N-SIDED IRREGULAR POLYGON?

The typical way to measure the area of a piece of land with straight-line boundaries is to note the 2D coordinates $[x_n, y_n]$ of the corners connecting neighboring lines. Thus if we have an irregular three sided area having corners at [0,0], [2,3] and [1,4], the side-lengths are L_1=sqrt(13), L_2=sqrt(2), and L_3=sqrt(17). One can then use the Heron formula to get the area-

$$A = \sqrt{s(s - L_1)(s - L_2)(s - L_3)} \quad where \quad s = \frac{(L_1 + L_2 + L_3)}{2} \quad is\ the\ semi-perimeter$$

An alternate, and much easier, way to get the area is to take half the absolute value of the vector product of the vectors representing two of the sides. Thus-

$$A = \frac{1}{2} abs \begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 1 & 4 & 0 \end{vmatrix} = \frac{5}{2}$$

This second approach of finding the area can be extended to any N sided irregular polygon which can always be broken up into N triangles. Then adding up the sub-areas A_n produces the total area.

Consider the following schematic of an N sided irregular polygon containing the coordinate origin (0,0) lying at some point within it-

SCHEMATIC FOR AN IRREGULAR N SIDED POLYGON
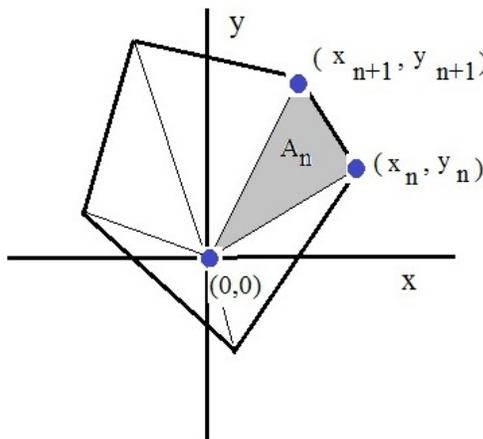
The sub-triangle shaded in gray has the area-

$$A_n = \frac{1}{2}abs[(x_n i + y_n j) \times (x_{n+1} i + y_{n+1} j)] = \frac{1}{2}abs\begin{vmatrix} i & j & k \\ x_n & y_n & 0 \\ x_{n+1} & y_{n+1} & 0 \end{vmatrix} = \frac{1}{2}abs(x_n y_{n+1} - x_{n+1} y_n)$$

Note that A_n also represents half the area of a rhombus having two of its sides have length –

$$L_n = \sqrt{x_n^2 + y_n^2} \quad and \quad L_{n+1} = \sqrt{x_{n+1}^2 + y_{n+1}^2}$$

Next, adding all N triangles making up the polygon produces the area-

$$A = \frac{1}{2}\sum_{n=1}^{N} abs[x_n y_{n+1} - x_{n+1} y_n]$$

This shows we only need the coordinates of each of the N corners of the polygon to find its total area. It should produce correct values for both convex polygons such as a hexagon or for concave polygons such as stars.

Let us begin by asking for the largest area an equilateral triangle can have and still fit into a circle of radius R. Clearly the three vertexes of this triangle must lie on the circle at 120° intervals. So one can choose the vertex coordinates, expressed in Cartesian coordinates, to be –

(R, 0), (-R/2, Rsqrt(3)/2), and (-R/2, -Rsqrt(3)/2)

The area will thus be given by-

$$A = \frac{1}{2}abs\begin{vmatrix} i & j & k \\ -3R/2 & R\sqrt{3}/2 & 0 \\ -3R/2 & -R\sqrt{3}/2 & 0 \end{vmatrix} = \frac{3\sqrt{3}}{4}R^2$$

The number 3sqrt(3)/4=1.299078.. is a little surprising since it says the equilateral triangle fills only 1.2990../π=0.41349..of the circle area which is given by πR^2.

Consider next the area of a hexagon of side-length s=1. Here the total area is given by six equal isosceles sub-triangles where the vertex coordinates of one of these can be-

(0, 0), (1,0) and (1/2, sqrt(3)/2)

The total area of the regular hexagon then is-

$$A = 6\{\frac{1}{2} abs \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \end{vmatrix}\} = \frac{3\sqrt{3}}{2}$$

Circumscribing the hexagon with a circle of unit radius shows that the hexagon fills $3sqrt(3)/(2\pi)$ or about 83% of the circle.

As already stated earlier the present area determination method will also work when part of the polygon boundary is concave. This will be the case for stars, Consider the area of the following five pointed star known as the pentagram-
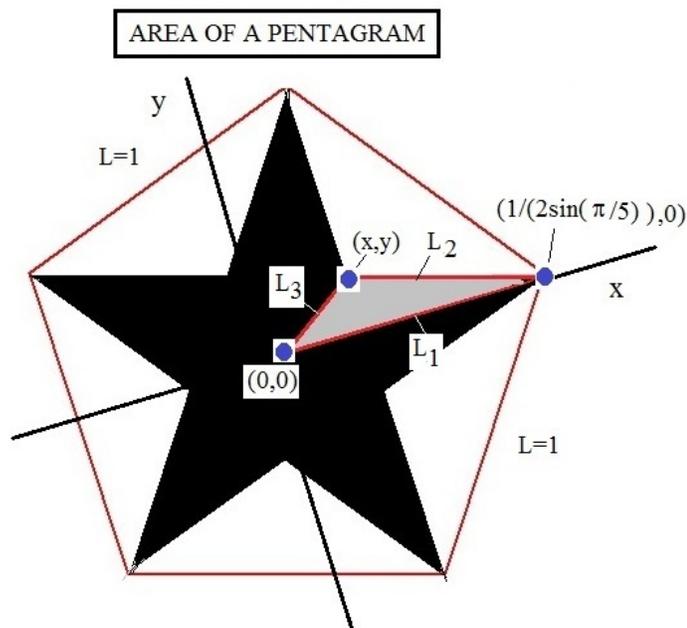
We show one of the ten equal sub-triangles which make up the pentagram in gray This sub-triangle has vertexes at –

$$(0, 0), \quad (L_1, 0), \quad \text{and} \quad (L_3\cos(\pi/5), L_3\sin(\pi/5))$$

The pentagram area will thus be-

$$A = 10\{\frac{1}{2} abs \begin{vmatrix} i & j & k \\ L_1 & 0 & 0 \\ L_3 \cos(\pi/5) & L_3 \sin(\pi/5) & 0 \end{vmatrix}\} = 5L_1 L_3 \sin(\frac{\pi}{5})$$

Looking at the pentagram geometry, using the law of sines, and having the side-length of the surrounding pentagon be s=1, we find-

$$L_1 = \frac{1}{2\sin(\pi/5)} \quad, \quad L_2 = \frac{1}{2\cos(\pi/5)} \quad and \quad L_3 = \frac{\sin(\pi/10)}{\sin(2\pi/5)}$$

Thus the area of a pentagram inscribed by a pentagon of side-length s=1 is-

$$A = (\frac{5}{2})(\frac{\sin(\pi/10)}{\sin(2\pi/5)}) = \frac{5}{8\cos(\pi/10)\cos(\pi/5)} = 0.8122992405...$$

We can check this answer by noting that the pentagram area just equals the area $A_p=(5/4)(1/\tan(\pi/5))$ of the circumscribing pentagon minus five times the area of the isosceles triangle whose sides have lengths L_2, L_2, and 1. The area of each of these sub-triangles is $A_n=(1/4)\tan(3\pi/10)$. The area of the pentagram thus becomes-

$$A = (\frac{5}{4})\left\{ \frac{1}{\tan(\pi/5)} - \frac{1}{\tan(3\pi/10)} \right\} = 0.81229924...$$

which checks with the earlier result. An interesting sideline of the pentagram is that the oblique straight line distance from one of its vertexes to one two away equals exactly-

$$\delta = \frac{\sin(3\pi/5)}{\sin(\pi/5)} = 3 - 4\sin^2(\pi/5) = \frac{1+\sqrt{5}}{2} = 1.680339...$$

You will recognize that δ is just the Golden Ratio already well known to the ancient Greeks.

Let us end the discussion by looking at the area of the four sided irregular polygon having corners at (3,0), (-1,2), (0,0), and(-2,-3) as shown-

AREA OF A FOUR SIDED IRREGULAR POLYGON
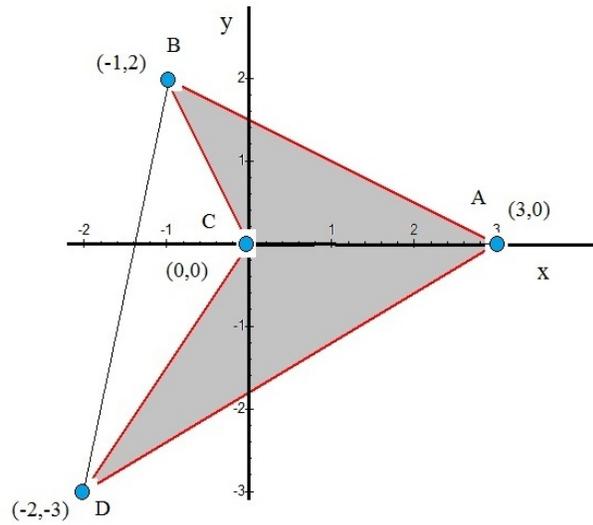
There are two ways to evaluate the area of this polygon. One can add together the areas defined by the sub-triangles ABC and DAC or one can subtract area DCB outside the polygon from the large triangle DAB. Using the second approach we find-

$$A = \frac{1}{2} \left\{ abs \begin{vmatrix} i & j & k \\ -4 & 2 & 0 \\ -5 & -3 & 0 \end{vmatrix} - abs \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -2 & -3 & 0 \end{vmatrix} \right\} = 7.5$$

The same result will be produced by taking the other route.