TEACHING PROBLEM SOLVING STRATEGIES IN THE 5 – 12 CURRICULUM
(Thank you George Polya)

GOAL

The students will learn several Problem Solving Strategies and how use them to solve non-traditional and traditional type problems. The main focus is to get students to **THIMK!** (I know it's supposed to be THI**N**K, but I just wanted to get your attention. I did. ☺)

OBJECTIVES

Upon completion of this unit, each student should:

- Know George Polya's four principles of Problem Solving
- Have an arsenal of Problem Solving Strategies
- Approach Problem Solving more creatively
- Attack the solution to problems using various strategies
- Acquire more confidence in using mathematics meaningfully

PREREQUISITES

The prerequisites for the students will vary. The teacher will need to read the examples and exercises to decide which problems are appropriate for your students and the level of mathematics that they understand. Most of these problems were originally written for elementary and middle school mathematics students. However, many of these problems are excellent for high school students also.

MATERIALS

- This document
- Calculators are encouraged (graphing or scientific is adequate)
- Option: Creative Problem Solving in School Mathematics by George Lenchner, 1983

SOURCES

- How To Solve It, George Polya, 1945
- Creative Problem Solving in School Mathematics, George Lenchner, 1983
- NCTM Principles and Standards, 2000
- Mathematical Reasoning for Elementary Teachers, Calvin T. Long and Duane W. DeTemple, 1996
- Intermediate Algebra and Geometry, Tom Reardon, 2001
- Problems Sets from Dr. G. Bradley Seager, Jr., Duquesne University, 2000
- Where ever else I can find good problems!

TEACHER BACKGROUND INFORMATION

> *"There is a poetry and beauty in mathematics and every student deserves to be taught by a person that shares that point of view."*
> *– Long and DeTemple*

Problem Solving is one of the five Process Standards of NCTM's Principles and Standards for School Mathematics 2000. The following is taken from pages 52 through 55 of that document.

Problem Solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and then be encouraged to reflect on their thinking.

By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. In everyday life and in the workplace, being a good problem solver can lead to great advantages. Problem solving is an integral part of all mathematics learning, and so it should not be an isolated part of the mathematics program. Problem solving in mathematics should involve all five content areas: Number and Operations, Algebra, Geometry, Measurement, and Data Analysis & Probability.

Problem Solving Standard

Instructional programs from prekindergarten through grade 12 should enable all students to:

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and in other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving

The teacher's role in choosing worthwhile problems and mathematical tasks is crucial. By analyzing and adapting a problem, anticipating the mathematical ideas that can be brought out by working on the problem, and anticipating students' questions, teachers can decide if particular problems will help to further their mathematical goals for the class. There are many, many problems that are interesting and fun but that may not lead to the development of the mathematical ideas that are important for a class at a particular time. Choosing problems wisely, and using and adapting problems from instructional materials, is a difficult part of teaching mathematics.

INTRODUCTION

PROBLEM SOLVING STRATEGIES FROM GEORGE POLYA

George Polya (1887 – 1985) was one of the most famous mathematics educators of the 20th century (so famous that you probably never even heard of him). Dr. Polya strongly believed that the skill of problem solving could and should be taught – it is not something that you are born with. He identifies four principles that form the basis for any serious attempt at problem solving:

1. **Understand the problem**
2. **Devise a plan**
3. **Carry out the plan**
4. **Look back (reflect)**

1. Understand the problem

- What are you asked to find out or show?
- Can you draw a picture or diagram to help you understand the problem?
- Can you restate the problem in your own words?
- Can you work out some numerical examples that would help make the problem more clear?

2. Devise a plan

A partial list of Problem Solving Strategies include:

Guess and check	*Solve a simpler problem*
Make an organized list	*Experiment*
Draw a picture or diagram	*Act it out*
Look for a pattern	*Work backwards*
Make a table	*Use deduction*
Use a variable	*Change your point of view*

3. Carry out the plan

- Carrying out the plan is usually easier than devising the plan
- Be patient – most problems are not solved quickly nor on the first attempt
- If a plan does not work immediately, be persistent
- Do not let yourself get discouraged
- If one strategy isn't working, try a different one

4. Look back (reflect)

- Does your answer make sense? Did you answer all of the questions?
- What did you learn by doing this?
- Could you have done this problem another way – maybe even an easier way?

PROCEDURE

The idea is to provide the students with several (12) different Problem Solving Strategies and examples of each. We will also supply a few exercises that encourage the student to use that particular Problem Solving Strategy (PSS).

Suggested Plan: Treat each one of these as a vignette. Present one Problem Solving Strategy and example for about 10 minutes as a class opener to augment the daily instructional plan. Then assign one problem for the following day in addition to the regular assignment. Present a different Strategy and example every few days, as it fits into the teacher's schedule. At the conclusion of the 12 Strategies, there will be some exercises that are "all mixed up", that is, the solutions require the use of any of the strategies that have been discussed, a combination of those strategies, or the students generate their own Strategy (Hurray! Success!) These exercises could be assigned at a rate of one or two per week, in addition to the teacher's regular assignments. The idea is "a little bit each day" and continuous spiraling of the different strategies.

Alternate Plan: Teach this as a unit. Do a few strategies and examples per day and assign the exercises that go along with those. At the conclusion of about four days of this, assign a problem or two every week as in the suggested plan.

ASSESSMENT

I do not recommend a full period test on just problem solving. That could be devastating. A few problems on a quiz or take home problems to be graded would be my suggestion. I would suggest that the explanations of the solution must be thorough and well-communicated in order to get full credit. Answers only without proper substantiation are worthless.

Quizzes given in pairs, triads, or groups of four may be an option also. Each student must write down the solution and explanation, however.

THE HEART OF THE MATTER

On the next several pages, you will encounter:

- A Problem Solving Strategy
- An example to illustrate that strategy
- Exercise(s) that use that particular strategy to solve it
- Teachers Notes and Solutions are included also that illustrate one or several ways to solve the problem.

DAY 0

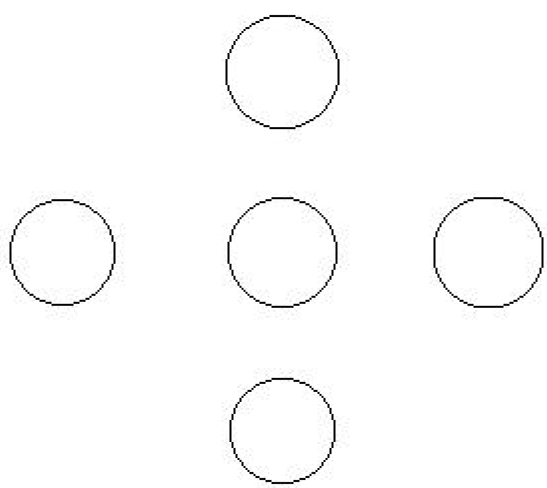
1. Copy page 3 of this document: PROBLEM SOLVING STRATEGIES FROM GEORGE POLYA and have it duplicated to give to each of your students. Also have the STUDENTS PROBLEMS duplicated for each student and distribute those. This "gift" includes the sample problems and exercises.

2. Discuss what Problem Solving is with your students (see page 2 of this document).

3. Discuss the page that lists the Problem Solving Strategies with your students. Tell them about good ol' George Polya, the Father of Problem Solving. Unfortunately he is dead now. Discuss his four principles for Problem Solving. See if students can come up with any other Problem Solving Strategies (PSS) than those that are listed on the page.

DAY 1 ***PSS 1*** ***GUESS AND CHECK***

EX. 1 Copy the figure below and place the digits 1, 2, 3, 4, and 5 in these circles so that the sums across (horizontally) and down (vertically) are the same. Is there more than one solution?



SOLUTION:
Emphasize Polya's four principles – especially on the first several examples, so that that procedure becomes part of what the student knows.
1st. Understand the problem. Have the students discuss it among themselves in their groups of 3, 4 or 5.
2nd. Devise a plan. Since we are emphasizing Guess and Check, that will be our plan.

3rd. Carry out the plan. It is best if you let the students generate the solutions. The teacher should just walk around the room and be the cheerleader, the encourager, the facilitator. If one solution is found, ask that the students try to find other(s).

Possible solutions:

	2			3			2	
1	3	5	2	1	5	1	5	4
	4			4			3	

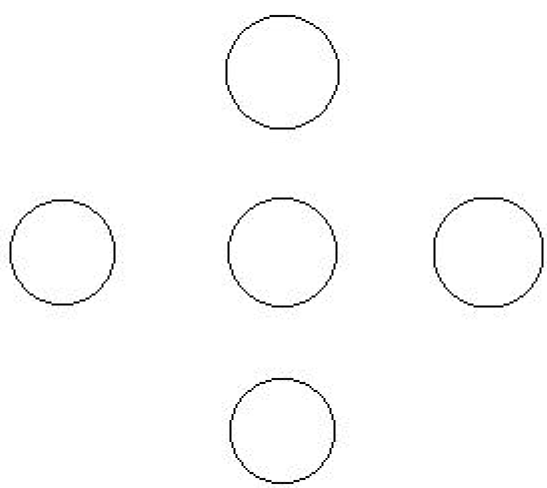
Things to discuss (it is best if the students tell you these things):

- Actually to check possible solutions, you don't have to add the number in the middle – you just need to check the sum of the two "outside" numbers.
- 2 cannot be in the middle, neither can 4. Ask the students do discuss why.

4th. Look back. Is there a better way? Are there other solutions?
Point out that "Guess and Check" is also referred to as "Trial and Error". However, I prefer to call this "Trial and Success", I mean, don't you want to keep trying until you get it right?

Below is an exercise to assign for the next day, which is also included in the STUDENT PROBLEMS.

1. Put the numbers 2, 3, 4, 5, and 6 in the circles to make the sum across and the sum down equal to 12. Are other solutions possible? List at least two, if possible.

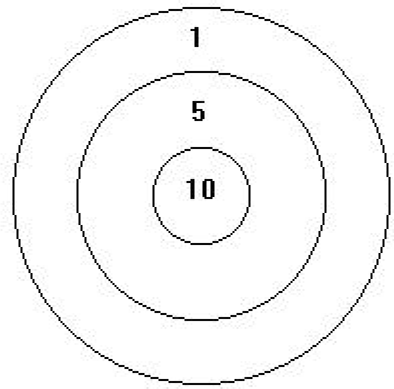


SOLUTION: One possibility 2
Other solutions possible. 3 4 5
Have students suggest those. 6

DAY 2 ***PSS 2*** ***MAKE AN ORGANIZED LIST***

EX. 2

Three darts hit this dart board and each scores a 1, 5, or 10. The total score is the sum of the scores for the three darts. There could be three 1's, two 1's and 5, one 5 and two 10's, And so on. How many different possible total scores could a person get with three darts?



SOLUTION:
1st. Understand the problem.
Gee, I hope so. ☺ But let students talk about it just to make sure.
2nd. Devise a plan. Again, it would be what we are studying: Make an organized or orderly list. Emphasize that it should be organized. If students just start throwing out any combinations, they are either going to list the same one twice or miss some possibilities altogether.
3rd. Carry out the plan.

# of 1's	# of 5's	# of 10's	Score
3	0	0	3
2	1	0	7
2	0	1	12
1	2	0	11
1	1	1	16
1	0	2	21
0	3	0	15
0	2	1	20
0	1	2	25
0	0	3	30

∴ There are 10 different possible scores.

4th. Look back. Point out the there are other ways to "order" the possibilities.

2. List the 4-digit numbers that can be written using each of 1, 3, 5, and 7 once and only once. Which strategy did you use?

SOLUTION:

1357	1735	3517	5137	5713	7315
1375	1753	3571	5173	5731	7351
1537	3157	3715	5317	7135	7513
1573	3175	3751	5371	7153	7531

24 possible 4-digit numbers.

DAY 3 ***PSS 3*** ***DRAW A DIAGRAM***

EX. 3 In a stock car race, the first five finishers in some order were a Ford, a Pontiac, a Chevrolet, a Buick, and a Dodge.

- The Ford finished seven seconds before the Chevrolet.
- The Pontiac finished six seconds after the Buick.
- The Dodge finished eight seconds after the Buick.
- The Chevrolet finished two seconds before the Pontiac.

In what order did the cars finish the race? What strategy did you use?

SOLUTION:

1st. Understand the problem.

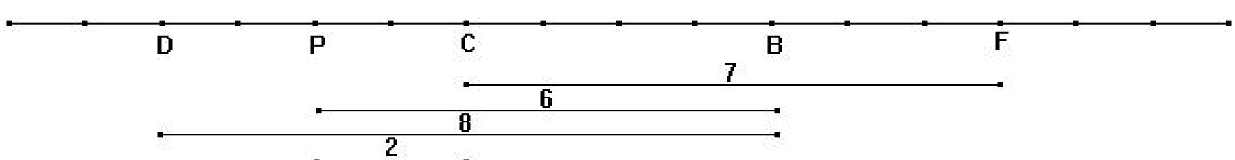
Let students discuss this.

2nd. Devise a plan.

We will choose to draw a diagram to be able to “see” how the cars finished.

3rd. Carry out the plan.

Make a line as shown below and start to place the cars relative to one another so that the clues given are satisfied. We are also using guess and check here.



The order is: Ford, Buick, Chevrolet, Pontiac, Dodge.

4th. Look back.

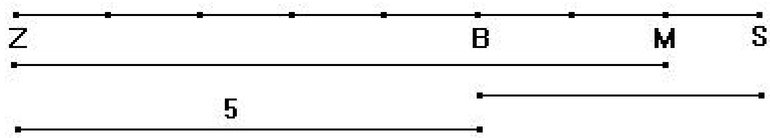
Not only do we have the order of the cars, but also how many seconds separated them.

Assign the following problem.
3. Four friends ran a race:

- Matt finished seven seconds ahead of Ziggy.
- Bailey finished three seconds behind Sam.
- Ziggy finished five seconds behind Bailey.

In what order did the friends finish the race?

SOLUTION:



The order was: Sam, Matt, Bailey, and Ziggy.

DAY 4 ***PSS 4*** ***MAKE A TABLE***

EX. 4 Pedar Soint has a special package for large groups to attend their amusement park: a flat fee of $20 and $6 per person. If a club has $100 to spend on admission, what is the most number of people who can attend?

SOLUTION:
1st. Understand the problem.
Students may need to discuss this a little before attempting to tackle the problem.
2nd. Devise a plan.
Make a table. But develop what should be in the table with the students. Let them assist how you make this table.
3rd. Carry out the plan.

# of people	Cost X $6	+ $20	Total fee	Result
10	60	20	80	Too low
15	90	20	110	Too high
13	78	20	98	Too low
14	84	20	104	Too high

Answer: At most, 13 people can attend for $100 and they will have $2 left over.
4th. Look back. Is there another way this could be done? Yes, guess and check (which is part of what we did). The difference is that we tried to do this in an orderly fashion – not just guess randomly. We tried to “surround” the solution.

Assign the following problem.
4. Stacey had 32 coins in a jar. Some of the coins were nickels, the others were dimes. The total value of the coins was $2.80. Find out how many of each coin there were in the jar. What problem solving strategy did you use?
SOLUTION: 8 nickels, 24 dimes

DAY 5 *PSS 5 LOOK FOR A PATTERN*

EX. 5 Continue these numerical sequences. Copy the problem and fill in the next three blanks in each part.

- 1, 4, 7, 10, 13, _____, _____, _____.
- 19, 20, 22, 25, 29, _____, _____ _____.
- 2, 6, 18, 54, _____, _____, ______.

SOLUTION:
1st. Understand the problem.
Students should realize that they are to be able to notice a pattern. It would be good if the pattern could be put into words
2nd. Devise a plan.
Look for a pattern.
3rd. Carry out the plan.

- 1, 4, 7, 10, 13, _____, _____, _____.

Hopefully the students will notice "add three to the previous term to generate the next term". The answer is 1, 4, 7, 10, 13, 16, 19, 22

- 19, 20, 22, 25, 29, _____, _____ _____.

The pattern is add one to the previous term, then add two to that term, then add three… The answer is 19, 20, 22, 25, 29, 34, 40, 47

- 2, 6, 18, 54, _____, _____, ______.

The pattern is to multiply the previous term by three to generate the next term.
The answer is 2, 6, 18, 54, 162, 486, 1458
4th. Look back.
Ask if students saw other patterns? Did they have different interpretations of the patterns?

Problems to assign:
5. Copy and continue the numerical sequences:
a) 3, 6, 9, 12, _____, _____, _____
b) 27, 23, 19, 15, 11, _____, _____, _____
c) 1, 4, 9, 16, 25, _____, _____, _____
d) 2, 3, 5, 7, 11, 13, _____, _____, _____

SOLUTION:

a) 3, 6, 9, 12, 15, 18, 21	multiples of three
b) 27, 23, 19, 15, 11, 7, 3, -1	subtract 4 from the previous term
c) 1, 4, 9, 16, 25, 36, 49, 64	perfect squares
d) 2, 3, 5, 7, 11, 13, 17, 19, 23	prime numbers

DAY 6 ***PSS 6*** ***SOLVE A SIMPLER PROBLEM***

Ex. 6 The houses on Main Street are numbered consecutively from 1 to 150. How many house numbers contain at least one digit 7?

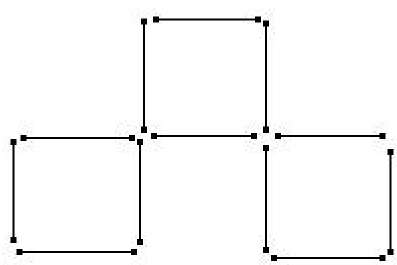
SOLUTION:
1st. Understand the problem.
Examples: 7, 73, 27, 117
2nd. Devise a plan.
Separate this into simpler problems.
3rd. Carry out the plan.
First consider: How many house numbers contain the digit 7 in the unit's place?
Answer: This occurs once in every set of 10 consecutive numbers. For houses numbered 1 to 150, there are 15 distinct sets of 10 consecutive numbers, so 15 house numbers contain the digit 7 in the unit's place.
Second consider: How many house numbers contain the digit 7 in the ten's place?
Answer: There are ten: 70 through 79. However we already counted the number 77 already so we can't count that twice.
Final answer: 24 house numbers contain at least one digit 7.
4th. Look back.
Are there other ways to do this? What if the house numbers are numbered up to 1000? Would it be much more work to count the ones that have at least one digit 7?

Assign:
6. The houses on Market Street are numbered consecutively from 1 to 150. How many house numbers contain at least one digit 4?

SOLUTION:
33 house numbers have at least one digit 4

DAY 7 ***PSS 7*** ***EXPERIMENT***

Ex. 7 The figure below shows twelve toothpicks arranged to form three squares. How can you form five squares by moving only three toothpicks?



SOLUTION:

1st. Understand the problem.

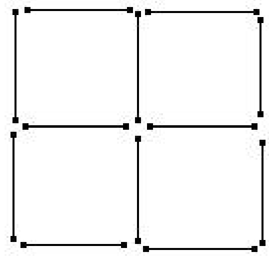
Students need to do this "hands on." Have toothpicks available for this in order to understand the problem.

2nd. Devise a plan.

Experiment. Even trial and error.

3rd. Carry out the plan.

This is a bit tricky. My answer is shown below:



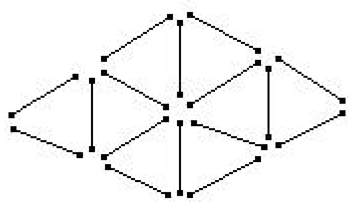
Notice that one of the squares is formed by the outer boundary of the arrangement. There was no requirement that each of the five squares must be congruent to each of the others (although must of us are locked into thinking that way ☺)

4th. Look back.

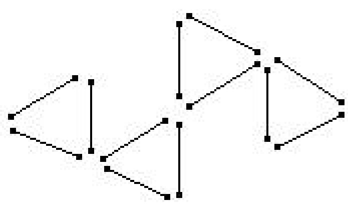
Are there other ways to do this?

Assign:

7. Sixteen toothpicks are arranged as shown. Remove four toothpicks so that only four congruent triangles remain.



SOLUTION:



DAY 8 ***PSS 8*** ***ACT IT OUT***

Ex. 8 Suppose that you buy a rare stamp for $15, sell it for $20, buy it back for $25, and finally sell it for $30. How much money did you make or lose in buying and selling this stamp?

SOLUTION:
1st. Understand the problem.
Note that the "most popular" wrong answer is that you make $15.
2nd. Devise a plan.
Act the situation out with another person.
3rd. Carry out the plan.
Give each of the two people slips of paper (or post-its) and have them make fake five-dollar bills – 10 of them for each. That is, each person starts with $50. Call them You and Friend. The Friend starts with the stamp.

You buy the stamp for $15 from your friend.	You $35	Friend $65
Your friend buys the stamp for $20.	You $55	Friend $45
You buy the stamp for $25	You $30	Friend $70
You friend buys the stamp for $30	You $60	Friend $40

Therefore your profit is $10.
4th. Look back.
Notice that You and the Friend's total is $100, as it should be.
Assign:
8. Suppose that you buy a rare stamp for $15, sell it for $20, buy it back for $22, and finally sell it for $30. How much money did you make or lose in buying and selling this stamp?
SOLUTION:
You made $13.

DAY 9 *PSS 9 WORK BACKWARDS*

Ex. 9 Ana gave Bill and Clare as much money as each had. Then Bill gave Ana and Clare as much money as each had. Then Clare gave Ana and Bill as much money as each had. Then each of the three people had $24. How much money did each have to begin with?
SOLUTION:
1st. Understand the problem.
This is a bit confusing and really needs to be discussed among the students.
2nd. Devise a plan.
We will work backwards here.
3rd. Carry out the plan.
There are four stages to this problem. I will number them 4 down to 1.

	Ana	Bill	Clare
4. Each has $24.	$24	$24	$24
3. Clare gives Ana and Bill as much money as each has.	$12	$12	$48
2. Bill gives Ana and Clare as much money	$6	$42	$24
1. Ana gives Bill and Clare as much money as each has.	$39	$21	$12

Answer: To begin with: Ana had $39, Bill had $21, and Clare had $12.

4th. Look back.
Is there another way to do this problem. Let me know if you find an easier way, please!

Assign:
9. I went into a store and spent half of my money and then $20 more. I went into a second store and spent half of my money and then $20 more. Then I had no money left. How much money did I have when I went into the first store?

SOLUTION: $120 to begin with.

DAY 10   *PSS 10*   *USE DEDUCTION*

Ex. 10 Three apples and two pears cost 78 cents. But two apples and three pears cost 82 cents. What is the total cost of one apple and one pear?

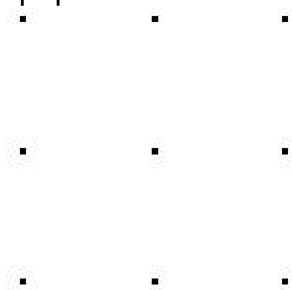
SOLUTION:
1st. Understand the problem.
This problem sounds fairly straightforward. However make sure that students notice that you are not required to find the cost of each apple and each pair.
2nd. Devise a plan.
Deduction is the process of reaching a conclusion through logic, or reasoning.
3rd. Carry out the plan.
By combining the two clues given, one can conclude that five apples and five pears cost 78 plus 82 cents, or 160 cents. Divide that by five and you can conclude that one apple and one pear costs 32 cents.
4th. Look back.
Certainly this problem could be done algebraically using two equations in two unknowns. But it would also require us to find the cost of each apple and each pear and we were not required to do all that. So don't. ☺

Assign:
10. Five oranges and a banana cost 87 cents. An orange and five bananas cost 99 cents. What is the total cost of two oranges and two bananas?

SOLUTION: 62 cents for two oranges and two bananas

DAY 11 ***PSS 11*** ***CHANGE YOUR POINT OF VIEW***

Ex. 11 Show how to draw four line segments through the nine dots shown below without lifting your pencil from the paper.



SOLUTION:

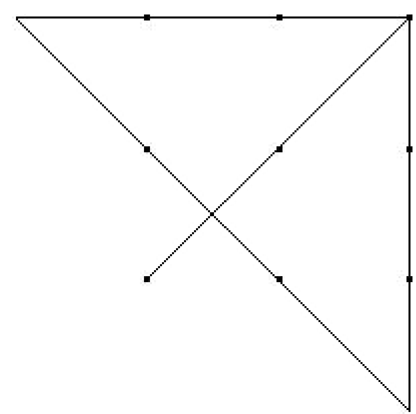
1st. Understand the problem.

Students need to discuss this.

2nd. Devise a plan.

Nearly everyone who attempts this problem becomes frustrated by assuming that the line segments must lie within the confines of the 3 by 3 array. But by removing this unnecessary restriction, it opens the door to the solution shown below.

3rd. Carry out the plan.



4th. Look back.

This is very hard for most people. But we want the students to consider other possibilities. Not just for this problem, but for any problem in which they get stuck.

Assign:

11. You have six sticks of equal length. Without altering the sticks in any way, show how to arrange them end-to-end to form four equilateral triangles.

SOLUTION:

It is not possible to solve this problem if you restrict yourself to a single plane surface. Arrange the sticks in a three-dimensional triangular pyramid. The four faces of the pyramid are the four equilateral triangles.

DAY 12 *PSS 12 USE A VARIABLE, WRITE AN EQUATION*

Ex. 12 Two apples weigh the same as a banana and a cherry. A banana weighs the same as nine cherries. How many cherries weigh the same as one apple?

SOLUTION:
1st. Understand the problem.
This is complicated since three quantities are being discussed.
2nd. Devise a plan.
We need to introduce three variables.
3rd. Carry out the plan.
A = the weight of an apple
B = the weight of a banana
C = the weight of a cherry

$2A = B + C$
$B = 9C$

Substituting: $2A = 9C + C$
$2A = 10C$
$A = 5C$

Answer: 5 Cherries weigh the same as 1 apple
4th. Look back.
Without algebra, this was pretty tough.

Assign:
12. Three pears weigh the same as a quince. A quince weighs as much as eighteen raspberries. How many raspberries weigh the same as a pear?

SOLUTION: Six raspberries weigh the same as one pear.

OTHER DOCUMENTS

For use in your classroom, please use the document:
STUDENT PROBLEMS USING PROBLEM SOLVING STRATEGIES.

The solutions to the problems above are found in the document:
PSS Hand solutions to student problems

Please feel free to add to the problems and continue to look for creative alternative ways to solve them. Do not be restricted by just the ones we talked about here.
Make George proud!
Also please pass along any ideas that you have to me.
Thanks,

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