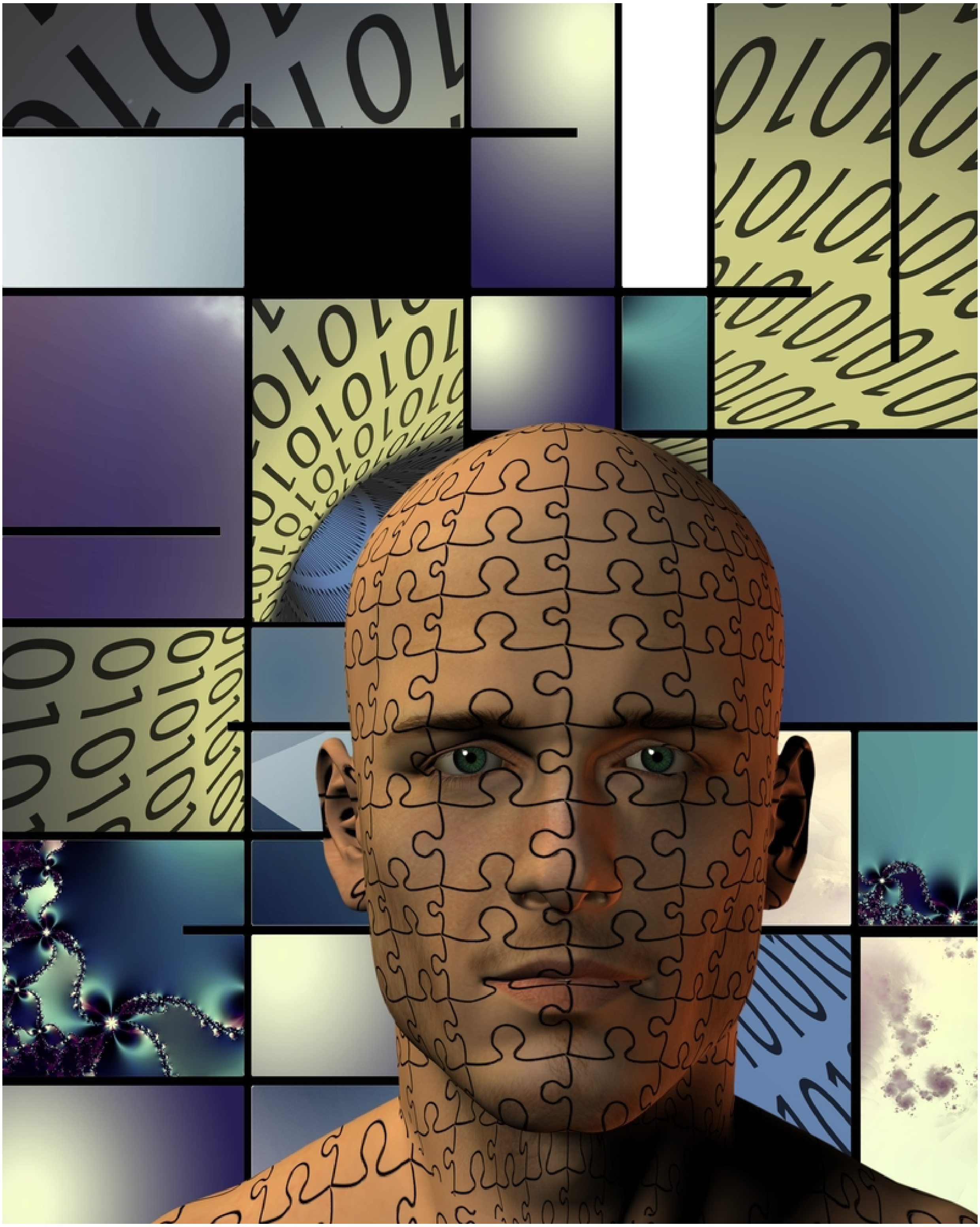
Pre-Algebra: A practical step-by-step approach

Gary S. Goldman, PhD



Dedicated to my 5th grade math teacher—Mr. Hivesly

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About the Author

I am 59 years old, but I still recall an event from 1964 that had a significant impact on my life. I was 10-years-old in 5th grade at Gardenhill Elementary School located in La Mirada, California. Two individuals in that grade especially influenced me back then: (1) my friend Jill Hagan, who slapped me in the face when I failed to vote for her as class president; and (2) my classroom teacher, Mr. Hivesly, who contributed greatly to my comprehension of math concepts. On the day that I received the slap (which occurred during recess), all students in Mr. Hivesly's classroom had voted for class president and had taken a math test that involved problems on simple division. Several days later, these tests had now all been graded. Students were in anticipation as the teacher returned each test paper with the letter grade shown prominently as a large letter at the top of the test page. I missed virtually every problem and could not stop focusing on the large "F" at the top of my test. I was near to tears, but was too embarrassed to cry in front of my classmates—so I was unusually quiet for the remainder of class trying to maintain a normal composure without any tears. For me, time seemed to be dragging on, until finally the bell rang all students were formally dismissed by the teacher. I purposely was slow in organizing my homework and to carry home—making certain that all students had exited the class when I finally approached my teacher. I think at this point, I did have tears in my eyes as I politely asked, "Mr. Hivesly, I just don't understand how to divide."

Mr. Hivesly discerned my tears (though I tried to fight them back by wiping my eyes as they teared up). He patiently took the time right then to show me an example problem on the blackboard: 43 divided by 8. I wiped back my tears and tried to focus as he took the chalk in his hands and kindly asked, "How many times does 8 go into 43?"

I told him, "I just don't know—I don't understand how to divide." He told me, "Then just make a guess, any guess."

I said, "How about 2?" He said, "Well, 2 times 8 is 16, that's too low, we're trying to get to 43, without going over that number. So guess again."

I said, "How about 4?" Mr. Hivesly said, "Well, 4 times 8 is 32—you're getting closer!"

I said, "Try 5." He said, "5 times 8 is 40. Now, 43 minus 40 is 3, so the answer is 5 remainder 3."

I said, "I think I am starting to understand, can we do another problem?"

Mr. Hivesly proceeded to show me two more division problems before I thanked him, dismissed myself, then walked home (without my "friend" Jill)—arriving just 10 minutes later than usual. But I learned a valuable lesson (at least in math—not about females) that lasted a lifetime! From that point on, I not only understood division; but I learned other math concepts often as quickly as they were presented to me. For the remainder of 5th grade, continuing through college, where I took Calculus, Vector Analysis, and other math courses, I always earned an "A". In fact, in an advanced course called Fourier Series and Laplace Transforms, I actually never missed a single test problem—math came easy to me. And in this book, I will try to make math easy for you!

It is not uncommon for a student in a math class to ask such questions as: "Why is this important? What benefit could this possibly have?" Where possible, in this book, math concepts will be put to use in real world problems and realistic applications.

In college, I had a double major: engineering and computer science. I received two Bachelor of Science degrees. Later, I received a Ph.D. (or doctorate) degree in Computer Science. In my employment, I developed the first microcomputer-based CAD (Computer Aided Drafting) System (prior to today's widely known AutoCAD system). I also patented a "Power Wheel"—an efficient motor in a wheel for electric cars. The CAD applications software allowed a user to create various engineering drawings. Writing that software required mastery of basic mathematics, algebra, geometry, and trigonometry. Later work involved some statistical methods and modeling of disease trends within a community. Math, to me, was not just variables and equations, it was a way to analyze and model real world applications. While some persons are satisfied recording data and tabulating figures, I have always been interested in understanding the underlying reasons for any trends or patterns that exist among the numbers.

Over the past 40+ years I have acquired real-world experience as a (1) designer and tester of electronic modules for Hughes Ground Systems, Fullerton, (2) contractor with a Class B General Contractor's License, (3) senior systems analyst in the Computer Aided Drafting (CAD) department at Holmes & Narver, Inc. (in Orange, CA), (4) computer consultant to CSUF faculty, staff, and students, (5) vice-president of Systems Development for Cascade Graphics Development (in Irvine, CA) producing the first microcomputer-based CAD system, (6) research analyst for the varicella surveillance project funded by the CDC, responsible for authoring medical journal papers on capture-recapture methods, varicella and herpes zoster, and other epidemiological subjects, (7) inventor of the "Power-Wheel" (U.S. Patent)—a highly efficient electric motor for vehicular transportation, (8) independent computer consultant working on a variety of projects ranging from the automation of the design of vertical cylindrical heaters to earth retainer walls with traffic loads (involving heuristic programming) and (9) substitute teacher for all subjects and all grades in three local school districts.

This real-world work experience combined with the answering of student math questions during the past twenty years as founder and director of Pearblossom Private School, Inc. has contributed to my ability to understand the math difficulties that students frequently encounter and direct students to a straightforward, practical, step-by-step procedure for solving problems.

It is highly recommended that you, the student, do every single exercise in this book. The repetition and variety of problems well help to re-enforce the concepts in each section so that you gain self-confidence and competency. I have always maintained that if a student received a poor grade in math, this was more a reflection on the qualities of the teacher—rather than the ability of the student. This course will provide you with algebra fundamentals that will help you to score well on most college entrance exams (such as S.A.T.) or placement tests. But it may also impact your ability to think logically and apply math concepts in other endeavors.

It is recommended that you *NOT* utilize a calculator for most problems in this book. As a prerequisite for this course, you should already be adept at reciting multiplication tables from memory and knowing the basic computational skills associated with addition, subtraction, multiplication, and division, as well as some basics working with fractions and decimals.

Sincerely,

Gary S. Goldman

Gary S. Goldman, Ph.D.

About this Book

This pre-algebra textbook (or ebook) is different from other math texts you have previously used in school where you were taught by a classroom teacher. This book, for the most part, *is* your teacher and tutor and as such, it will guide your step-by-step learning. Just as you should pay strict attention to a teacher's lecture in a classroom, you will need to carefully read each and every sentence of the book narrative, Your learning will be enhanced by doing the problems in the exercise presented at the end of each section.

You may not actually have read any of your previous math texts—perhaps only opening them to do various even- or odd-numbered problems assigned as homework by the teacher. But more is required of you as an independent learner studying pre-algebra. First, you must closely follow the logic or reason behind each and every step that is used to solve various example problems that are worked in extreme detail. Try to comprehend the specific solution to the problem as it is explained in the narrative. Then after considering all the example problems, try to grasp the general approach and method of solution that might be applicable to solve other similar problems.

Reinforce your mastery of each newly introduced concept by doing all the problems that follow each section. These problems provide further learning opportunities and reinforce the concepts that were previously introduced. Oftentimes, concepts previously learned in earlier sections of the book will need to be recalled and applied to achieve a solution to a current problem. By having to recall knowledge previously acquired, this serves as a sort of ongoing review. While some problems will be straightforward and vary little from the example problems discussed in the narrative, you will find other problems progressively more difficult to solve—perhaps requiring more innovative or challenging thinking that is often characteristic of the mindset of a mathematician or scientist.

Answers are provided to each and every question so that you can confirm each of your answers is correct. If your answer is incorrect, review the step-by-step procedures used to solve the example problems in the narrative and try to determine at what step you deviated from the correct procedure or method of solution.

At the conclusion of this course, how well you understand pre-algebra concepts and maintain pre-algebra skills will directly depend on how closely you have followed the above suggestions. To develop mathematical insight and gain an understanding of abstract concepts and their application takes time. If taken seriously, this course can profoundly influence your perception of things and equip you with analytical skills that may be useful in your future pursuits and endeavors—especially with respect to taking your next math course in Algebra I. If you master the concepts presented in this pre-algebra course, you will find Algebra I to be largely a review with the introduction of only a few new topics.

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Chapter 1: Expressions and Integers

Before you start... did you read the introductory section on page v, ***About the Book***? Please do not proceed further without going back and first reading that section.

Pre-Algebra Defined

Pre-algebra is a branch of mathematics marked chiefly by the use of symbols to represent numbers. Pre-algebra is a generalization of arithmetic.

In about the year 830 C.E., Mohammad ibn-Musa al-Khwarizmi of Baghdad (Iraq) wrote a book in Baghdad with the Arabic title *Hisab al-jabr w'al-muqabala*, (or *Science of the Reunion and the Opposition*). The reunion, or *al-jabar*, became what we presently call *algebra*, which deals not so much with numbers themselves (or arithmetic) but with solving equations that consist of operations (such as addition and multiplication) and relationships (such as equality).

1.1 Constants, Variables, Terms, Expressions, and Equations

I promised to make pre-algebra easy despite the fact that pre-algebra is somewhat more abstract than general math. In general math you learned very early that 2 + 2 equals 4. Any positive or negative whole number is called an **integer** (pronounced IN'TA-JEAR). So numbers such as -103, -12, 0, 8, 25 are integers—they are also called **constants** (but constants can also include fractions, mixed numbers, and decimals). Perhaps you did not realize it at the time, but in the process of performing the indicated addition or calculating the answer to two plus two, you were actually using one of the most basic concepts of pre-algebra—that of solving for a number or quantity that is unknown. When given the problem:

two plus two equals what number
or written another way, $2 + 2 = ?$

we are actually doing pre-algebra since we are trying to obtain an answer which is unknown or at least not given in the stated problem. The unknown value, in pre-algebra (and later in algebra), is called a ***variable*** and often some letter of the alphabet is used to represent an unknown value. In the above problem, if we select n as the unknown value or variable, we could write: $2 + 2 = n$. To solve for n you simply perform the addition (2 + 2) and obtain the result of 4. So we could say $n = 4$ in the equation $2 + 2 = n$.

One or more different variables can actually appear anywhere in an equation. An ***equation*** or equality, by definition, always contains an equal sign. Consider another related pre-algebra problem: two plus what number equals 4? If we represent the unknown number as the variable x, then we could write the equation:

$$2 + x = 4$$

If you told me that x is 2 in the above equation, you would be correct and at this point you have solved a typical, basic pre-algebra problem and so you are actually doing pre-algebra. Now was that difficult? Of course not!

If we had written simply $2 + x$ (without the equal 4), this is called an algebraic ***expression*.** We cannot know the value of a variable (such as x) that appears in an expression until it is set equal to some value or another expression. In other words, we cannot solve for the value of a variable that is shown only in an expression; but we can solve for the value of a variable once it appears within an equation. In mathematics, $2 + 2$ is called an arithmetic (air'ith-me'tic) expression that consists of two ***terms*** which happen to both be the same—the number 2—with a addition or subtraction operator connecting the two terms. Often, we call numbers ***constants***, since they are fixed and do not change like variables can. The expression $2 + x$ is an algebraic expression (since it contains one or more variables) and it also has two terms, one is the constant 2, the other is the variable x. The variable x represents an unknown number whose value we can only determine when the expression is set equal to some value. So, for example, if $2 + x = 20$, then we find that setting the value of x to 18 (or when $x = 18$) makes that equation true. If, instead, $2 + x = 1002$, then we find that $x = 1000$ satisfies the equation.

It is important to grasp the meanings of the vocabulary used in pre-algebra (and later algebra), so you should remember the difference between an ***expression*** and an ***equation***; you should also know that expressions are made up of terms that can be ***constants*** and ***variables***. ***Arithmetic expressions*** are comprised strictly of numeric (or number) terms while ***algebraic expressions*** contain one or more variables.

Before we continue our discussion, let's consider a problem (actually this is more of an experiment) that I think you may find intriguing.

Suppose I have a rather large, empty container with a lid. I open the container and fill it full of white marbles all of which are the same size. I actually know the exact number of marbles that I have poured into the container. My challenge to you is this: without counting all the marbles, which could be quite time-consuming, could you determine, or at least estimate, the total number of marbles in the container? In other words, if we let m be the total number of marbles in the container, what is your estimate for the value of m? Remember, you are not allowed to directly count the marbles one-by-one.

First, let's assume that all the marbles are uniform (the same size). So now what additional information would be helpful to solve this challenge and estimate the total number of marbles?

One idea might be to weigh the container full of marbles and then weigh a single marble. By dividing the container weight (with all the marbles inside) by the weight of a single marble, we could compute the total number of marbles assuming the weight of the container was negligible. In pre-algebra (and algebra), so far we have used x as the name of a variable. Variables may be labeled with different letters of the alphabet, so that y and z are other suitable names for variables. In some situations, many variables may be required and since there are only 26 different letters in the alphabet, it is indeed possible to run out of names if we limit the names of variables to a single letter. Another common practice with regard to naming variables is to use a single letter with a number immediately below and to the right of the number. Thus, variables can take the form a_0, or a_1, a_2, etc., where the 0, 1, and 2 are called the **subscripts** associated with the variable a. On some occasions it makes sense to use a letter with actually some type of word label as the subscript so as to clearly identify the meaning of the variable. Thus, variables can be named in various ways, so that all variables in the following list are valid variable names:

$$a,\ b,\ c,\ x,\ y,\ z,\ x_1,\ y_2,\ x_{356},\ n_{\text{total}},\ n_{\text{t}},\ v_i,\ v_f,\ v_{initial},\ v_{final}$$

We next provide an example of using variables by returning to our marble problem. If we knew the weight of the empty container, we could accurately determine the number of marbles by subtracting the weight of the empty container, call this weight $W_{container}$, from the combined weight of both the container and marbles, call this weight $W_{combined}$, to give the actual weight of the marbles in the container, call this weight $W_{marbles}$. This wordy statement could be written mathematically as follows:

$$W_{combined} - W_{container} = W_{marbles}$$

Let's quickly double check our formula by assigning some sample values to the variables. Let's say the empty container weighs 5 pounds, or $W_{container} = 5$. The combined weight of the marbles and container is 15 pounds, or $W_{combined} = 15$. Then, using pre-algebra, we next substitute the values for the corresponding variables in equation above and this yields:

$$15 - 5 = W_{marbles}$$

Thus, the weight of only the marbles (without the container) is 10 pounds (15 pounds minus 5 pounds). If we knew that each marble weighed 0.1 pound we could now accurately compute the total number of marbles, *m*, by dividing the weight of all the marbles by the weight of a single marble (let's call this W_{single}), or

$$m = \frac{W_{marbles}}{W_{single}}$$

Using the sample figures we have chosen, the total number of marbles is 10 pounds and the weight of a single marble is 0.1 pound, *m* is given by

$$m = \frac{10}{0.1} = 100 \text{ marbles}$$

Unfortunately, there is no scale available to determine the weight of (a) the empty container, (b) empty container plus all marbles, and (c) a single marble; otherwise, our method of solution would have been a success.

Did you find the equation $W_{combined} - W_{container} = W_{marbles}$ difficult to understand—that the weight of just the marbles is equal to the combined weight of the container and marbles minus the weight of the container?

If you found this circumstance difficult to visualize, here is another example that you might be more familiar with. Let's say you want to determine your actual weight. You know your clothes weigh 5 pounds, and we call this $W_{clothes}$. You weigh yourself on a scale with your clothes on. We will call this $W_{clothes\ on}$. Then your actual weight without any clothes on, W_{actual}, is given by the equation: $W_{actual} = W_{clothes\ on} - W_{clothes}$
In other words, your actual weight is simply equal to your weight with your clothes on minus the weight of the clothes you are wearing. Now that wasn't too difficult, was it?

Now let's return to the challenge of determining the number of marbles in the container. Since no scale is available (or at least the problem does not allow you to use one), how can you determine the total number of marbles without counting each and every one? Having some experience in probability and statistics, I provide you with a special formula that is derived from a well-known technique called "capture-recapture". Here is the novel way you can determine the total marble count—without counting all the marbles. Follow these steps:

1. Scoop up a handful of white marbles from the container and count these marbles. Using a permanent black marker (you have permission to write on the marbles), draw a circle on each marble that you scooped up. We will call the number of scooped and marked marbles B.

2. Place these marked marbles back into the container and mix the container of marbles thoroughly so that all of the marked marbles become essentially evenly distributed throughout the other unmarked marbles in the container.

3. Now scoop up another handful of marbles from the container and count these marbles. We will call the number of marbles in this second scoop N. However, you will see at least some marbles that we previously marked in Step #1. Let's call the number of marked marbles that exist in this scoop a.

Then an estimate for the total number of marbles in the container, m, can be found by using the following capture-recapture formula:

$$m = \frac{B \bullet N}{a}$$

The dot between B and N indicates multiplication or B times N. In arithmetic, often a small "x" is used to designate multiplication; however, B x N in pre-algebra can be confusing since the letter x (in italics) is often used as the name of a variable. In pre-algebra, when multiplying variables, generally no sign, not even the dot, is used between the variables. So, from this point forward, we will simply type, for example, BN, to indicate B times N. At this point, for the purposes of this discussion, it does not matter if you understand how this formula or equation was derived (in fact, you are not expected to know this). The above formula is provided simply for your use to see how numeric values can be substituted for the various variables, B, N, and a to obtain the estimated total number of marbles, m. We will return to the marble problem in the next **Section 1.2**.

Some Conventions used in Pre-Algebra (and Algebra)

In pre-algebra, we may encounter a numeric constant multiplied by a variable, such as 2 times x. The convention is to simply write $2x$, where the constant is written first, followed immediately by the variable. The 2 in the term $2x$, is also called a coefficient. Just as no multiplication sign (or operator) is directly written when two variables are multiplied together, no multiplication sign is used between a constant (also called a coefficient) and a variable. So it is not necessary to write $2 \bullet x$. So, $2x$ is proper for 2 times x. As a rule, we never write the variable first and then the coefficient—so that $x2$ is not proper. Please note also that in pre-algebra we would never write 2 x x for $2x$, since writing the multiplication operator (x) can be confusing. In the special case where we have the constant (or coefficient) 1 times the variable x, we do not write $1x$; instead, we just write x since the 1 is always implied. This is logical since one times any number is that number (or $1x = x$). Another special case occurs if we have 0 times x. Since 0 times any number (or variable) is zero (or $0x = 0$), we never write $0x$, since this simplifies to 0.

It is noteworthy that 2 times the sum of x and 1 is also written without using a multiplication operator (sign), so that we write $2(x+1)$ instead of using a dot such as in $2•(x+1)$ or instead of using an "x" as the multiplication sign as in 2 x $(x+1)$. Please note that $2(x + 1) = 2x + 2$ using the distributive property of multiplication.

Definitions of key terms

Term	Definition
constant	A fixed number that may be expressed as an integer (whole number), fraction, or real number. Pi (π) is also considered to be a constant that is approximately equal to 3.1415926535… Usually a positive constant, such as +51.4 is written as simply 51.4 (without the plus sign).
coefficient	A constant that is multiplied times a variable, for example, given the term $5x$, 5 is the coefficient

variable	A letter used to represent an unknown value in an expression or equation. Variables may also be represented by a letter and a numerical subscript or alphabetical or word subscript. These variables with subscripts 1 to 1000: $a_1, a_2, a_3, \ldots, a_{1000}$ are commonly used in computer programs that can easily handle a large number of variables. For example, instead of having to add 26 different variable names such as $a + b + c + d + e + \ldots + z$, a computer program can use an array structure and easily perform the addition of 26 variables named $a_1, a_2, \ldots a_{26}$. For a computer, it is just as easy to add 10,000 different variables as it is 26. Here is just a 3-line computer program written in the programming language called "PHP" that performs the sum of 1000 different variables (that are stored in what is called an array—a[1], a[2], … a[1000])—you do not need to understand this.
	`sum = 0;`
	`for (i=1; i<=1000; i++)`
	`sum = sum + a[i];`

term	A term may be either numeric (arithmetic) or variable (algebraic). The combination of two or more terms with an operator (such as addition or subtraction) is referred to as an expression.
expression	An expression is comprised of two or more terms combined through use of an operator. An expression that contains only numbers and no variables is called an arithmetic expression. An expression that contains at least one variable is called a variable expression.
equation	An equation contains an equal sign and may have one or more terms or expressions on either side of the equal sign.

Some Conventions in Algebra		Some Conventions in Basic Math	
Improper	**Correct**	**Improper**	**Correct**
$0x$	0	123.400000000	123.4
$1x$ $x7$	x $7x$	000.1234	0.1234 or .1234
$3•x$ or 3 x x	$3x$	0012.34	12.34
$5•(x + 3)$	$5(x + 3) = 5x + 15$	12.00034000	12.00034
$x•y$	xy	1/-2	-1/2 or $-\frac{1}{2}$
$3\frac{3}{5}x$	$\frac{18}{5}x$ or $\frac{18x}{5}$		

Note: $1/2x$ is ambiguous since it can indicate either $\frac{1}{2x}$ or $\frac{1}{2}x$ (which is the same as $\frac{x}{2}$).

Exercise 1.1

In problems 1–10 below, identify whether the term is a

A. constant
B. variable

1. x
2. 25
3. -25.2
4. a_1
5. W_{total}
6. W_t
7. ½
8. π (which is approx. 3.14…)
9. z
10. 0 (this is a zero, not the letter "O")

In problems 11–20, write the algebraic expression corresponding to each word expression.

11. 7 times a number x
12. a number x times one
13. the product of a number x and zero
14. 3 less than a number x (see note below)
15. a number x minus 5
16. the sum of 5 and a number x
17. a number x increased by 10
18. four times a number x decreased by 2
19. six times the sum of a number x and 4
20. 3 divided by a number x

In problems 21–25 below, evaluate each term or expression. Use $x = 5$ and $y = 7$ as necessary and simplify your answer by combining any and all like terms.

21. $5x$
22. x
23. $8 - x$
24. $x + x$
25. $100/x$ (or $100 \div x$ or $\frac{100}{x}$)
26. $\frac{3}{x}$
27. $x - 1$
28. xy
29. $\frac{x}{y}$
30. $\frac{y}{x}$
31. $3x + 2y$
32. $4(2x + 5y)$
33. $2(3 + 4x + 5y)$
34. $\frac{5y}{x}$
35. $x + \frac{x}{y}$

Note: In problem 14, some students will incorrectly answer $3 - x$ as if the stated problem were 3 minus a number x. But thinking about the problem more carefully, "3 less than a number x" means that we start with a number x and want 3 less; so the correct answer is $x - 3$.

1.2 Applying Capture-Recapture Methods

To solve the marble problem presented earlier, let's say that in your first scoop, you count 20 marbles, mark these, and return them to the container which is thoroughly mixed. In your second scoop, you count 30 marbles, 6 of which are marked. Thus, we have the values for the variables in the equation $m = \frac{BN}{a}$ given as $B = 20$ (the number marked in the first scoop), and $N = 30$ (the number in the second scoop), and $a = 6$ (the number in the second scoop that had a mark). We compute (actually estimate) the total number of marbles by substituting these values into the corresponding variables given in the above equation:

$$m = \frac{(20\, marbles)(30\, marbles)}{6\, marbles}$$

After some simplifying, we have the result:

$$m = \frac{600\, marbles^2}{6\, marbles} = 100 \text{ marbles}$$

It turns out that my container did indeed have exactly 100 marbles!

<u>The remaining narrative in Section 1.2 is entirely optional.</u>

While this section is optional, it is intended to give you a feel for how algebra is applied in the real world and demonstrate the various thought processes that characterize a scientific investigation.

Note in the calculation of the number of marbles above, that when units of "marbles^2" is divided by units of "marbles", the resulting unit is marbles. This observation is part of what is called dimensional analysis. While it is important to obtain a correct numerical result, it is equally important that the units or dimensions of the result be correctly identified. Other values for the variables B, N, and a that also yield 100 marbles when substituted into the equation ($m = BN/a$) are presented in Table 1.2a below. In the table, case No. 4 shows the values of the variables that we used above.

Table 1.2a Capture-Recapture results for various theoretical values of variables *B*, *N*, *a*

Case No.	***B*** *(number of marbles on 1st scoop)*	***N*** *(number of marbles on 2nd scoop)*	***a*** *(number of marked marbles on 2nd scoop)*	***m = BN/a*** *(total estimated number of marbles)*
1	10	20	2	100
2	10	30	3	100
3	20	20	4	100
<u>4</u>	**<u>20</u>**	**<u>30</u>**	**<u>6</u>**	**<u>100</u>**
5	25	20	5	100
6	25	24	6	100
7	50	30	15	100

Notice, that at least in theory, it does not matter how many marbles are selected in the first and second scoops. The chart shows (assuming the marked marbles are well mixed with the other marbles in the container) that the estimated 100 marbles is obtained whether or not your first scoop consists of 10 marbles or 50 marbles (or any values in between). The same is true for the 2nd scoop of marbles.

If we were to actually perform the experiment with marbles in the real world, it would not be unrealistic to obtain values such as those shown in Table 1.2b below. Notice in Trial No. 1, we obtain B=21, N=32, and a = 5, which yields an estimated value of m=134, which is higher than the actual number of 100 marbles in the container. In Trial No. 2, we obtain B=24, N=26, a=7, which yields an estimated value of m=89, which is lower than the actual number. After performing the experiment two times, the average the number of estimated total marbles is 112 [(134 + 89)/2]. The average of multiple trials will tend to approach the actual number of marbles in the container.

Table 1.2b Capture-Recapture results for various actual values of variables *B*, *N*, and *a*

Trial No.	***B*** *(number of marbles on 1st scoop)*	***N*** *(number of marbles on 2nd scoop)*	***a*** *(number of marked marbles on 2nd scoop)*	***m = BN/a*** *(total estimated number of marbles)*
1	21	32	5	134
2	24	26	7	89
Average				112

It is critical that the marked marbles are thoroughly mixed inside the container following the first selection. If the marbles are not thoroughly mixed, this would result in bias and could make the estimate (based on the capture-recapture formula) invalid. Bias would occur especially if all the marbles in the first selection were placed together at the very bottom of the container where perhaps just a few marked marbles are scooped up on the recapture or second scoop. The capture-recapture estimate in this case would be unusable and much too high. In fact, if no marked marbles were selected, it would not be possible to apply the formula with a zero denominator (or a = 0), since the denominator would be undefined. Whereas, if all the drawn marbles were placed near the top of the container, this would result in a capture-recapture estimate that would be biased too low, since many of the marked marbles might be scooped up or recaptured.

The capture-recapture technique is gradually becoming more widely used in different fields. Initially, it was used to estimate the population of specific animals in a specific region. First, animals would be trapped and marked, then set free back into the study region. Next, animals in the study region would be captured again. It was then possible to estimate the total number of animals in the study region—similar to what we did previously in the marble experiment. It is assumed that animals did not die (or travel outside the study area) or were not born (or invade the study area from outside the boundaries) between the capture and re-capture events. If animals such as fish were caught and released, it is assumed that those fish that were caught the first time would have the same likelihood of being caught a second time. This assumption does not always hold true, however, because some fish remember how they were caught and are not likely to return to be captured a second time because they are "trap shy". More recently, capture-recapture has been used in population census studies and in the field of epidemiology (the study of the transmission of disease within a community).

In general, it is always good to consider the assumptions and any limitations or biases that are associated with calculations or estimates involving real world situations. Also, probability and statistics, subjects beyond the scope of pre-algebra, are often utilized to define a range of likely or expected values, rather than reporting a single estimate or value. In our marble experiment, for example, the larger the scoops of marbles that we select from the container, the greater the accuracy of our estimate. For example, with only 100 marbles in the container and selecting scoops of 20 and 30 marbles, our estimate of the total marbles could be as much as twice the actual number in the container. Whereas, if there were 1000 marbles in the container and we selected larger scoops of say 200 to 300 marbles, our estimated number of marbles would differ from the actual number by less than 10%.

Why the Capture-Recapture Formula Works (Optional)

In the marble experiment, let's work backward to try to discern why our capture-recapture formula, $m = BN/a$, served as the estimate for the total number of marbles in the container.

I had 100 marbles in the container. You first scooped up 20 marbles, marked and returned them to the container. The container was thoroughly mixed so that the marked marbles were uniformly distributed among the other unmarked marbles. The probability of selecting a marked marble from the container was 20/100 or 1/5 (one out of five marbles). However, you did not know this probability because the total number of marbles was unknown to you.

Next, you selected a second scoop that consisted of 30 marbles. Since 1 out of 5 marbles is expected to be marked, we would expect (1/5)(30) or 6 marked marbles in this scoop, which, in fact, was the number of marked marbles that appeared in the second scoop. At this point you knew that the probability of selecting a marked marble from the marbles in your second scoop was 6/30 or 1/5 (or one out of five marbles). Due to the even distribution of unmarked marbles among the marked marbles in the container, we would expect the ratio (1/5) of marked to total marbles in the 2nd scoop to be the same ratio of marked to total marbles in the container. Since we marked 20 marbles from the 1st scoop, we could expect a total of 5•20 or 100 marbles in the container, since the total number of marbles is five times the number of marked marbles.

Let's now derive the formula. Since this 2nd scoop of marbles was from a larger set of marbles that was thoroughly mixed, we could expect that the ratio of the total marbles in the second scoop (N) to the marbles in the 2nd scoop that were marked (a) must be equal to the same ratio of the total marbles in the container (m) to those that were marked on the first scoop (B), or

$$\frac{m}{B} = \frac{N}{a}$$

To eliminate the variable B in the denominator on the left side of the equation so that we isolate the variable m (a technique that we will learn later), we multiply both sides of the equation by B, giving us a new equation that is equal to the previous equation (since we multiplied both sides by the same value, B): $B \bullet \frac{m}{B} = \frac{N}{a} \bullet B$

Since on the left side of the equation, the B in the numerator cancels the B in the denominator (or $B/B = 1$, leaving 1 times m which is the same as m), we obtain the formula in terms of m, the (estimated) total number of marbles in the container $m = \frac{BN}{a}$

Exercise 1.2

Solve the following word problems for the estimated size of the population, *m*, using the capture-recapture formula $m = \frac{BN}{a}$, where *B* is the number captured and marked in the first group, *N* is the number captured in the second group, of which *a* were found marked or the duplicates found in both groups. The raw number captured (or size of the population captured or reported), r, is given by the equation: $r = B + N - a$. Finally, the percentage of reporting completeness, *c*, is given by the equation: $c = 100r/m$.

1. In the city of Lancaster, CA, schools reported 500 children 1 to 9 years old that had chickenpox during 1995. During that same year, healthcare providers in that same city reported 850 children of the same age that had chickenpox, of which 100 were duplicate cases (reported by the schools also). Estimate the total number of children aged 1to 9 years old that had chickenpox in Lancaster, CA.

2. Using the data provided in question #1, what is the raw number of children with chickenpox reported to schools and healthcare providers?

3. Using the data provided in question #1 and the result calculated in #2, what is the percentage of reporting completeness rounded to the nearest whole number?

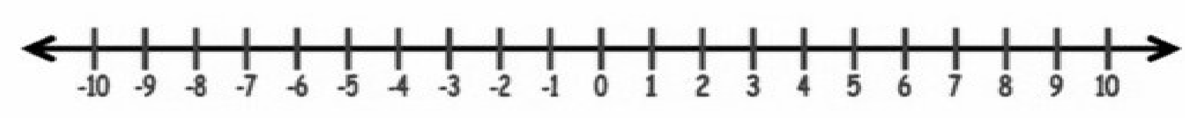
4. The Fish and Game Commission sent out a representative to determine the number of fish in Fin Lake. He set traps that caught 80 fish from the lake which were marked and then released back into the lake. Two days later, the traps caught 65 fish, of which 8 fish were marked. Estimate the total number of fish in Fin Lake.

5. Using the data provided in question #4, what is the raw number of fish captured by the Fish and Game Commission representative?

6. Using the data provided in question #4 and the result calculated in #5, what is the percentage of reporting completeness rounded to the nearest whole number?

1.3 Adding and Subtracting Integers

Integers are whole numbers that can be negative, positive, or zero. Negative integers are less than zero while positive integers are greater than zero. Thus, integers can be …, $^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2, 3, … and they are shown on the number line below:



As you move to the right on the number line, the numbers get larger. Conversely, as you move to the left on the number line, the numbers get smaller. Therefore $^{-}8$ is less than $^{-}6$ (or $^{-}8 < {}^{-}6$) and $^{-}8.5 < {}^{-}8.49$ while $^{+}8 > {}^{+}6$ and $^{+}8.5 > {}^{+}8.49$

Previously, in general math, you learned to add positive numbers to obtain their sum; thus, for example, 7 + 3 is 10. You also learned how to subtract two numbers, for example, 7 – 3 is 4. However, subtraction can be thought of as addition—the adding of a negative number—so that the subtraction 7 – 3 is equivalent to the addition 7 + ($^{-}3$).

Now let's consider an expression that you are perhaps less familiar with: $^{-}3$ + ($^{-}7$) or equivalently, $^{-}3$ – 7. When we encounter two negative numbers ($^{-}3$ and $^{-}7$), we first add the numbers ignoring the negative signs and compute 3 + 7 which is 10, but assign a negative sign to the result; thus,

$$^{-}3 - 7 = {}^{-}10.$$

Try to remember that just as the sum of two positive terms such as 3 + 7 is positive 10; the sum of two negative terms such as –3 – 7 is negative 10. Perhaps you recall that multiplication of two negative numbers yields a positive result; thus, ($^{-}3$)($^{-}7$) = 21. Do not confuse the positive result obtained when multiplying two negative numbers with the negative result obtained by adding two negative numbers. In this text, numbers that are expressed as – 7 (minus 7) or $^{-}7$ (negative 7), are considered identical. Also considered identical are positive terms such as these: 3, +3, and $^{+}3$. Our preference will be to write 3 instead of +3 or $^{+}3$.

Also, in elementary school, you learned to subtract numbers such as 7 – 3 which is 4. Now, consider another expression that you are perhaps less familiar with:

$$3 - 7$$

To simplify the above expression or any expression that contains a positive number and a negative number (since 3 – 7 is technically equal to 3 + $^{-}7$; where 3 is the positive number and 7 is the negative number), we notice that of the two numbers 3 and 7, 7 is the larger number and it is preceded by a minus sign (or negative sign). We first subtract the small number from the large number: 7 – 3 = 4 and then assign the result the same sign that is associated with the larger number. Since the larger number, 7, is preceded by a negative sign in the original problem, we assign a negative sign to the result to obtain 3 – 7 = $^{-}4$.

Table 1.3a provides a summary of the addition/subtraction rules we have just considered. Technically, addition and subtraction are closely related operations and become virtually indistinguishable from each other when negative numbers are encountered in sum and difference expressions. The importance of knowing addition/subtraction rules cannot be over emphasized. Properly applying these rules as the circumstance demands will be necessary to obtain correct solutions to the pre-algebra problems presented in upcoming chapters. Mastery of addition/subtraction is essential to making further progress in pre-algebra.

Table 1.3a Addition (or Subtraction) Rules

Case	Rule	Example	Result
Two positive numbers	Add both numbers and the result is positive.	$3 + 7$	10
One positive number and one negative number	Ignoring the sign on each number, subtract the lower number from higher number and assign the result from the above subtraction the sign that is associated with the higher number.	$3 – 7$ $^{-}7 + 3$	$^{-}4$
Two negative numbers	Add both numbers and the result is negative. Please note: – 7 can be referred to as minus 7 or negative 7; thus, we have the equivalent of “two negative numbers” in this example.	$^{-}3 – 7$	$^{-}10$

It is unfortunate, but some other texts and discussions of addition/subtraction complicate matters by presenting problems where addition and subtraction operators appear adjacent to the plus or minus sign of the second term being added or subtracted. In other words, addition/subtraction problems are expressed in forms such as these:

$$^{+}3 + {}^{+}7$$
$$^{-}3 + {}^{-}7$$
$$^{-}3 - {}^{+}7$$
$$^{+}3 - {}^{-}7$$

Each of these expressions above can be simplified by combining the addition or subtraction operator with the plus or minus sign of the adjacent number. An addition operator next to a positive sign (shown with yellow highlighting), can first be simplified to a single plus sign (also shown with green highlights) and then the terms can be combined according to the rules given previously, or

$$^{+}3 + {}^{+}7 = 3 + 7 = 10$$

An addition operator adjacent to a negative sign (shown with yellow highlighting), or subtraction operator adjacent to a positive sign, can be simplified to a single minus signand then the terms can be combined according the rules given previously, or

$$^{-}3 + {}^{-}7 = {}^{-}3 - 7 = {}^{-}10 \quad \text{or} \quad {}^{-}3 - {}^{+}7 = {}^{-}3 - 7 = {}^{-}10$$

Finally, a subtraction operator adjacent to a negative sign (shown with yellow highlighting) can be simplified to a plus sign, or

$$^{+}3 - {}^{-}7 = 3 + 7 = 10$$

Generalization of the three simplification rules we have just considered is summarized in Table 1.3b. Having simplified addition/subtraction expressions with redundant signs or operators, these reduced expressions can be evaluated by applying the three fundamental rules previously given in Table 1.3a.

Table 1.3b Simplification Rules for Addition/Subtraction

Case	Rule	Example	Reduction	Result
An addition operator next to a plus sign	Replace with +	$^{+}3 + {}^{+}7$	$3 + 7$	10
A subtraction operator next to a plus sign, or addition operator next to a negative sign	Replace with –	$^{+}3 + {}^{-}7$ $^{+}3 - {}^{+}7$	$3 - 7$	$^{-}4$
A subtraction operator next to a negative sign	Replace with +	$^{+}3 - {}^{-}7$	$3 + 7$	10

As a final observation, since addition is commutative, that is $a + b = b + a$, notice that if we switch the order of the terms $3 - 7$ and write $^{-}7 + 3$, the result is $^{-}4$ in both instances.

In this book, a negative seven (or minus seven) will sometimes appear as -7, $^{-}7$, or – 7. All of these forms are equivalent—even though the minus or negative sign appears slightly different in each case.

Exercise 1.3

In the problems 1 - 10, first apply the simplification rules for addition/subtraction to obtain a reduced expression, then evaluate the expression.

1. $^{+}2 + {}^{-}5$
2. $^{-}8 - {}^{-}3$
3. $^{-}8 + {}^{-}3$
4. $^{+}6 - {}^{+}1$
5. $^{-}6 + {}^{-}1$
6. $^{-}7 + {}^{+}2$
7. $^{-}7 - {}^{-}2$
8. $^{-}2 - {}^{+}7$
9. $^{+}4 + {}^{-}10$
10. $^{-}4 - {}^{-}10$

In problems 11 - 20, evaluate the expression.

11. $18 - 25$
12. $^{-}65 + 7$
13. $21 - 5$
14. $^{-}12 - 13$
15. $12 + 5$
16. $8 - 12$
17. $^{-}8 - 12$
18. $7 - 17$
19. $^{-}1 + 3$
20. $6 - 4$

In problems 21 – 30, evaluate the expression when $x = {}^{-}9$ and $y = 5$ as necessary.

21. $8 - x$
22. $8 + x$
23. $-x + 5$
24. $^{-}7 + x$
25. $-x + 9$
26. $x + 4$
27. $-y - x$
28. $-y + x$
29. $y + x$
30. $y - x$
31. $-2y - 3x$
32. $4y + 3x$
33. $-x + y$
34. $-5x - 9y$
35. $5x - 9y$

1.4 Multiplying and Dividing Integers

Early in general math you learned repeated addition of the same numeric term over and over can be expressed in terms of multiplication. For example, suppose you are in a room with 4 rows of chairs and in each row there are 5 chairs. You could compute the total chairs by adding the number of chairs in each row:

$$5 + 5 + 5 + 5 = 20$$

Or, as a shortcut, you could multiply 4 times 5 (sometimes written as 4 x 5 or 4 • 5), or

$$(4)(5) = 20$$

In general, if a term x occurs repeatedly n times in a sum expression, we can reduce the expression to:

$$n \cdot x = \underbrace{x + x + \ldots + x}_{n}$$

Notice that the multiplication of two positive numbers always yields a positive result. The multiplication of a positive number and a negative number always yields a negative result; and finally the multiplication of two negative numbers always yields a positive result. The sign of the result for division is similarly based on the signs of the numerator and denominator. A summary of these rules is given in Table 1.4 below.

Table 1.4 Determining the sign of a product (multiplication) or quotient (division)

Case[1]	**Sign of Result**[1]	**Division Expression**	**Multiplication Expression**	**Result**
2 positive numbers	+	12 ÷ 2	(3)(2)	6
1 positive and 1 negative number	–	12 ÷ (-2) -12 ÷ 2	(3)(-2) (-3)(2)	-6
2 negative numbers	+	-12 ÷ (-2)	(-3)(-2)	6

[1] **Note:** It is extremely important to memorize these three cases and the sign of the result. Only when one term is positive and one term is negative, do you obtain a negative result when multiplying or dividing two numbers.

The division operator may take on a variety of different forms. For example, suppose we have 8 divided by $x + 3$. This can be expressed in any of the three ways shown here:

1. $8 \div (x + 3)$
2. $\dfrac{8}{x+3}$
3. $8/(x + 3)$

We can similarly convert sums that contain the same repeated variable, such as

$$x + x + x + x$$

to $4x$. Noting that x is common to each term in the above expression, using the distributive property we can write $x(1 + 1 + 1 + 1) = 4x$. By definition, the numeric constant, or 4, in this example, that is multiplied times x is called a coefficient. A convention in pre-algebra is always to write the coefficient first then the variable, so that $x4$, while valid, is not standard practice.

Let's consider another expression that can be reduced by combining like (or similar) terms:

$$8 + x + 3x + 4x - 5y - 2y + y + 3z + z - 7$$

Recall that in the above expression that the x implies $1x$ (but we never write the one in front of a variable). As an intermediate step we can use the distributive property on like terms and write:

$$(8 - 7) + x(1 + 3 + 4) + y(-5 - 2 + 1) + z(3 + 1)$$

The simplified expression obtained by combining like terms thus becomes

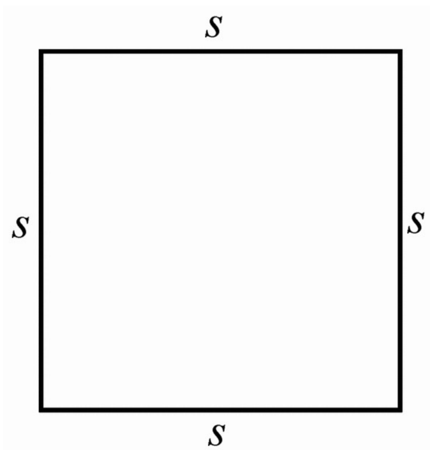
$$1 + 8x - 6y + 4z$$

Again, by definition, 8, –6, and 4 are each coefficients of the variables x, y, and z, respectively. Notice how we have reduced the original expression simply by combining like (or similar) terms—all the constants, then all the x terms, y terms, and z terms.

One common student error is to attempt to reduce the expression $2 + 5x$. The answer is ***not*** 7 or $7x$ since unlike terms such as constants and variables cannot be combined. Therefore, $2 + 5x$ is already in simplest form.

Also different variables cannot be combined, so that $2x + 7y$ is ***not*** 9; but instead, is already in simplest form since x and y are two different variables, and thus, not similar terms.

Let's put what we have learned above to some use. Let's say we have a square lot and desire a fence around the perimeter. If each side of the square has a length s**,** what is the perimeter or what is the lineal length of fencing needed?



The perimeter, p, of the square is given by the sum of each of the sides, or

$$p = s + s + s + s = 4s$$

Exercises 1.4

In problems 1–6 evaluate the given products.

1. (-8)(3)

2. (-4)(5)

3. (6)(2)

4. (7)(-8)

5. (-1)(0)

6. (-1)(-1)

In problems 7–12 evaluate the given quotients.

7. 18 ÷ (-3)

8. -15 ÷ (-2)

9. 3 ÷ (-4)

10. -27 ÷ 3

11. 15 ÷ (-1)

12. -30 ÷ 6

In problems 13–25 simplify the expression.

13. $3x + 4x$

14. $-9x + 11$

15. $3x - 4y$

16. $-7x - x$

17. $8 + 3x$

18. $-4y + y$

19. $3 - 12 + x - 6x$

20. $-4 + 2x - 3 - 4x$

1.5 Exponentiation

Just as multiplication corresponds to repeated addition,

$$n \cdot x = \underbrace{x + x + \ldots + x}_{n}$$

exponentiation corresponds to repeated multiplication:

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n}$$

In the above generalized formula for exponentiation, x is called the base and the superscript (raised letter) n is called the exponent. x^n is read "x raised to the n^{th} power" which is often abbreviated as "x to the n^{th}". When the exponent is 2, this is treated as a special case so that x^2 is commonly referred to as x squared (instead of x to the 2nd power). Interestingly, the area of a *square* with side x is x^2 and this is the origin of the term "x squared." Similarly, the exponent 3 is treated as a special case so that x^3 is commonly referred to as "x cubed" (instead of x to the 3rd power). The volume of a *cube* with side x is x^3 and this is the origin of the term "x cubed".

Now, as will often be the case in this book, let's apply what we have learned thus far and then consider several other specific problems that involve exponentiation along with their generalizations.

When the number 10 appears 4 times in repeated multiplication, we can write 10•10•10•10 which equals 10000. This expression can also be written as a power, 10^4, where 10 is the base and 4 is the exponent. Any number can be expressed in terms of powers of 10. For example, 32,586.49 (or more correctly $32{,}586.49_{10}$—where the 10 subscript designates the base of the number) can be expressed as $3 \cdot 10^4 + 2 \cdot 10^3 + 5 \cdot 10^2 + 8 \cdot 10^1 + 6 \cdot 10^0 + 4 \cdot 10^{-1} + 9 \cdot 10^{-2}$. Since you normally work in base 10, writing numbers that have the digits 0 through 9, it is usually assumed that all numbers are automatically base 10 unless otherwise specified using a different base. Next, consider a base 2 number (having the digits 0 and 1), for example 111101_2 (note that the 2 subscript to the right of the number designates the base of the number; 2 indicates the number is base 2 or binary). You can express this binary number in terms of powers of 2, or $1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$ (or $32 + 16 + 8 + 4 + 0 + 1 = 61_{10}$) .

We can also use exponents to simplify the repeated product of the variable x:

$$x \cdot x \cdot x \cdot x = x^4$$

Exponents may also be negative. By definition, $x^{-1} = \frac{1}{x}$, so that $2^{-1} = \frac{1}{2}$. In general, any base raised to a negative exponent, x^{-n} can be re-written with the same base having a positive exponent as $\frac{1}{x^n}$. So that we have

$$x^{-n} = \frac{1}{x^n}$$

Thus, 10^{-4} is equivalent to: $10^{-4} = \frac{1}{10^4} = \frac{1}{10 \cdot 10 \cdot 10 \cdot 10} = \frac{1}{10000} = 0.0001$

Similarly if we have a negative exponent in the denominator, $\frac{1}{x^{-n}}$, we can write the same base with a positive exponent in the numerator; thus, $\frac{1}{x^{-n}} = x^n$.

As another example, we can rewrite an expression containing terms with negative exponents, $\frac{x^{-4}y^2z^{-3}}{w^{-5}}$ in terms of only positive exponents: $\frac{y^2w^5}{x^4z^3}$.

Zero raised to a negative power would imply division by 0, so this case is considered undefined. Any number that is raised to the 1st power (or a base with an exponent of 1), is equal to that number. Any number raised to the 0 power is defined to be 1. There remains an ongoing controversy as to whether 0^0 should be considered 1.

Also note that since $3^2 \bullet 3^4 = (3\bullet3)(3\bullet3\bullet3\bullet3) = 3^6$, this leads to the property that $x^a \bullet x^b = x^{a+b}$. Also note that $3^4/3^2$ is equivalent to $\frac{3^4}{3^2} = \frac{3\bullet3\bullet3\bullet3}{3\bullet3} = 3\bullet3 = 3^2$. This leads to the property that $\frac{x^a}{x^b} = x^{a-b}$.

Finally, we have the case of $(3^2)^3$ which equals: $(3 \bullet 3)^3 = (3 \bullet 3)\,(3 \bullet 3)\,(3 \bullet 3) = 3^6$

This leads to the property that $(x^a)^b = x^{a\bullet b}$.

Now, let's consider a combination of these exponentiation properties by considering: $(4x^{-3}y^{-4})^{-2}$.

This is equal to $4^{-2}(x^{-3})^{-2}(y^{-4})^{-2} = \frac{1}{4^2}x^6y^8 = \frac{x^6y^8}{16}$ (using the last two formulas in Table 1.5 below).

Alternatively, $(4x^{-3}y^{-4})^{-2} = \frac{1}{(4x^{-3}y^{-4})^2} = \frac{1}{4^2x^{-3\bullet2}y^{-4\bullet2}} = \frac{1}{16x^{-6}y^{-8}} = \frac{x^6y^8}{16}$.

Table 1.5 Some exponentiation properties

Property	Examples	Restrictions
$x^{-1} = 1/x$	$5^{-1}=\frac{1}{5}$; $(\frac{5}{2}x^2)^{-1} = \frac{1}{\frac{5}{2}x^2} = \frac{2}{5x^2}$	for nonzero x
$x^{-n} = 1/x^n$	$(2x^{-2})^{-4} = \frac{1}{(2x^{-2})^4} = \frac{1}{2^4x^{-2\bullet4}} = \frac{1}{16x^{-8}} = \frac{x^8}{16}$; $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$	
$1/x^{-n} = x^n$	$\frac{1}{3x^{-4}} = \frac{x^4}{3}$	for positive n and nonzero x
$x^1 = x$		
$x^0 = 1$		Controversial: x may be zero
$x^{a+b} = x^a \bullet x^b$	$(x^{-3})(x^2) = x^{-3+2} = x^{-1} = 1/x$	When multiplying same bases, the exponents add.
$x^{a-b} = x^a/x^b$	$4x^{-3}/x^2 = 4x^{-3-2} = 4x^{-5} = \frac{4}{x^5}$	When dividing same bases, the exponents subtract.
$x^{a\bullet b} = (x^a)^b$	$(x^{-3})^{-2} = x^{-3(-2)} = x^6$	
$(xy)^n = x^ny^n$	$(5x)^{-3} = 5^{-3}x^{-3} = \frac{1}{5^3x^3} = \frac{1}{125x^3}$	

Exercise 1.5

In problems 1-5, evaluate and write the answer in simplest form.

1. $\frac{10^5}{10^2}$
2. $3^2 \bullet 3^3$
3. 5^1
4. 9^0
5. 7^{-1}

In problems 6-14 simplify the expression by using exponentiation properties.

6. $(3)(3)(3)(3)(3)$
7. $(-8)(-8)(-8)$
8. $x \bullet x \bullet x \bullet x$
9. $\frac{1}{y \bullet y \bullet y}$
10. $x^2 \bullet x^5$
11. $\frac{x^5}{x^2}$
12. $\frac{x^2}{x^5}$
13. $(5^2)^3$
14. $(x^3)^4$

In problems 15-20, rewrite the expression in terms of positive exponents.

15. $w^2x^{-3}y^{-4}z^{-5}$
16. $3x^{-4}$
17. $\frac{1}{x^{-4}}$
18. $\frac{a^2c^{-6}xd^{-3}}{b^{-4}e^{-5}}$
19. $\frac{1}{(5x^{-3})^2}$
20. $\frac{(3x^{-2}y^{-3})^{-3}}{z^{-4}}$

1.6 Rounding

Likely, in general math you learned the basic skill of rounding. So that 34.6 rounded to the nearest whole number (or to the one's place) is 35—since 34.6 is closer to 35 than it is to 34; or 372 rounded to the nearest 10 (or the ten's place) is 370—since 372 is closer to 370 than 380. What about rounding 3.4627 to the nearest thousandth. First we must be able to identify the places of the digits to the right of the decimal point: the first number (4) is the tenths digit, the second number (6) is the hundredths digit, the third number (2) is the thousandths digit, and the fourth digit (7) is the ten thousandths digit. If we are rounding to the nearest thousandth, we must inspect the digit immediately to the right of the thousandth digit. If that digit (immediately to the right of the thousandth digit) is 5 or greater, then we round the thousandth digit up 1; otherwise, it remains unchanged. In our example, since the digit to the right of the thousandth digit is 7, we round the thousandth digit up one to obtain 3.463; notice, that when rounding decimal digits we do not include any additional digits to the right of the digit that we are rounding. In the example that we just did, since we were rounding to the thousandths place, the final result contains three significant digits after the decimal point instead of 4 decimal digits that existed in the original number. A similar approach can be applied to rounding 824,397 to the nearest ten-thousands. The ten-thousands digit is 2, so we inspect the digit to the immediate right (4) and since it is less than 5, the ten-thousands digit remains unchanged so that the rounded number is 820,000.

Notice how the number, 83,527.19438 given in Table 1.6 is rounded to various places.

Table 1.6 Rounding example

Ten Thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-Thousandths	Hundred-Thousandths
8	3	5	2	7	**.**	1	9	4	3	8

8352**7**.19438 rounded to the nearest whole number (or ones): 83527 (since the digit to the right of the ones place is less than 5, the ones digit remains unchanged at 7).

835**2**7.19438 rounded to the nearest tens: 83530 (since the digit to the right of the tens place is 5 or greater, the tens digit is rounded up from the 2 to 3).

8**3**527.19438 rounded to the nearest thousands: 84000 (since the digit to the right of the thousands place is 5 or greater, the thousands digit 3 is rounded up to 4).

83527.**1**9438 rounded to the nearest tenths: 83527.2 (since the digit to the right of the tenths place is 5 or greater, the tenths digit is rounded up from 1 to 2).

83527.19**4**38 rounded to the nearest thousandths: 83527.194 (since the digit to the right of the thousandths place is less than 5, the thousandths digit remains unchanged at 4.

Notice that we always ignore any of the additional digits that may exist beyond the next digit to the right of the digit that we are rounding. Some students have difficulty with rounding 0.995 to the nearest hundredth. Notice that since the thousandth digit is a 5, we add one to the hundredth's digit. But since this digit is a 9, we change the 9 to a 0 and add or carry the one to the adjacent digit to the left, or tenths digit. But since that digit is also a 9, we change that 9 to a 0 and carry the one over to the next adjacent digit to left, or one's digit. Thus, 0.995 rounded to the nearest hundredth is 1.00; similarly, 0.995 rounded to the nearest tenth is 1.0.

Exercise 1.6

In problems 1-10, identify the digit in the ones place.

1. 18.4
2. 5.8
3. 26.4
4. 58.9
5. 19.6
6. 0.4999
7. 0.51
8. 9.99
9. 9.499
10. 14.45

In problems 11-20, identify the digit in the hundredths place.

11. 6.0585
12. 0.0001
13. 0.9995
14. 75.5555
15. 75.55549
16. 32.0130
17. 9.090909
18. 0.666667
19. 54.0004
20. 18.98719

In problems 21-30, round to the nearest whole number (or to the ones place).

21. 18.4
22. 5.8
23. 26.4
24. 58.9
25. 19.6
26. 0.4999
27. 0.51
28. 9.99
29. 9.499
30. 14.45

In problems 31-40, round to the nearest tenth.

31. 4.24
32. 3.96
33. 8.449
34. 25.05
35. 86.66
36. 41.917
37. 41.95
38. 0.0196
39. 0.045
40. 35.551

In problems 41-50, round to the nearest thousandth.

41. 6.0585
42. 0.0001
43. 0.9995
44. 75.5555
45. 75.55549
46. 32.0130
47. 9.090909
48. 0.666667
49. 54.0004
50. 18.98719

1.7 Order of Operations

In mathematics, there is a well-defined order in which operations are performed. First we evaluate any expressions in parentheses (sometimes brackets are also used), then perform any exponentiation, followed by any multiplication and division (in any order), and finally addition and subtraction (in any order).

Thus, using the above rules, consider this expression which involves both addition and multiplication:

$$5 + 3 \bullet 7.$$

We would start with the highest order operation that the expression contains. So, first we multiply and evaluate $3 \bullet 7$ to obtain 21. This leaves us with the reduced expression $5 + 21$ which evaluates to 26. Notice, that if we do not follow the order of operations, we might improperly first evaluate $5 + 3$ which is 8, leaving us with the reduced expression $8 \bullet 7$ which evaluates to 56—a wrong result.

Now, if the problem had been given as $(5 + 3) \bullet 7$, it would be entirely proper to give the addition operation the priority since expressions in parentheses are higher order than multiplication. In this case we first compute $5 + 3$ which is 8, then simplify the expression to $8 \bullet 7$ which evaluates to 56.

As another example, consider the expression:

$$8 + 3[5 - 2 \bullet (7+3)]$$

Properly employing the order of operations, we first perform the inner most expression that is found within the parentheses; thus, $7 + 3 = 10$. We now substitute this value into the original expression to obtain the simplified expression:

$$8 + 3[5 - 2 \bullet 10]$$

Next, looking within the brackets, we first must multiply before we subtract; thus, $2 \bullet 10 = 20$. Again we substitute this value into the previous expression to further simplify the expression again:

$$8 + 3[5 - 20]$$

Now we perform the subtraction because it occurs within brackets, so that $5 - 20$ is -15. Again substituting this value into the previous expression, we have reduced the original expression to

$$8 + 3(-15)$$

Note that at this point, we have substituted parentheses for the brackets since this does not alter the problem. Next, we must multiply before we add, so that $3(-15) = -45$, leaving us with the expression

$$8 + {}^{-}45$$

Applying what we learned earlier, the consecutive + and – signs can be rewritten as a single minus sign:

$$8 - 45 \qquad \text{(which is the same as } 8 - 45\text{)}$$

Since one number (8) is positive and one negative (45), we first subtract the small number from the larger number, or evaluate 45 – 8 = 37; then we assign the result the sign associated with the large number, so that the final result is -37.

Evaluate Expressions Based on this Order of Operations

Order of Operation	Description
1	Expressions in parentheses
2	Expressions that have exponents
3	Expressions that have multiplication and/or division in any order
4	Expressions that have addition and/or subtraction in any order

Exercise 1.7

Evaluate the expression, writing the answer in simplest form.

1. $3 + 8 \bullet 4$
2. $4(3 - 1)^2$
3. $8 \bullet 2 + 3 \bullet 6$
4. $9 \div 3 + 2 \bullet 5$
5. $6 + 32 \div 4^2$
6. $3[2 + (4 \bullet 4)^2]$
7. $8[7 + 6(4 - 2 \bullet 3^3)]$
8. $15 \div 3 + 6$
9. $15 \div (3 + 2)^2$
10. $-9 - 3^2$

1.8 Word Problems

Unfortunately, many students are intimidated by word problems when they should not be. The key to working a word problem is to first read over the entire problem and then: (1) extract the key information that is given, (2) assign variables to any quantities that are unknown, and (3) state what the problem is asking you to find. Often the word problem will provide clues as to how the given values and variables are related. Based on these relationships you will often be able to determine which operations are appropriate: addition, subtraction, multiplication, division, exponentiation, or a combination of these. In some word problems, drawing a picture or diagram can serve to clarify the details.

Let's take the case of the following word problem: Gary has a 5 pound bag of cashews. If Gary gives x pounds of cashews to Dora, then how many pounds of cashews does Gary have remaining?

So, let's break down this problem. We are given:

Gary—Initially has 5 pounds of cashews

Dora—Given x pounds of cashews

And we are asked to find:

The number of pounds of cashews Gary has remaining.

Sometimes, it is helpful to try specific numbers in an attempt to understand the mechanics or the way in which the various facts in the word problem are related. If Gary were to give 2 pounds of cashews to Dora, then Gary would have 5 – 2, or 3 pounds remaining; if Gary were to give 3 pounds of cashews to Dora, then Gary would have 5 – 3, or 2 pounds remaining. We might construct a table so that we can observe this relationship even more clearly:

Initial amount of cashews (pounds)	Amount of cashews given away (pounds)	Initial amount minus amount given away	Remaining amount of cashews (pounds)
5	0	5 – 0	5
5	1	5 – 1	4
5	2	5 – 2	3
5	3	5 – 3	2
5	4	5 – 4	1
5	5	5 – 5	0

When we use specific numbers or examples as we have done in the above table, we do so in the hopes of observing a pattern within the word problem. Did you notice that since Gary gives away some of his cashews, the operation that determines the amount of cashews that Gary has remaining involves subtraction? Our power of observation must be exercised one step further: instead of subtracting a specific amount of cashews as we have done in the table, we must now let the variable x represent the amount of cashews (in pounds) given away to Dora. Since we always subtracted the specific amount of cashews given away from the initial 5 pounds, we now generalize the procedure by subtracting the variable x pounds from the initial 5 pounds. Generalizing the procedure in terms of a variable in order to obtain the solution is perhaps the most difficult mental process in the solving of a word problem. Hopefully, with practice, you will not find it such a great leap to go from using specific numerical values to using a variable in place of those values. Thus, Gary has $\mathbf{5-\boldsymbol{x}}$ pounds remaining and that is the answer to the above word problem. We can check this answer by looking at the various limits (which are shown as the first and last rows of numbers in the table): suppose Gary gives no cashews (0 pounds) to Dora, then Gary has $5 - 0 = 5$, or the full 5 pound bag of cashews—which is the correct result; suppose Gary gives all 5 pounds of cashews to Dora, then $5 - 5 = 0$, or in this case, Gary has 0 pounds of cashews remaining—which is also correct.

Word problems can appear in a wide variety of forms. For example, suppose you have a stick 5 feet long and you cut off x feet. How long is the stick that you have remaining?

Do you recognize this problem as being similar to the previous cashew problem? Interestingly, the length of the stick after x feet is cut off is $\mathbf{5-\boldsymbol{x}}$, the very same expression we determined previously.

Exercise 1.8

Write the algebraic expression necessary to solve each of the following word problems:

1. Rusty makes 19 laps around the arena with her horse Sambra each day of the year. Write a variable expression for the number of laps that Rusty makes in d days.

2. Casondra has drawn 400 cartoons that she wishes to place in display books. If she can mount n cartoons in each display book, how many display books will she need to accommodate all her cartoons?

3. The cost of a new car, including the tax, is c. The tax that was paid was \$985. What is an expression for the cost of the car without the tax?

4. Johnny was sick with a highly contagious disease that spread to every person he contacted. At the end of 1 week, Johnny came in contact with 3 persons. At the end of the next week, each of these 3 persons came in contact with 3 other persons, who all became infected. And this pattern continued week after week. How many people became infected at the end of the nth week.

5. Dora had some change consisting of quarters, dimes, nickels, and pennies. How much change (in cents) did she have if she had q quarters, d dimes, n nickels, and p pennies?

6. Gary drove due north at a constant speed for 5 hours. If x is the speed in miles per hour, what is the distance (in miles) that Gary traveled.

7. Janet had three books in a stack, each having consecutive volume numbers. If the first book was volume n, what was the volume number of the second book?

8. A rectangle has a long side, or length, that measures 4 feet less than 6 times its shorter side, or width. What is the perimeter of the rectangle if the width is w?

9. What is the area of the rectangle in problem 8?

10. A building contractor would like to divide an 18-foot long piece of lumber into n equal pieces. What is the length of each piece?

Answers to Chapter 1 Exercises

Exercise 1.1

1. B; **2.** A; **3.** A; **4.** B; **5.** B; **6.** B; **7.** A; **8.** A; **9.** B; **10.** A; **11.** $7x$; **12.** x; **13.** 0; **14.** $x - 3$; **15.** $x - 5$; **16.** $5 + x$; **17.** $x + 10$; **18.** $4x - 2$; **19.** $6(x+4)$; **20.** $3/x$; **21.** 25; **22.** 5; **23.** 3; **24.** 10; **25.** 20; **26.** 3/5; **27.** 4; **28.** 35; **29.** 5/7; **30.** 1 2/5; **31.** 29; **32.** 180; **33.** 116; **34.** 7; **35.** 5 5/7

Exercise 1.2

1. (500 • 850)/100 = 4250 children aged 1 to 9 had chickenpox in Lancaster, CA during 1995; **2.** 500 + 850 – 100 = 1250; **3.** 100(1250)/4250 = 29%; **4.** (80 • 65)/8 = 650 fish are estimated to be in Fin Lake; **5.** 80 + 65 – 8 = 137; **6.** 100(137)/650 = 21%.

Exercise 1.3

1. 2 -5 = -3; **2.** -8 + 3 = –5; **3.** -8 – 3 = –11; **4.** 6 – 1 = 5; **5.** -6 – 1 = -7; **6.** -7 + 2 = -5; **7.** -7 + 2 = -5; **8.** -2–7 = -9; **9.** 4 – 10 = –6; **10.** -4 + 10 = 6; **11.** -7; **12.** -58; **13.** 16; **14.** -25; **15.** 17; **16.** -4; **17.** -20; **18.** -10; **19.** 2; **20.** 2; **21.** 17; **22.** -1; **23.** 14; **24.** -16; **25.** 18; **26.** -5; **27.** 4; **28.** -14; **29.** -4; **30.** 14; **31.** 17; **32.** -7; **33.** 14; **34.** 0; **35.** -90

Exercise 1.4

1. -24; **2.** -20; **3.** 12; **4.** -56; **5.** 0; **6.** 1; **7.** -6; **8.** 7½; **9.** -3/4; **10.** -9; **11.** -15; **12.** -5; **13.** $7x$; **14.** $-9x + 11$; **15.** $3x - 4y$; **16.** $-8x$; **17.** $8 + 3x$; **18.** $-3y$; **19.** $-9 - 5x$; **20.** $-7 - 2x$;

Exercise 1.5

1. $10^3 = 1000$; **2.** $3^5 = 243$; **3.** 5; **4.** 1; **5.** $\frac{1}{7}$; **6.** 3^5; **7.** $(-8)^3$; **8.** x^4; **9.** $\frac{1}{y^3}$ or y^{-3}; **10.** x^7; **11.** x^3; **12**. $\frac{1}{x^3}$ or x^{-3}; **13.** $5^6 = 15625$; **14.** x^{12}; **15.** $\frac{w^2}{x^3y^4z^5}$; **16.** $\frac{3}{x^4}$; **17.** x^4; **18.** $\frac{a^2b^4e^5x}{c^6d^3}$; **19.** $\frac{x^6}{25}$; **20.** $\frac{x^6y^9z^4}{27}$

Exercise 1.6

1. $1\underline{8}.4$; **2.** $\underline{5}.8$; **3.** $2\underline{6}.4$; **4.** $5\underline{8}.9$; **5.** $1\underline{9}.6$; **6.** $\underline{0}.4999$; **7.** $\underline{0}.51$; **8.** $\underline{9}.99$; **9.** $\underline{9}.499$; **10.** $1\underline{4}.45$; **11.** $6.0\underline{5}85$; **12.** $0.0\underline{0}01$; **13.** $0.9\underline{9}95$; **14.** $75.5\underline{5}55$; **15.** $75.5\underline{5}549$; **16.** $32.0\underline{1}30$; **17.** $9.0\underline{9}0909$; **18.** $0.6\underline{6}6667$; **19.** $54.0\underline{0}04$; **20.** $18.9\underline{8}719$; **21.** 18; **22.** 6; **23.** 26; **24.** 59; **25.** 20; **26.** 0; **27.** 1; **28.** 10; **29.** 9; **30.** 14; **31.** 4.2; **32.** 4.0; **33.** 8.4; **34.** 25.1; **35.** 86.7; **36.** 41.9; **37.** 42.0; **38.** 0.0; **39.** 0.0; **40.** 35.6; **41.** 6.059; **42.** 0.000; **43.** 1.000; **44.** 75.556; **45.** 75.555; **46.** 32.013; **47.** 9.091; **48.** 0.667; **49.** 54.000; **50.** 18.987;

Exercise 1.7

1. 35; **2.** 16; **3.** 34; **4.** 13; **5.** 8; **6.** 774; **7.** –2344; **8.** 11; **9.** 3/5 (or 0.6); **10.** –18

Exercise 1.8

1. Solution Strategy:
Given: 19 laps per day
Find: laps in d days
Next, we must find the relationship between the information that is given and the value that is requested. It is helpful to create a chart and look at the total laps after 1 day, 2 days, and 3 days:

Days	Laps each day	Days times laps each day	Total Laps
1	19	1 x 19	19
2	19	2 x 19 or 19 + 19	38
3	19	3 x 19 or 19 + 19 + 19	57

At this point, notice the pattern 1 x 19, 2 x 19, 3 x 19… in other words, the day is multiplied by the number of laps per day (which is fixed at 19 laps) to get the total laps. Thus the expression is

$$19d$$

2. Solution Strategy:
Given: 400 cartoons
Find: number of display books if each display book holds n cartoons

Next, we must find the relationship between the information that is given and the value that is requested. It is helpful to make a table showing the number of display books that would be required if they each held for example 100 cartoons, or if they each held 50 cartoons, of if they each held perhaps 25 cartoons:

No. of Total Cartoons	No. of Cartoons that each display book holds	Total number of display books	Operation Performed
400	100	4	400/100
400	50	8	400/50
400	25	16	400/25

Notice that the fewer the cartoons that a display book can hold, the more display books are needed for the 400 cartoons. In fact, the number of total cartoons divided by the no. of cartoons that each display book holds, yields the total number of display books needed; or

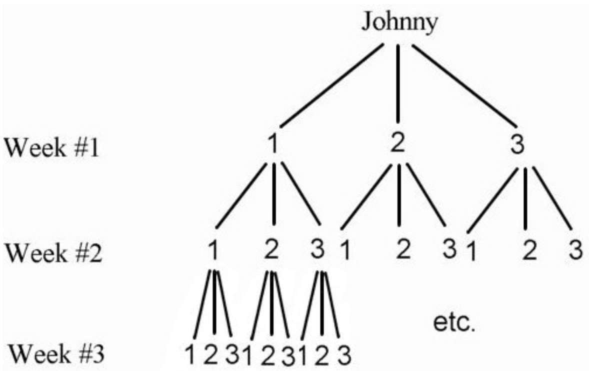
$$\frac{400}{n}$$

3. Solution Strategy:
Given: Cost of new car, including tax is c.
Tax is \$985
Find: Cost of car only.
Now we do not know what the tax rate is, but let's just pick a number and say \$20,985 is the cost for the new car plus tax. If we now subtract the amount of tax, \$985 from the total cost, \$20,985, we observe that the cost of the car alone was \$20,000 in this specific case. Now let's generalize the problem. Now, if instead of \$20,985, the cost of the car plus tax is c, then the cost of the car only is obtained by substituting the variable for the specific value we selected, or

$$c - 985$$

4. Solution Strategy:
Given: Johnny's disease spreads to every contact.
Johnny in week 1 spreads the disease to 3 persons.
Each of the 3 persons in week 2 spread the disease to 3 more persons (or 9 become Infected)
This pattern repeats
Find: Number of persons infected at the end of the n^{th} week.

This problem lends itself to drawing some type of diagram so that we can observe what is really happening in the above scenario: Since each person transmits the disease to three others, we can draw a tree diagram in which each row of the tree represents the next sequential week:



We see clearly from this figure that at the end of week 1, 3 are infected; at the end of week 2 a total of 9 are infected; and at the end of week 3 a total of 27 are infected. We observe that the numbers increase as a power of 3, so that at the end of weeks 1, 2, and 3, we have 3^1, 3^2, and 3^3 persons infected. Now we generalize this specific pattern to the number of people infected at the end of the n^{th} week:

$$3^n$$

5. Solution Strategy:
Given: Change consisting of q quarters, d dimes, n nickels, and p pennies
Find: Change in cents

Well, let's plug in some specific numbers for q, d, n, and p and say we had 1 of each coin. We would then have a value of change (in cents) of a quarter, dime, nickel, and penny, or

$$25 + 10 + 5 + 1$$

Notice the expression for the value of the change if we had two of each coin:

$$2(25) + 2(10) + 2(5) + 2(1) = 50 + 20 + 10 +2$$

We see a more general pattern developing where the value of the quarters is simply the number of quarters times the value of a quarter (25 cents); similarly the value of the dimes is simply the number of dimes times the value of a dime (10 cents), etc. Thus, the general expression becomes

$$25q + 10d + 5n + p$$

6. Solution Strategy:

Given: Driving constant speed for 5 hours. Traveling at a constant speed of x (miles per hour).
Find: Total distance traveled.

Again, we can make a table that shows the relationship between some specific speeds and the total distance traveled:

No. of Hours	Speed (miles/hour)	Total distance	Operation
5	50	250	5 x 50
5	40	200	5 x 40
5	30	150	5 x 30

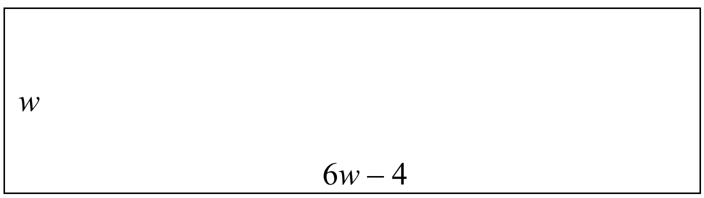
From the table, we see that the distance traveled is always 5 hours times the speed. If the speed is x, then the distance traveled will be $5x$

7. Solution Strategy:
Given: 3 book volumes numbered consecutively, starting with number n
Find: The volume number of the 2nd book in the stack

Let's plug in a sample volume number and say the books started with the number 1000; then the next book volume number would be 1001. Had the books started with the volume number 50; then the next book volume number would be 51. The pattern in this problem is established quickly, if the books start with volume n, then the next consecutively numbered volume is $n + 1$

8. Solution Strategy:
Given: width of rectangle is w
length of rectangle is $6w – 4$
Find: Perimeter of rectangle

This word problem lends itself to a drawing of the rectangle with the length and width labeled:



The perimeter of a rectangle is obtained by adding the lengths of each of the 4 sides or

$$w + (6w - 4) + w + (6w - 4)$$

Alternatively, we could use the formula that the perimeter is 2 times the length plus 2 times the width, or

$$2w + 2(6w\text{-}4) = 2w + 12w - 8$$

Regardless of which formula for perimeter we used, next we combine like terms (the terms containing the variable w and the terms that are constants) to obtain the simplified expression for the rectangle perimeter:

$$14w - 8$$

9. Solution Strategy:

Given: width of rectangle is w
Length of rectangle is $6w - 4$
Find: Area of rectangle

The area of a rectangle is defined to be the length times the width, thus we can write that the area is

$$w(6w - 4)$$

Using the distributive property, we have the simplified expression for the rectangle area:

$$6w^2 - 4w$$

10. Solution Strategy:

Given: 18-foot long piece of lumber
Divide lumber into n equal pieces
Find: Length of each piece.

Let's make a table and consider several specific cases. We will consider the length of each piece if we divide the lumber into 2 equal pieces, 3 equal pieces, and 6 equal pieces:

Length of full lumber (feet)	**No. of pieces that lumber is to be divided into**	**Length of each piece (feet)**	**Operation Performed**
18	2	9	$18 \div 2$
18	3	6	$18 \div 3$
18	6	3	$18 \div 6$

We notice that the full lumber length (18 feet) is divided by the number of pieces to get the length of each piece; thus, if the lumber is divided into n pieces, we have the length given as

$$18/n$$

Chapter 2: One-Step Equations and Decimals

It is now time to expand on what you have learned. Previously you evaluated expressions; now you will learn to solve algebraic equations. Previously you worked only with integers, now you will learn to work with decimal numbers.

To solve an algebraic equation, the goal is to always isolate the variable—this means to get the variable all by itself on one side of the equation. This can usually be accomplished by performing inverse operations that result in an equivalent equation that has the same solution as the original equation. This is demonstrated in sections 2.1 and 2.2 below.

2.1 Solving Equations using Addition and Subtraction

Consider the algebraic equation,

$$x + 2 = 4$$

Likely, without having to resort to any written calculations, doing mental math you know that if x is 2 (or when $x = 2$), the above equation is satisfied, since $2 + 2 = 4$. However, we will soon be doing problems that involve more than one step or operation to solve for the unknown variable. In order to prepare for these multiple step problems, you need to consider how you might more formally find the value of x. Our goal is always to isolate the variable x; however, observe that the constant 2 has been added to the variable. You can isolate the variable I by subtracting 2 from both sides of the equation (which is the opposite or inverse of adding). Thus, as a result of doing the opposite operation (or inverse operation), we have

$$x + 2 - 2 = 4 - 2$$

Now on the left side the equation, since $2 - 2 = 0$, only the variable x remains—which is to be expected since you did an inverse operation; on the right side of the equation we evaluate $4 - 2$ which is 2. Thus, after simplifying both sides of the equation we have solution,

$$x = 2$$

In general, if you have an algebraic equation of the form $x + \mathrm{a} = \mathrm{b}$, where a and b are constants, you can use the subtraction property of equality to form the new equivalent equation $x + \mathrm{a} - \mathrm{a} = \mathrm{b} - \mathrm{a}$. This effectively isolates the variable x, yielding the desired solution $x = \mathrm{b} - \mathrm{a}$.

Consider another algebraic equation—this time with the variable on the right side of the equation:

$$35 = x + 18$$

Again, the goal is to isolate the variable x. Notice that the variable appears on the right side of the equation and since the constant 18 is added to x, you must perform the inverse operation by subtracting 18 from each side. Showing the details of this step, the equivalent equation becomes

$$35 \mathbf{- 18} = x + 18 \mathbf{- 18}$$

Simplifying the left side of the equation, we have $35 - 18 = 17$; simplifying the right side of the equation we have the variable x. Thus, the desired solution to the initial equation is

$$17 = x$$

Actually, it is more conventional to write $x = 17$ as the solution (with the variable written first and set equal to the value that makes the initial equation true). In the above example, you could have considered the equation as $x + 18 = 35$ (instead of $35 = x + 18$) and the solution would have again been $x = 17$. In general, the fact that any equation of the form $a = b$ can be written as $b = a$ is called the symmetric property of equality. When you subtract the same number from both sides of the equation, you are using the subtraction property of equality.

Now, consider another equation,

$$x - 2 = 4$$

You observe that the constant 2 is subtracted from x on the left side of the equation. To solve for x, you must do the inverse operation by adding 2 to both sides of the equation, or

$$x - 2 + 2 = 4 + 2$$

After doing the inverse operation and simplifying, we have

$$x = 6$$

Let's try one more problem similar to the ones we have considered previously:

$$25 = x - 18$$

You observed that the variable x appears on the right side of the equation; but since 18 is subtracted from x, you must do the inverse by adding 18 to each side of the equation:

$$25 \mathbf{+ 18} = x - 18 \mathbf{+ 18}$$

Evaluating the left side of the equation we have $25 + 18 = 43$; evaluating the right side of the equation, since $-18 + 18 = 0$ which is expected since we performed an inverse operation, we have only the variable x remaining, thus

$$x = 43$$

In general, if you have an algebraic equation of the form $x - \mathrm{a} = \mathrm{b}$, where a and b are constants, you can use the addition property of equality to form the new equivalent equation

$$x - a \boldsymbol{+ a} = b \boldsymbol{+ a}.$$

This effectively isolates the variable x, yielding the desired solution

$$x = b + a.$$

Therefore, if you are given an equation where there is a constant that is added to or subtracted from the variable, remember to do the inverse operation and subtract or add this constant to both sides of the equation to effectively isolate the variable.

Exercise 2.1

Solve for the unknown variable in each equation.

1. $x - 7 = 10$
2. $x + 7 = 10$
3. $x - 7 = -10$
4. $x + 7 = -10$
5. $-4 = x + 3$
6. $-4 = x - 3$
7. $x + 5 = 17$
8. $x + 9 = 17$
9. $x + 25 = 17$
10. $x - 9 = 14$
11. $x - 9 = -14$
12. $x - 18 = -14$
13. $x - 18 = 14$
14. $35 = x + 12$
15. $35 = x - 12$
16. $35 = x + 42$
17. $-35 = x - 12$
18. $-35 = x + 42$
19. $x - 85 = 49$
20. $64 = x + 55$
21. $y - 39 = 42$
22. $a - 83 = 27$
23. $c + 37 = 35$
24. $w + 84 = 84$
25. $z - 56 = -56$
26. $-41 = \mathrm{f} + 30$
27. $68 = q - 77$
28. $-49 = r + 66$
29. $28 = n - 91$
30. $-27 = g + 27$

2.2 Solving Equations using Multiplication and Division

Just as addition and subtraction are inverse operations of each other, so multiplication and division are inverse operations of each other. Given the equation $2x = 4$, you can solve for the unknown variable x in one step using mental math, noting that $2 \cdot 2 = 4$, therefore, $x = 2$. Since later you will encounter problems with more than one step, it is important to learn a more formal approach to solving this problem. First, notice that the variable x appears on the left side of the equation $2x = 4$; however, the variable is multiplied by 2. In order to isolate the x, we must do the inverse operation and divide both sides of the equation by 2, thus we have

$$\frac{2x}{2} = \frac{4}{2}$$

On the left side of the equation, the expression $\frac{2x}{2}$ can be reduced to x, since $\frac{2}{2} = 1$ and $1x$ is always written simply as x. On the right side of the equation you can compute $\frac{4}{2} = 2$, so that we have the solution,

$$x = 2$$

Suppose you are given the equation,

$$3x = 27$$

Again, notice that the variable x is on the left side of the equation. Since the variable is multiplied by 3, you must isolate the variable by performing the inverse operation—using the division property of equality—and divide both sides of the equation by 3 to obtain,

$$\frac{3x}{3} = \frac{27}{3}$$

Simplifying both sides of the equation you have $x = 9$. Likewise, you would have obtained the exact same solution had the initial equation been $27 = 3x$ due to the symmetric property of equality that you learned earlier in Section 2.1.

Now consider one more example equation,

$$54 = -6x$$

Here you observe that the variable x is on the right side of the equation and it is multiplied by -6. To accomplish your objective of isolating the variable, you must do the inverse operation and divide each side of the equation by -6. Do not make the mistake of dividing by $+6$. Using the division property of equality, you now have this equation:

$$\frac{54}{-6} = \frac{-6x}{-6}$$

Simplifying the left side of the equation, you have $\frac{54}{-6} = -9$; simplifying the right side of the equation you have $\frac{-6x}{-6}$ which is equal to $1x$, which is written as simply x. Therefore, the solution is

$$x = -9$$

Please note that had you incorrectly chose the wrong inverse and divided by a positive 6, on the right side, you would have had the expression

$$\frac{-6x}{6}$$

which simplifies to $-x$ instead of the desired x which is achieved by dividing by the proper inverse (or diving by -6 since x is multiplied by -6).

In general, you can solve any equation of the form

$$ax = b$$

where x is the unknown multiplied by the constant "a", by performing the inverse operation and dividing by "a", or

$$\frac{ax}{a} = \frac{b}{a}$$

which simplifies to $x = \frac{b}{a}$.

Finally, we consider a new equation involving division:

$$\frac{x}{3} = 24$$

Remember, your goal is to isolate the variable x; however, the variable is divided by 3. So what is the next step? If you thought "do the inverse operation and multiply both sides of the equation by 3", you are correct. This step yields the equation

$$3 \bullet \frac{x}{3} = 24 \bullet 3$$

You now simplify the left side of the equation and only the variable x remains since $3x/3 = 1x$ or x (which was expected since you performed an inverse operation); simplifying the right side of the equation we have 24 times 3, or 72. Thus, the solution is

$$x = 72$$

Finally, consider the equation:

$$\frac{-x}{3} = 24$$

Since, we want to solve for x and x is multiplied by $\frac{-1}{3}$, in other words, $\frac{-x}{3} = -\frac{1}{3}x$, you perform the inverse operation by multiplying both sides of the equation by $-\frac{3}{1}$ (or simply -3) . This step yields:

$$(-3)\left(\frac{-x}{3}\right) = (24)(-3)$$

Performing the indicated multiplication, we have x on the left side of the equation and on the right side of the equation we have

$$x = 24 \bullet (-3) \quad \text{or} \quad x = -72$$

In general, you can solve any equation of the form

$$\frac{x}{a} = b$$

where x is the unknown divided by the constant "a", by performing the inverse operation and multiplying by "a", or

$$a \bullet \frac{x}{a} = b \bullet a$$

Notice the numerator and denominator cancel on the left side, which simplifies to $x = ab$.

Table 2.1 Summary of Inverse Operations used to Solve an Algebraic Equation

Sample Equation	Inverse Operation Applied to each side	Equivalent Equation	Result
$x + 5 = 35$	Subtract **5**	$x + 5 - \mathbf{5} = 35 - \mathbf{5}$	$x = 30$
$x - 5 = 35$	Add **5**	$x - 5 + \mathbf{5} = 35 + \mathbf{5}$	$x = 40$
$5x = 35$	Divide by **5**	$\frac{5x}{\mathbf{5}} = \frac{35}{\mathbf{5}}$	$x = 7$
$x/5 = 35$	Multiply by **5**	$\mathbf{5} \bullet x/5 = 35 \bullet \mathbf{5}$	$x = 175$
$-5x = 35$	Divide by **-5**	$\frac{-5x}{\mathbf{-5}} = \frac{35}{\mathbf{-5}}$	$x = -7$
$\frac{-x}{5} = 35$ $\frac{x}{-5} = 35$ $-\frac{x}{5} = 35$ $-\frac{1}{5}x = 35$	Each of the 4 equations are equivalent so we Multiply by **-5** (Note: the inverse of the coefficient of x in each of sample equations is $-\frac{5}{1}$ or simply -5, not 5)	$\mathbf{-5} \bullet \frac{x}{-5} = 35 \bullet (\mathbf{-5})$	$x = -175$

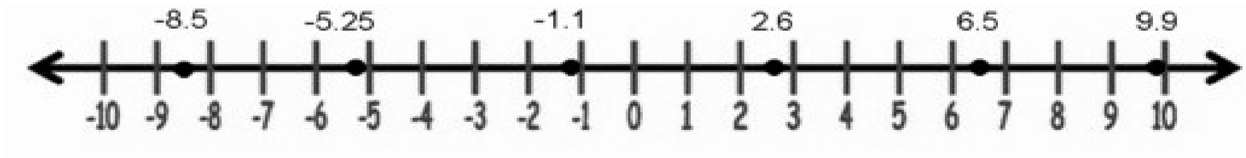
Exercise 2.2

Solve for the unknown variable in each equation.

1. $7x = 63$
2. $48 = 2x$
3. $-100 = -5x$
4. $-4x = 32$
5. $6x = -48$
6. $-33 = 3x$
7. $-43x = 0$
8. $3z = 81$
9. $35 = -5g$
10. $-2 = 8k$
11. $9n = 36$
12. $-4a = -32$
13. $42 = 7c$
14. $10d = 0$
15. $-10f = -10$
16. $\frac{x}{8} = 5$
17. $4 = \frac{-x}{3}$
18. $-9 = \frac{x}{-8}$
19. $\frac{-x}{2} = 13$
20. $\frac{x}{7} = 0$
21. $15 = \frac{x}{2}$
22. $-2 = \frac{c}{3}$
23. $\frac{1}{3} = \frac{-n}{3}$
24. $\frac{w}{6} = -15$
25. $\frac{-k}{9} = 4$
26. $-12 = \frac{j}{-2}$
27. $\frac{e}{7} = 9$
28. $\frac{y}{-8} = 6$
29. $-15 = \frac{z}{15}$
30. $\frac{1}{6} = \frac{r}{6}$

2.3 Adding and Subtracting Decimals

Previously, you considered operations with positive and negative integers or whole numbers. You will now consider operations with positive and negative decimal numbers as illustrated by the dots on the number line below with their associated values given directly above the dots:



To add numbers that contain decimal digits, you must align the decimal points of all numbers to be added. Thus, to add 417.5 + 0.008 you first align the decimal points and then add:

$$\begin{array}{rr} & 417.5 \\ + & 0.008 \\ \hline & 417.508 \end{array}$$

Now, try this subtraction 417.5 – 0.008. First, you align the decimal points of the numbers and then subtract. Notice how you must add two zeros to 417.5 so that the number of decimal digits is the same in each number that is involved in the subtraction:

$$\begin{array}{rr} & 417.500 \\ - & 0.008 \\ \hline & 417.492 \end{array}$$

Some students erroneously forget the zero placeholders and this often leads to their obtaining an incorrect result.

The same rules that you previously learned for addition/subtraction of integers (Section 1.7) also apply to decimal numbers. So when you compute –417.5 + 0.008, you notice the one number is negative and one is positive, so you first subtract the small number from the larger number, 417.5 – 0.008, which is 417.492; then, because the sign of the larger number is negative, we assign a negative sign to the answer, thus,

$$\begin{array}{rr} & -417.500 \\ + & 0.008 \\ \hline & -417.492 \end{array}$$

Exercise 2.3

In problems 1–10, evaluate the arithmetic expression.

1. $3.1 + 0.06$
2. $7.2 - 4.008$
3. $6.5 - 8.49$
4. $2.017 - 3$
5. $-7.75 - 9.1$
6. $1.07 - 25$
7. $-14.48 + 29.7$
8. $318.15 - 21.38$
9. $3.99 - 3.990$
10. $4.09 - 4.99$

In problems 11–20, simplify the algebraic expression.

11. $9.1 + 8.4x$
12. $7.3x + 10.7x$
13. $7.6 + 3.1x + 9.2 + 4.8x$
14. $1.001x + 1.01x + 1.1x$
15. $3.3x - 8.54x$
16. $-18.56x + 9.432 - 12.4x$
17. $-2.8x^2 + 4.3x - 0.04x^2 - 5.91x$
18. $3.85x^3 + 0.4x^2 - 0.23x - 15.4 + 6.7x^3$
19. $4.1x^3 + 5.2x^2 + 6.3x$
20. $3.82x - 45.2 - 3.8x - 0.02x$

In problems 21–26, evaluate the expression using $x = -1.23$ and $y = 2.4$ as necessary.

21. $x - y$
22. $-x - y$
23. $-x + y$
24. $x + y$
25. $-25.66 - x$
26. $x + y - 8$

In problems 27–30, solve the equation for the unknown.

27. $k - 4.5 = 20.15$
28. $n + 54.44 = -22.9$
29. $34.01 = s + 35.1$
30. $-28.5 = 74.66 + r$

2.4 Multiplying and Dividing Decimals

Previously, you learned how to multiply and divide integers; now it is time to learn how to multiply and divide numbers that contain decimal digits.

To multiply two numbers (called factors) that contain decimal digits, you first multiply the two numbers essentially ignoring the decimal points. Then you count the number of decimal digits in each factor and that is the number of decimal digits that should be applied to the answer or product. For example, you will now perform this multiplication 20.3 • 0.1 or alternatively we could have indicated the same multiplication using these formats: 20.3 x 0.1 or (20.3)(0.1). As the first step, you multiply the two factors ignoring the decimal points in each of the factors. At this intermediate step you have:

$$\begin{array}{r} 20.3 \\ \underline{\text{x}\ \ 0.1} \\ 203 \end{array}$$

Next, since there is 1 decimal digit in the first factor (20.3) and 1 decimal digit in the second factor (0.1)—for a total of two decimal digits—your result must contain two decimal digits, or

$$\begin{array}{r} 20.3 \\ \underline{\text{x}\ \ \ 0.1} \\ 2.03 \end{array}$$

Consider another example, 20.3 • 0.14. As an intermediate step, you perform the multiplication ignoring the decimal points:

$$\begin{array}{r} 20.3 \\ \underline{\text{x}\ \ 0.14} \\ 812 \\ \underline{+\ \ 2030} \\ 2842 \end{array}$$

Next, since there is 1 decimal digit in the first factor (20.3) and 2 decimal digits in the second factor (0.14) —for a total of three decimal digits—the product must contain three decimal digits, so that you must appropriately add the decimal point to 2842 to obtain 2.842 as the correct result. Notice that the decimal points of the two factors being multiplied do not have to be aligned as is the case with addition and subtraction of decimal numbers.

In some instances, when multiplying numbers with decimals, you must be careful where to place decimal point in your result. Consider this final example, 0.01 • 0.01; first, we ignore the decimal points and multiply as shown here:

$$\begin{array}{r} 0.01 \\ \underline{\text{x}\ \ 0.01} \\ 1 \end{array}$$

Since there are two decimal digits in each of the factors being multiplied, the product must have 4 decimal digits. Since our result has just 1 digit, we must add zero placeholder digits to fill out the result to obtain 4 decimal digits, or 0.0001.

Multiplying a number by 10 (or 10^1) shifts the decimal point one digit to the right, so that 10 • 0.14 = 1.4. Multiplying a number by 100 (or 10^2) shifts the decimal point two digits to the right so that 100 • 0.14 = 14. In general, when you multiply a decimal number by a positive power of 10—such as 10^n—you simply shift the decimal point of the number "n" digits to the right.

Similarly, multiplying a number by 0.1 (or $1/10 = 10^{-1}$) shifts the decimal point one digit to the left, so that 0.1 • 0.14 = 0.014. Multiplying a number by 0.01 (or $1/100 = 10^{-2}$) shifts the decimal point two digits to the left, so that 0.01 • 0.14 = 0.0014. In general, when you multiply a decimal number of a negative power of 10, you shift the decimal point of the number "n" digits to the left.

Division operators can be specified in several different forms:

$$\frac{a}{b} = c$$

$$a \div b = c$$

$$a/b = c$$

$$\begin{array}{r} c \\ b\overline{)a} \end{array}$$

Regardless of which form of the division operator is used, the number represented by "*a*" is the dividend, "*b*" is the divisor, and "*c*" is the quotient (or result of a division problem).

Consider finding the quotient that is represented by stating either "1.2 divided into 2.46" or "2.46 divided by 1.2"—both wordings are equivalent and specify that the divisor is 1.2; thus, the both wordings indicate the following expression:

$$\frac{2.46}{1.2} \text{ or } 2.46 \div 1.2.$$

When the divisor (1.2) is a number that contains decimal digits, you must effectively convert the divisor to a whole number by multiplying by a power of 10 such that the decimal is moved completely to the right of the number; thus, 1.2 is first converted to 12. Because we multiplied the divisor by 10, we must also multiply the dividend, 2.46 by 10 to obtain 24.6. Alternatively, you can count the number of places you moved the decimal in the divisor, and then move the decimal in the dividend the same number of places. Effectively, you have converted this division problem from this initial problem:

$$1.2\overline{)2.46}$$

into this modified (but equivalent) problem where the divisor is now an integer:

$$12\overline{)24.6}$$

You must keep the decimal point in the quotient aligned with the decimal point in the dividend (24.6) to obtain the result or quotient of 2.05 shown below:

```
     2.05
12)24.60
  - 24
    0 6
    - 0
     60
   - 60
      0
```

Note in this example that a zero, shown in italics, was added to the dividend 24.6 so that the long division could be completed yielding a quotient of 2.05.

An example of what is called a repeating decimal is obtained in the quotient by performing 4/3. The details of this division are shown below, where the zeros in italics have been added after the decimal point in the dividend (4) so that the calculation can be carried out to additional decimal places (or greater precision), as shown below

```
   1.333...
3)4.000
 -3
  10
  -9
   10
   -9
    10
    -9
     1
```

This result can be approximated by rounding, for example, to the hundredths place or 1.33; or it can be written as $1.\overline{3}$ (note the horizontal line placed over the repeating digit 3). Alternatively, you can write the quotient as an exact mixed number 1⅓ (one and one-third).

In a similar manner, to find the quotient of 345 ÷ 1.04, since the divisor contains two decimal digits, we must multiply the dividend by 100 and then perform this division: 34500 ÷ 104. If you carefully perform this division you will find the result contains a repeating sequence of decimal digits shown alternating below:

$$34500 \div 104 = 331.7\mathbf{307692}\underline{307692}\mathbf{307692}\ldots$$

When such a result occurs, normally the situation will call for you to round your answer to perhaps the nearest tenths, hundredths, or thousandths place—depending upon the precision desired. Alternatively, a line placed above the repeating digits indicates the decimal digits that repeat forever:

$$34500 \div 104 = 331.7\overline{307692}$$

If you would like to give a “friend” a challenge, ask him or her to convert the fraction $^1/_{97}$ to a decimal. The result is a sequence of 96 repeating digits:

$$0.\overline{010309278350515463917525773195876288659793814432989690721649484536082474226804123711340206185567}$$

Instead of rounding the decimal number or using a line to designate the repeating digits, sometimes it is useful to express the exact result as a mixed number such as

$$331\frac{76}{104} = 331\frac{19}{26}$$

where the fractional part, $\frac{76}{104}$, has been reduced to lowest terms, $\frac{19}{26}$.

As a shortcut, when dividing a number by 10 (or 10^1), the decimal point of the dividend is shifted to the left one place to obtain the quotient, thus 264.359 ÷ 10 = 26.4359. Similarly, notice the outcome when we divide by 100 in this problem: 264.359 ÷ 100 = 2.64359 where the decimal point of the dividend was shifted two places to the left. In general, when a dividend is divided by a power such as 10^n, the resulting quotient is the same as the dividend with the decimal shifted “n” places to the left.

As a final example, consider 0.0021/1.05; since there are two decimal digits in the divisor we must move the decimal place over two places and now have the equivalent problem 0.21/105. When dividing 105 into 0.210, notice below that the decimal point in the answer is aligned with the decimal point in 0.210 and you must use zero place holders to obtain the answer 0.002.

$$\begin{array}{r} 0.002 \\ 105\overline{)0.210} \\ -210 \\ \hline 0 \end{array}$$

Exercise 2.4

In problems 1–7, perform the indicated. product.

1. (3)(8.7)
2. (-1.1)(3.5)
3. (-1.1)(1.1)
4. (3.85)(-2.65)
5. (100)(45.8976)
6. (100)(0.01)
7. (3.765)(0.01)

In problems 8–12, perform the indicated quotient

8. $3 \div 2$
9. $2 \div 3$
10. $-0.05 \div 100$
11. 45.6/0.24
12. -0.37/5

In problems 13–16, evaluate the expression.

13. $(1.1)^4$
14. $(-0.13)^2$
15. $4.87 + 8.1 \cdot 9.3$
16. $1.6 \cdot 8.4 + 3.5 \cdot (-4.7)$

In problems 17–22, solve for the unknown. variable.

17. $1.1x = 20.5$
18. $\frac{x}{2.36} = -5.2$
19. $10.9 = -3.3x$
20. $4.24x - 0.003x = 13.1347$
21. $24.58a = 24.58$
22. $3w = 1$
23. $\frac{-t}{2.36} = 7.4$
24. $-23.2 = \frac{p}{-0.02}$
25. $\frac{1}{3} = \frac{g}{3}$
26. $32.159h = 0$
27. $3b = \frac{1}{3}$
28. $q = \frac{1}{9^2}$
29. $-4z = -6.6$
30. $0.1 = 10^{-1}w$

2.5 Properties

There are three basic properties that can help increase your speed of computation in certain cases: commutative property, associative property, and distributive property.

The commutative property of addition states that changing the order of the addends does not change the resulting sum, so that 1 + 2 = 2 + 1. Interestingly, since 1 – 2 is not equal to 2 – 1, math books correctly state that only addition is commutative—not subtraction; however, if you consider that subtraction is simply the addition of a negative number, then the subtraction 1 – 2 is, in reality, the same as performing this addition: 1 + (-2). Thus, the commutative property of addition is satisfied when you write

$$1 + (-2) = -2 + 1$$

where 1 + (-2) = -1 and -2 + 1 = -1.

Similarly, the commutative property of multiplication states that changing the order of the factors does not change the resulting product, so that 2 • 3 = 3 • 2. Again since 2 ÷ 3 is not equal to 3 ÷ 2, division is not said to be commutative.

The associative property of addition states that changing the grouping of the addends does not change the resulting sum, so that (1 + 2) + 3 = 1 + (2 + 3). Similarly, the associative property of multiplication states that changing the grouping of the factors does not change the resulting product, so that (2 • 3) • 4 = 2 • (3 • 4). Notice that you can distinguish the associative property from the commutative property by observing that in the associative property, the order of each of the terms is not changed.

The distributive property involves both multiplication (or division) and addition or (subtraction) and can sometimes be used to simplify certain algebraic expressions. It states that 2(3 + 4) = 2 • 3 + 2 • 4, or 2(3 – 4) = 2 • 3 – 2 • 4. Notice that in both these cases, the factor 2 is is applied or distributed to each term given in parentheses.

In addition to the three basic properties given above, there is an identity property for addition and multiplication. For addition the identity property is also called the adding zero property which states the sum of any number and zero is that number, or 3 + 0 = 3. For multiplication the identity property is also called the multiplying by one property which states that the product of any number and 1 is that number, or 3 • 1 = 3.

Now, at this point you may be wondering how you can put your knowledge of the commutative, associative, and distributive properties to good use. Frankly, their use is limited; however in certain instances using these properties can allow you to evaluate expression mentally and eliminate the need to resort to needless calculations.

As an example, consider evaluating the sum 23 + 148 + 27. You could perform the sum in these steps:

$$(23 + 148) + 27$$
$$= 171 + 27$$
$$=198$$

Alternatively, using the commutative and associative properties, you could mentally perform this sum (perhaps quicker) by recognizing that 23 + 27 is 50, so that you instead perform these steps:

$$(23 + 27) + 148$$
$$= 50 + 148$$
$$= 198$$

Likewise, you could perform the multiplication of 25 • 7 • 4 by performing the product in these steps:

$$(25 \bullet 7) \bullet 4$$
$$= 175 \bullet 4$$
$$= 700$$

However, if you use the commutative properties of multiplication, you can perform the product (perhaps quicker) by recognizing that 25 • 4 is 100, and instead perform these steps:

$$(25 \bullet 4) \bullet 7$$
$$= 100 \bullet 7$$
$$= 700$$

Finally, consider the expression

$$4 \bullet 8 + 6 \bullet 8$$

Which can be evaluated by performing these steps:

$$(4 \bullet 8) + (6 \bullet 8)$$
$$= 32 + 48$$
$$= 80$$

Alternatively, using the distributive property by recognizing that 8 is a common factor in each term, you can evaluate the expression mentally by performing these steps:

$$4 \bullet 8 + 6 \bullet 8$$
$$= 8(4 + 6)$$
$$= 8(10)$$
$$= 80$$

The distributive property, in certain circumstances, can be used in the reverse direction to assist the evaluation of 18 • 29 by recognizing 29 is 30 – 1. Using the distributive property you can compute 18(30 – 1) = 18 • 30 – 18 • 1 = 540 – 18 = 522. As another example, consider the evaluation of 18 • 21 by recognizing that 21 is 20 + 1. Again, using the distributive property you can compute 18(20 + 1) = 360 + 18 = 378

The distributive property is most often utilized to simplify algebraic expressions by effectively combining like terms. Noticing that the variable x is common to each term, x can be used as a factor so that the expression:

$$5x + 6\mathrm{x} - x$$
$$= x(5 + 6 - 1)$$
$$= 10x$$

Recall that x actually implies (or is the same as) $1x$ and $-x$ implies $-1x$ when combining like terms.

As another, more complicated example of using the distributive property, consider the expression $27x^3 - 6x^2$. Notice carefully that the above expression is equivalent to

$$3 \bullet 9 \bullet x \bullet x \bullet x - 3 \bullet 2 \bullet x \bullet x$$

since $x^3 = x \bullet x \bullet x$ and $x^2 = x \bullet x$, notice that $x \bullet x$ or x^2 is in common with both the terms $27x^3$ and $-6x^2$. Furthermore, since the coefficient of the first term, 27, is $3 \bullet 9$ and the coefficient of the second term, 6, is $3 \bullet 2$, notice that both terms also share a factor of 3 in common. Thus, using both common factors, you have $3x^2$ common to both terms. These common factors ($3x^2$ or $3 \bullet x \bullet x$) are shown underlined in each term of the original expression:

$$\underline{3} \bullet 9 \underline{\bullet x \bullet x} \bullet x - \underline{3} \bullet 2 \underline{\bullet x \bullet x}$$

Notice that when you factor out (or remove) the common terms, then you have remaining this expression:

$$9 \bullet x - 2$$

Thus, you can now use the distributive property and write $27x^3 - 6x^2 = \mathbf{3x^2}(9x - 2)$.

Finally, the distributive property which is generally $a(b + c) = ab + ac$, has several different equivalent forms, for example, $(a - b)c = ac - bc$. Consider this equation

$$(x - 5)(x + 3) = x(x + 3) - 5(x + 3)$$

At first glance, you might not recognize this equation as the application of the distributive property. However, notice the common term of $(x + 3)$ in each of the two terms on the right side of the equation. If you make the following substitutions: $c = (x + 3)$, $a = x$, and $b = 5$, you can observe that this is another application of the distributive property $(a - b)c = ac - bc$.

As our final example of the distributive property, consider this example:

$$-(3x - 4) = -3x + 4,$$

$$\text{since } -1 \bullet 3x - (-1)4 = -3x - (-4) = -3x + 4$$

Notice that the minus sign at the beginning of the expression is treated as if it were the factor -1 and it as well as any other negative factor has the effect of changing the sign of each term within the parentheses:

$$-1(3x - 4) = -3x + 4.$$

Summary of Properties

Property	General Property	Description	Algebraic Example
Commutative Property of Addition	$a + b = b + a$	Change in the order of the terms	$3x + 4y = 4y + 3x$
Commutative Property of Multiplication	$a \bullet b = b \bullet a$		$xy = yx$
Associative Property of Addition	$(a + b) + c = a + (b + c)$	Change in the grouping of the terms; order is unchanged	$(3x + 5y) + z = 3x + (5y + z)$
Associative Property of Multiplication	$(a \bullet b) \bullet c = a \bullet (b \bullet c)$		$(xy)z = x(yz)$
Distributive Property	$a(\underline{b} + c) = ab + ac$ $(a - b)c = ac - bc$ $-(a + b) = -a - b$	Common factor is distributed over each of the addends	$3x(x + 2) = 3x^2 + 6x$ $(x - 7)(x + 1) = x(x+1) - 7(x+1)$ $-1(3x + 4) = -3x - 4$
Identity Property of Addition	$a + 0 = a$	Adding zero results in the same number	$5x + 0 = 5x$
Identity Property of Multiplication	$a \bullet 1 = a$	Multiplying by one results in the same number	$1x = x$

2.6 Mean, Median, Mode, and Range

The mean, median, mode, and range all are values that help characterize a data set or compilation of observations. The mean is technically called the arithmetic mean. The mean, median, and mode each represent different types of averages of the data set.

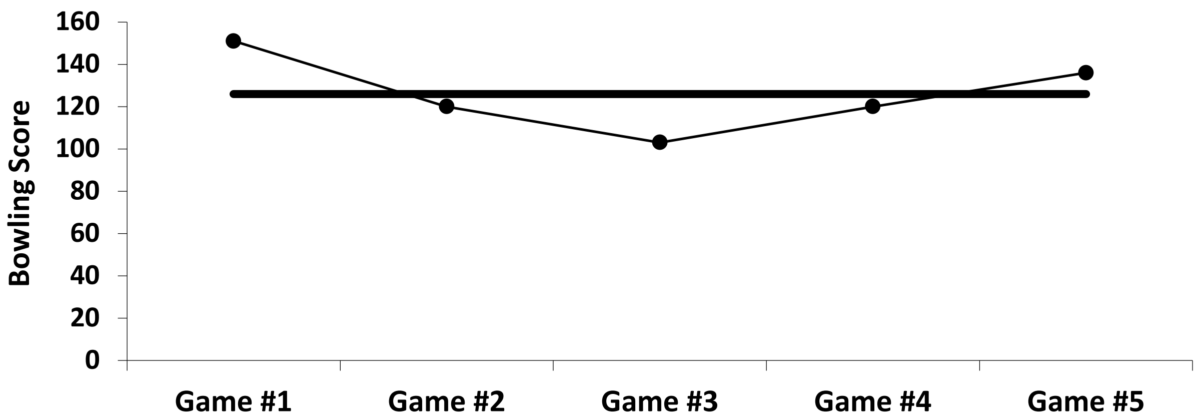
Let's consider an example. Carl went bowling and each of his scores for 5 games is shown in the table below. Thus, these 5 scores represent the data set of observations.

Game	Score
#1	151
#2	120
#3	103
#4	120
#5	136

To determine the mean, you add the data values (or scores) and divide by the total number of values (scores). Thus,

$$\text{Mean} = \frac{151+120+103+120+136}{5} = \frac{630}{5} = 126$$

Since the mean is 126, this indicates that your expectation is that if Carl bowled a 6$^{\text{th}}$ game, his score would be 126. Let's look at this visually in the graph or figure below where a dot has been placed in the chart corresponding to the bowling score reported for each of the 5 games:



The bold horizontal line represents the mean score of 126. Notice that the vertical distance from each of the game scores higher than the average to the average line are balanced with the vertical distances from each of the game scores lower than the average to the average line. Another type of average is called the median which separates the lower half of the observations from the upper half of the observations. To determine the median, you must arrange the observations in numerical order from lowest to highest and select the middle number. Thus, after having arranged the scores in numerical order, you have:

$$103,\ 120,\ 120,\ 136,\ 151$$
$$\uparrow$$

Notice that the arrow points to the middle number, 120. Had there been an even number of scores, there would have been two numbers in the middle of the sequence. For example, given an even number of scores as follows:

103, 120, 120, 136, 151, 158

↑ ↑

The median is the average of the two numbers in the middle of the sequence, or

$$\text{median} = \frac{120+136}{2} = \frac{256}{2} = 128$$

The mode is the number that occurs the most frequently among the observations. Since there are two scores of 120, 120 is the mode. Had all scores been different, no mode would have existed. If two additional bowling scores were identical, then there would have been two modes and the data set would be said to be bimodal.

Often, a single mean is not very meaningful when a data set is bimodal. Suppose there are 5 uniform sections in a pool that are 1-foot deep (very shallow) and 5 sections in the same pool that are 7-feet deep. Thus our data set is as follows:

1, 1, 1, 1, 1, 7, 7, 7, 7, 7

The mean depth of the pool is computed as follows:

$$\text{mean} = \frac{1+1+1+1+1+7+7+7+7+7}{10} = \frac{40}{10} = 4$$

If Daniel, a 5-foot tall teenager, drowned in this pool, you might hear on the news this report: "Daniel drowned in a pool that was only an average of 4-feet deep." However, this would be misleading, since the mean of bimodal data is not meaningful—it is more likely Daniel drowned in one of the sections of the pool that was 7-feet deep!

Finally, returning to our original bowling example with 5 scores, the range is the difference between the highest and lowest observations. Thus, the range is 151 – 103 or 48.

Summary of Statistical Measures

Statistical Measure	**Definition**
mean	Sum of the observations divided by the number of observations.
median	Middle number of the observations ordered from lowest to highest.
mode	The observation that occurs the most often.
range	Difference between the highest and lowest observation.

Exercise 2.6

For each data set below, find the mean, median, mode(s), and range.

1. 321, 112, 200
2. 3, 12, 7, 2
3. 8, 15, 8, 14, 10
4. 0.18, 0.21, 0.18, 0.23, 0.2
5. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Answers to Chapter 2 Exercises

Exercise 2.1

1. $x = 17$; **2.** $x = 3$; **3.** $x = -3$; **4.** $x = -17$; **5.** $x = -7$; **6.** $x = -1$; **7.** $x = 12$; **8.** $x = 8$; **9.** $x = -8$; **10.** $x = 23$; **11.** $x = -5$; **12.** $x = 4$; **13.** $x = 32$; **14.** $x = 23$; **15.** $x = 47$; **16.** $x = -7$; **17.** $x = -23$; **18.** $x = -77$; **19.** $x = 134$; **20.** $x = 9$; **21.** $y = 81$; **22.** $a = 110$; **23.** $c = -2$; **24.** $w = 0$; **25.** $z = 0$; **26.** $f = -71$; **27.** $q = 145$; **28.** $r = -115$; **29.** $n = 119$; **30.** $g = -54$

Exercise 2.2

1. $x = 9$; **2.** $x = 24$; **3.** $x = 20$; **4.** $x = -8$; **5.** $x = -8$; **6.** $x = -11$; **7.** $x = 0$; **8.** $z = 27$; **9.** $y = -7$; **10.** $k = -\frac{1}{4}$; **11.** $n = 4$; **12.** $a = 8$; **13.** $c = 6$; 1**4.** $d = 0$; **15.** 1; **16.** $x = 40$; **17.** $x = -12$; **18.** $x = 72$; **19.** $x = -26$; **20.** $x = 0$; **21.** $x = 30$; **22.** $c = -6$; **23.** $n = -1$; **24.** $w = -90$; **25.** $k = -36$; **26.** $j = 24$; **27.** $e = 63$; **28.** $y = -48$; **29.** $z = -225$; **30.** $r = 1$;

Exercise 2.3

1. 3.16; **2.** 3.192; **3.** -1.99; **4.** -0.983; **5.** -16.85; **6.** -23.93; **7.** 15.22; **8.** 296.77; **9.** 0; **10.** -0.9; **11.** $9.1 + 8.4x$; **12.** $18x$; **13.** $16.8 + 7.9x$; **14.** $3.111x$; **15.** $-5.24x$; **16.** $-30.96x + 9.432$; **17.** $-2.84x^2 - 1.61x$; **18.** $10.55x^3 + 0.4x^2 - 0.23x - 15.4$; **19.** $4.1x^3 + 5.2x^2 + 6.3x$; **20.** -45.2; **21.** -3.63; **22.** -1.17; **23.** 3.63; **24.** 1.17; **25.** -24.43; **26.** -6.83; **27.** $k = 24.65$; **28.** $n = -77.34$; **29.** $s = -1.09$; **30.** $r = -103.16$

Exercise 2.4

1. 26.1; **2.** -3.85; **3.** -1.21; **4.** -10.2025; **5.** 4589.76; **6.** 1; **7.** 0.03765; **8.** 1.5 (or $1\frac{1}{2}$); **9.** $0.\overline{6}$ (or $\frac{2}{3}$); **10.** –0.0005; **11.** 190; **12.** –0.074; **13.** 1.4641; **14.** 0.0169; **15.** 80.2; **16.** -3.01; **17.** $x = 18.\overline{63}$ (or 18 7/11); **18.** $x = -12.272$; **19.** $x = -3.\overline{30}$ (or –3 1/33); **20.** $x = 3.1$ **21.** $a = 1$; **22.** $w = 0.\overline{3}$ (or $\frac{1}{3}$); **23.** -17.464; **24.** $p = 0.464$; **25.** g = 1; **26.** $h = 0$; **27.** $b = 0.\overline{1}$ (or 1/9); **28.** $q = 0.\overline{012345679}$ (or 1/81) **29.** $z = 1.65$; **30.** $w = 1$;

Exercise 2.6

Problem	Mean	Median	Mode	Range
1	211	200	none	209
2	6	5	none	10
3	11	10	8	7
4	0.2	0.2	0.18	0.05
5	5.5	5.5	none	9

Chapter 3: Two-Step Equations and Scientific Notation

In the previous chapter (sections 2.1 and 2.2) you learned to solve one-step equations by isolating the variable. This required doing an inverse operation—either (1) adding or subtracting a constant or (2) multiplying or dividing by the divisor or factor associated with the variable. Now you are ready to learn to solve algebraic equations that require using both operations.

3.1 Solving Two-Step Equations

Consider the algebraic equation,

$$3x - 5 = 4.$$

You draw your attention to the left side of the equation where you find the term $3x$ which contains the variable x. You isolate the $3x$ term containing the variable by first noting that 5 is subtracted. Thus, you perform the inverse operation by adding 5 to each side of the equation, yielding

$$3x - 5 + 5 = 4 + 5$$

After this first step, the equivalent equation is simplified to

$$3x = 9$$

You are already familiar with solving similar (one-step) equations discussed in Section 2.2. Since x is multiplied by 3, you do the inverse and divide each side of the equation by 3,

$$\frac{3x}{3} = \frac{9}{3}$$

This simplifies to the solution,

$$x = 3$$

Now try this next example which involves an equation requiring a 2-step solution:

$$\frac{x}{2} + 3 = 8$$

What is your first step? Yes, obviously you must isolate the term $x/2$ which contains the variable x. But more specifically, what must you do? Since 3 is added to this term, you must do the inverse and subtract 3 from each side of the equation,

$$\frac{x}{2} + 3 - 3 = 8 - 3$$

After having performed the first step, the equivalent equation now becomes

$$\frac{x}{2} = 5$$

Again, you have solved similar equations previously in Section 2.2. To isolate the variable, what is your second step? Yes, since x is divided by 2, you perform the inverse operation and multiply each side of the equation by 2, or

$$2 \bullet \frac{x}{2} = 5 \bullet 2$$

This simplifies to the solution,

$$x = 10$$

Sometimes, it may be necessary to combine like terms and then apply the two-step approach. This is seen in the equation,

$$25 = 3x - 5x + 15$$

We first combine the like terms on the right side of the equation. Since $3x - 5x = -2x$, we can simplify the equation to

$$25 = -2x + 15$$

Now, as before, we have a two-step equation. Since the term with the variable is on the right side of the equation, you notice the addition of 15 and therefore subtract 15 from each side of the equation,

$$\mathbf{-15} + 25 = -2x + 15 \mathbf{- 15}$$

We now have,

$$10 = -2x$$

Since x is multiplied by -2, you perform the inverse operation by dividing each side of the equation by -2 to obtain the solution, $-5 = x$, or as is more conventional to write,

$$x = -5$$

Now consider an algebraic equation that is just a bit tricky,

$$4(x + 4) - 4x = 6.$$

When you expand the $4(x+4)$ term using the distributive property this yields,

$$4x + 16 - 4x = 6.$$

Next, combining like terms on the left side of the equation, you note that $4x - 4x$ is 0, so you have

$$16 = 6$$

which is not a true statement. Thus, there is no solution to the initial problem given. I had warned you that the problem was tricky!

One more algebraic equation that is worthy of our consideration is as follows:

$$4(x + 4) - 4x = 16$$

This time, when you expand the $4(x + 4)$ term using the distributive and simplify, you obtain $4x + 16 - 4x = 16$ and this simplifies to the true statement

$$16 = 16$$

In this particular example, because (1) the variable in the equation ultimately cancels out of the equation when like terms are combined and (2) the resulting equation is true, it does not matter what value you assign to x; that is, the initial equation holds true for all values of x.

A Special Note Regarding Entering Answers to the Exercises and Test Questions

Let's say you are given the subtraction problem: $\frac{3}{4} - \frac{2}{3} =$ ______

There are several different ways that you might format your answer. Notice that since the common denominator is 12, we can convert each fraction to an equivalent fraction, to obtain

$$\frac{9}{12} - \frac{8}{12} = \frac{1}{12}$$

However, if we convert each fraction to the equivalent decimal number, we obtain an answer that is a repeating decimal:

$$0.75 - 0.66666\ldots = 0.083333\ldots \text{ or } 0.08\overline{3}$$

Since, typing anything less than an infinite number of threes would represent a rounded answer (such as 0.083 which is the answer rounded to the nearest thousandth, or 0.0833 which is the answer rounded to the nearest ten-thousandth), it is best to provide the answer in the form of a fraction $\frac{1}{12}$.

So, please type or enter your answer as a fraction which is completely accurate without any rounding rather than attempting to type a repeating decimal (since typing a decimal number that repeats forever is impossible anyway)! Now if the problem specifies that the answer is to be rounded to the nearest thousandth, only then should a decimal approximation be used.

Exercise 3.1

1. $3x + 5 = 17$
2. $17 - 5x = 42$
3. $-5 + 2x = 11$
4. $4x - 7 = 29$
5. $9x - 4 = -58$
6. $55 = -8x - 9$
7. $-40 = 2 + 6x$
8. $-56 = -7 - 7x$
9. $3x + 10 - 7x - 12 = 18$
10. $-18 = -3x - 4 - 4x - 14$
11. $3(2x - 7) = 21$
12. $2(x - 4) - 4(3x + 7) = 24$
13. $16 = 4(n - 8)$
14. $57 = 3e\,(5 - \frac{1}{e})$
15. $5h^2(\frac{2}{h} + 3) - 15h^2 = 150$
16. $-3.5(3x - 4) = -20$
17. $0.4 = -0.2x + 0.08$
18. $4w + 6 = 8.2$
19. $4z + 6 = -8.2$
20. $-0.009 = 8.2m - 3.33$
21. $0.1(3s + 5) = 12.5$
22. $\frac{p}{4} - 5 = -8$
23. $-18 = -\frac{f}{1.1} - 12.3$
24. $-6.3 - \frac{g}{1.1} = 0$
25. $\frac{1}{7.5} - \frac{n}{3.75} = 0$
26. $3.1(x - 5) = 0$
27. $(x - 6.4)3 = -6$
28. $0.001x = 12345.67$
29. $10000y = 123.4567$
30. $4(x - 1) - 4x = 0$

3.2 Solving Equations When the Variable is on Both Sides

Up to this point, you have solved algebraic equations that have been carefully generated with the unknown or variable present either on the left side or the right side of the equation. But, what if an algebraic equation has variable terms present on both sides of the equal sign. For example, consider this equation:

$$220 - 50x = 60x$$

Notice that the variable term, $50x$, appears on the left side of the equation and another variable term, $60x$, appears on the right side of the equation. Before you can isolate the variable, you must first get the variable terms on one side of the equation and the constant terms on the other side. Basically, you have two approaches: you can either
1) add $50x$ to both sides of the equation to eliminate the $-50x$ on the left side of the equation and obtain

$$50x + 220 - 50x = 60x + 50x$$

which reduces to

$$220 = 110x$$

or

2) subtract $60x$ from both sides of the equation to eliminate the $60x$ on the right side of the equation to obtain

$$\mathbf{-60x} + 220 - 50x = 60x \mathbf{- 60x}$$

which reduces to

$$-110x + 220 = 0$$

Of the two approaches shown above, approach (1) facilitates a faster solution since adding $50x$ to both sides of the equation accomplishes having only the constant 220 on the left side of the equation and the variable term on the right side of the equation: $220 = 110x$. To solve for x you do the inverse operation and divide both sides of the equation by 110 to obtain $x = 2$.

Using approach (2), to solve $-110x + 220 = 0$, we isolate the variable as usual by doing the inverse operation and first subtracting 220 from both sides of the equation, yielding

$$-110x = -220.$$

Next, since x is multiplied by -110, you do the inverse and divide both sides of the equation by -110 to obtain the solution, $x = 2$. Notice that regardless of the approach you choose—removing the variable term from the left side of the equation or removing the variable term from the right side of the equation—you derive the same solution; however the route to the solution is faster by taking the approach that leaves the constant on one side and the variable on the other side (as was the case using approach 1 above).

Consider another example of an equation with the variable on both sides,

$$3(x - 5) + 3 = 4(-2x + 3) + 7x$$

You first use the distributive property to obtain

$$3x - 15 + 3 = -8x + 12 + 7x$$

and then combine like terms, yielding the equivalent equation,

$$3x - 12 = -x + 12$$

At this point, you have the option of either eliminating (1) the variable term $-x$ on the right side of the equation by adding x to each side of the equation or (2) the variable term $3x$ on the left side of the equation by subtracting $3x$ from each side of the equation. It really does not matter, but simply due to a preference to work with positive variable terms, you will add x to each side of the equation, thus you now have a typical two-step equation,

$$4x - 12 = 12$$

Since the left side of the equation contains the variable term and 12 is subtracted from this term, you do the inverse and add 12 to each side of the equation,

$$4x = 24$$

Finally, since x is multiplied by 4, you do the inverse and divide each side of the equation by 4, yielding the solution,

$$x = 6$$

Had you taken the alternate approach and subtracted $3x$ from the equation $3x - 12 = -x + 12$, you would have obtained the two-step equation,

$$-12 = -4x + 12.$$

Since the right side of the equation contains the variable term, we must first subtract 12 from each side of the equation to obtain $-24 = -4x$, then dividing each side of the equation by -4 you have the same solution as before, $x = 6$.

Exercise 3.2

1. $3k - 4 + k = 8 + 8k$
2. $8 + 7x = -4x$
3. $-12 - 5t = t$
4. $16 + 7s = -s$
5. $8f = 3f + 10$
6. $-14 + s + 3s - 10 = 0$
7. $-v - 4 - 3v = 5v + 5$
8. $-2c + 2 = 4c - 16$
9. $-12x = -8x + 25 - x - 10$
10. $-7b - 27 + 11b = -5b$
11. $3w - 7w = -14w + 0.013$
12. $3e = -8e$
13. $31z - 8 - 3 - 25z = -3z + 9z - 11$
14. $-4 + 5y - y = 4y$
15. $-q = q + 8$
16. $24r + 48 - 12r = -6r$
17. $5 - m = m + 3$
18. $-2d + 6 - 4d = 3(d + 5)$
19. $7(u - 33) = 6(-3u + 7)$
20. $3.1a^2 + 4a - 6.4 = 3.1a^2 + 3a$
21. $-5x - 20.5 + 7x = 32.8$
22. $-18 + 9x - 37 = 20x$
23. $48 - x = 3x$
24. $12x + 9 - 2x = 0.0123$
25. $7x - 14 - 10x = -17$
26. $-25 + 32 + 8x = -x$
27. $-2x + 18 - 3x = -2$
28. $3 - 14x = 3x$
29. $-8 + 7x = 4x + 1$
30. $3(-8 + 7x) = -4(4x + 1)$

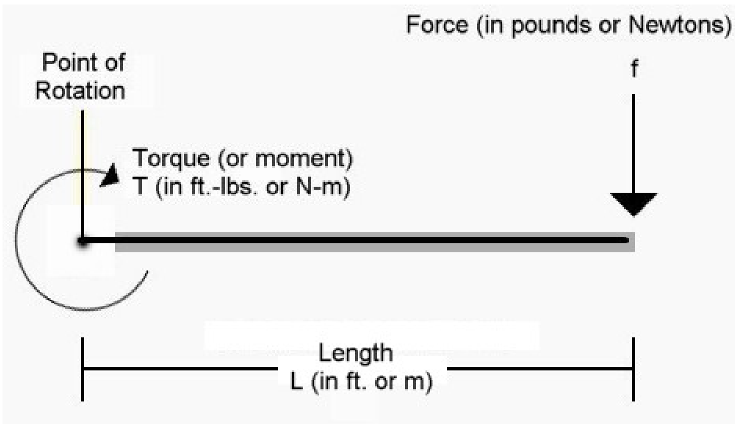
3.3 Torque, Lever Arms, Momentum, and Distance

At this point, likely you have wondered how the solution to algebraic equations can be used in the "real world"—or how you might make application of the material you have covered thus far. There are literally thousands of practical applications, but we will now consider the topic of torque in an automotive application, followed by both lever arms and momentum in a physics application, and then finally, distance calculations. As you will soon see, each of these varied topics share a common operation: torque is defined as the product of force and distance; lever arms involve the concept of work which is defined as the product of weight and distance; momentum is defined as the product of mass and velocity; and distance is defined as the product of rate and time.

Torque is defined as the force to rotate an object about an axis. Torque is computed by taking the force times the length of the lever arm. In the U.S. torque is often expressed in foot-pounds (or ft.-lbs.) and in metric units as newton-meters (or N-m). In general, the torque, T, is the product of the force, f, and the length, L, or mathematically:

$$T = f \bullet L$$

The figure below shows the torque (or moment) that is developed by a force, f, at a length, L, from the pivot point.



Alternatively, if we know the torque and length, we can treat T and L as constants and solve for the force, f, by dividing each side of the above equation by L, or

$$f = \frac{T}{L}$$

Similarly, if we know the torque and the force, we can treat T and f as constants and solve for the length, L, by dividing each side of the basic equation, $T = f \bullet L$, by f, or

$$L = \frac{T}{f}$$

Thus, given any two of the three parameters of torque, length, or force, we can solve for the remaining unknown quantity.

On an automobile wheel it is important to apply the proper torque to the lug nuts (which hold the wheel to its support). If you over tighten (or over torque) a wheel, you can strip the lug nut or break a stud or bolt. If you under tighten (or under torque) a wheel, the lug nut could loosen over time and fall off. Torque specifications for any given wheel and vehicle can usually be found by consulting the vehicle's owner's manual or the shop manual.

Typically, for a ½" diameter bolt or stud size, the proper torque is specified as having a range between 75 to 85 ft-lbs. So let's say the mean value is 80 ft.-lbs. You might ask, how many pounds of force would it take to apply 80 ft.-lbs. of torque to a lug nut using a 6-inch (or 0.5 foot) long torque wrench? Translating this word problem into an algebraic equation, you know that the length times the force, f, must equal 80, so you could write:

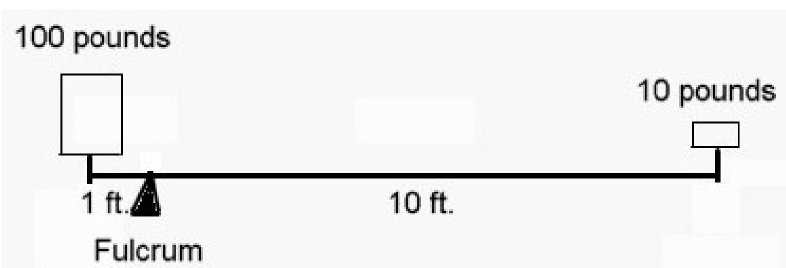
$$0.5f = 80$$

Solving for the unknown by dividing both sides of the equation by 0.5 feet, you have the solution, $f = 160$; thus, you must apply a force of 160 pounds at a distance of 0.5 feet to achieve the desire torque of 80 ft.-lbs. What if you had a longer torque wrench, say 24-inches (or 2 feet) long and wanted to achieve the same 80 ft-lbs. of torque on the lug nut. You could write:

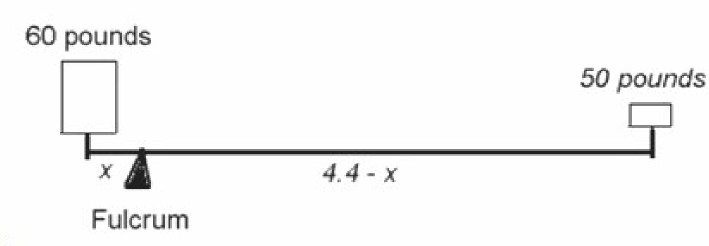
$$2f = 80$$

Solving for the unknown, you have $f = 40$, or you must apply a force of 40 pounds. Notice that the longer the length of the torque wrench, the less force that is needed. This explains why it is easier to tighten (or loosen) a nut with a longer wrench or pliers rather than a shorter tool. In theory, if you had a torque wrench 8-feet long, you would need to apply a force of only10 pounds to achieve a torque of 80 ft.-lbs.

Next, consider a horizontal balancing bar or lever arm with a weight at each end (see illustration below). The bar is balanced at the fulcrum (or pivot) point when the weight at left end times its distance to the fulcrum point (or the work on the left side) equals the weight at the right end times its distance to the fulcrum point (or the work on the right side). Thus, the 100 pound weight located 1 foot from the fulcrum is balanced by the 10 pound weight located 10 feet from the fulcrum since $100 \cdot 1 = 10 \cdot 10$. From this example you can understand how levers since ancient times might be used to move heavy loads a small distance through use of a light load at a greater distance from the fulcrum. Levers were first described by the Greek mathematician Archimedes in the third century BC. Examples of simple levers include the see-saw (teeter-totter), the claws on a hammer, a crow bar, a screw driver used to pry open a can of paint, a bolt cutter, scissors, shoehorn, etc.



Now, suppose you have a 4.4-foot long balancing bar (of negligible weight) with a 60-pound weight at the left end and a 50 pound weight at the right end. An illustration, not to scale is shown below.



How far from the 60-pound weight will the fulcrum (or balancing) point be located? This unknown distance is labeled x in the figure. You can write an equation knowing that 60 pounds times the distance x to the fulcrum must equal 50 pounds times its distance to the fulcrum. Since the total length of the bar is 4.4 feet and the 60-pound weight is a distance x to the fulcrum, the remaining length of the bar from the fulcrum to the 50-pound weight must be $4.4 - x$, thus,

$$60x = 50(4.4 - x)$$

Solving the equation for x, we first use the distributive property to expand the right side of the equation:

$$60x = 220 - 50x$$

This equation is exactly equivalent (due to the symmetric property of equations) to the equation you solved at the beginning of section 3.2. Thus, you have encountered a practical application of formulating an algebraic equation where the variable appears on each side. As you have determined previously for this equation, $x = 2$, or the fulcrum point is located 2-feet from the 60 pound weight. This solution seems reasonable since if both weights were 50 pounds, we would have expected the fulcrum point to be located exactly at half the distance of the balance bar or $4.4/2 = 2.2$ feet. Since the left weight (60 pounds) is greater than the right weight (50 pounds), you would expect the distance from the 60 pound weight to the fulcrum to be slightly less than half the balance bar length.

Now imagine you are an engineer and your next project will require that you solve many of these fulcrum problems, however, each problem will have a different weight on the left, W_1, a different weight on the right, W_2, and a different balancing bar length, L. You can treat W_1, W_2, and L as if they were constants and write the equation in terms of the unknown distance x from the weight on the left to the fulcrum point by balancing the work performed on both the left and right, or

$$W_1 \bullet x = W_2(L - x)$$

Now your goal is to solve this equation for the variable x. Even though there are the additional "variables" W_1, W_2, and L, try to think of these and treat these as if they were constants instead. First you must expand the right side of the equation using the distributive property,

$$W_1 \bullet x = W_2 \bullet L - W_2 \bullet x$$

Next, since you have the variable x on both sides of the equation, you add $W_2 \bullet x$ to both sides of the equation to obtain

$$W_1 \bullet x + W_2 \bullet x = W_2 \bullet L$$

Now using the distributive property on the left side of the equation with x as the common factor,

$$x\,(W_1 + W_2) = W_2 \bullet L.$$

Finally, since x is multiplied by $(W_1 + W_2)$, you do the inverse operation and divide both sides of the equation by $(W_1 + W_2)$. This yields the solution,

$$x = \frac{W_2 \bullet L}{W_1 + W_2}$$

The above formula has the advantage of allowing you to compute directly the distance x from the left weight to the fulcrum point, given any balance bar length (assumed to be of negligible weight), and any values for the right and left weights. This is not a formula you need memorize since you know the basis of its derivation. Notice as a quick check that if the left and right weights are the same, or $W_2 = W_1$, the formula simplifies to what you would expect as shown below:

$$x = \frac{W_1 \bullet L}{W_1 + W_2} = \frac{W_1 \bullet L}{2W_1} = \frac{L}{2}$$

In other words, the fulcrum point is located exactly at the bar's half-way point—exactly as expected.

Finally, consider another physics problem involving momentum which is defined as mass times velocity. Suppose we have a train of mass 2,000 pounds traveling 60 miles per hour. It crashes into a stationary car of mass 200 pounds that is initially at rest on the railroad tracks. What will be the new speed following the crash assuming that momentum is conserved and that the car is now mangled and attached to the train?

You know that the initial momentum must equal the final momentum so that the mass of the train times its speed (or technically, its velocity) must equal the mass of the train plus the car times the speed after the crash which is the unknown we will call v, or

$$2000(60) = 2200v$$

You isolate the variable v by doing the inverse operation and dividing both sides of the equation by 2200, to obtain $v = 54.\overline{54}$ or 54 6/11. Notice, that since momentum is conserved, the combined mass of the train and car travel down the tracks at a slower velocity (speed) following the collision.

Distance, d, is defined as the product of the rate (or speed) r, and time t as follows:

$$d = rt$$

Treating distance and rate as if they were constants, we can solve the equation $d = rt$ for the time, t, by dividing both sides of the equation by r; thus, $t = d/r$.

Treating distance and a time as if they were constants, we can solve the equation $d = rt$ for the rate, r, by dividing both sides of the equation by t; thus, $r = d/t$

If a car is travelling at an average rate (speed) of 50 miles/hr for a time of 4 hours, you can substitute the values of $r = 50$ and $t = 4$ into the above equation

$$d = (50)(4)$$

and compute the distance traveled is 200 miles.

Two cars leave from a certain point A at precisely the same time. One travels due east at 55 mi/hr and the other car travels due west at 70 mi/hr. When will the two cars be 500 miles apart? Since the cars started from the same point at the same time and are travelling in opposite directions, the sum of their distances travelled must total 500 miles. Algebraically, you could write

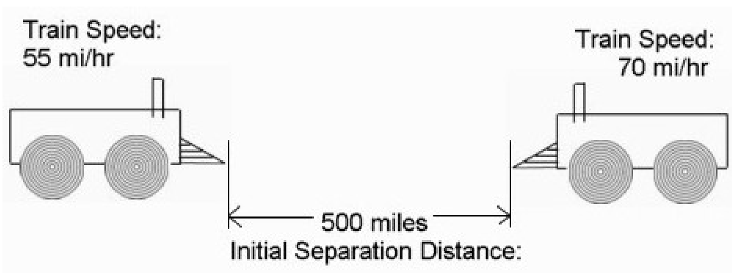
$$d = r_1 t + r_2 t$$

where r_1 and r_2 denote the speed of the first and second car, respectively; "t" is the unknown time of travel that we are interested in determining. Substituting for $r_1 = 55$ and $r_2 = 70$, and knowing the distance $d = 500$, you can write,

$$500 = 55t + 70t$$

You combine like terms to obtain 500 = 125t, then divide both sides of the equation by 125. This yields the solution, $t = 4$ hours.

Consider an alternative presentation of the above scenario. There are two trains, separated by a distance of 500 miles. Both leave their stations at the same and are on the same track, headed toward one another. One is traveling at 55 mi/hr and the other is traveling at 70 mi/hr. The scenario is shown in the figure below.



When (or in how many hours) will the trains collide with each other? Time, t, is the unknown variable. The distance traveled by one train is $55t$ and the distanced traveled by the other train is $70t$. The collision of the two trains will occur at a point in time when the sum of the distances traveled by both trains totals 500 miles, or algebraically, when $500 = 55t + 70t$, or in $t = 4$ hours.

Now suppose the 2nd car left 2 hours after the first car left. Then if "t" represents the total time traveled by the first car, then $t - 2$ would be the time traveled by the second car—since it is on the road 2 hours less than the first car. You now have

$$d = r_1 t + r_2(t - 2)$$

Again, substituting the known values into the equation, you have

$$500 = 55t + 70(t - 2)$$

Using the distributive property, we expand the expression on the right side of the equation to obtain the equivalent equation,

$$500 = 55t + 70t - 140$$

You solve this equation in two steps by first adding 40 to each side of the equation,

$$\mathbf{140} + 500 = 55t + 70t - 140 \mathbf{+ 140}$$

This simplifies to $640 = 125t$. Next, you divide each side of the equation by 125 to obtain the solution, $t = 5.12$ hours.

As another example of a distance problem, suppose a car leaves from a given point and travels east at 50 mi/hr. Another car leaves 1 hour later from the same point and travels the same route at 60 mi/hr. When (or at what time) will the second car meet the first car? Here again, the unknown variable is time, t. The distance traveled by the first car is the product of its speed and time, or 50t. This must equal the distance traveled by the second car which starts 1 hr. later, or $60(t - 1)$. Thus, algebraically, you can write the equation

$$50t = 60(t - 1)$$

Using the distributive property on the right side of the equation you have $50t = 60t - 60$. To obtain the variable on the left side of the equation, you subtract 60t from each side of the equation giving you the equivalent equation,

$$-10t = -60$$

Dividing each side of the equation by -10 you obtain the solution, $t = 6$ hours.

Review of Basic Torque, Distance, Balance Bar, and Momentum Formulas

	Main Formula	Other Variations	
Torque, *T* in terms of force, f and length l	$T = f \bullet L$	$f = \frac{T}{L}$	$L = \frac{T}{f}$
Distance, *d* in terms of rate, r and time t	$d = rt$	$r = \frac{d}{t}$	$t = \frac{d}{r}$
Balance Bar[a] of length L, weight W_1 to fulcrum distance is x.	$W_1 \bullet x = W_2(L - x)$	$x = \frac{W_2 \bullet L}{W_1 + W_2}$	$L = \frac{x(W_1 + W_2)}{W_2}$
Conservation of Momentum in terms of masses m_1 *and* m_2 and velocities v_1 and v_2	$m_1 v_1 = m_2 v_2$	$v_2 = \frac{m_1 v_1}{m_2}$	$m_2 = \frac{m_1 v_1}{v_2}$

[a]Balance bar is considered to be of negligible weight.

Exercise 3.3

In problems 1–3, find the torque given values for the force and length.

1. 18.1 pounds, 3.6 feet

2. 12.4 Newtons, 4.7 meters

3. 6.0 pounds, 0 feet

In problems 4 – 6, find the force given values for the torque and the length.

4. 80 ft-lbs, 0.6 feet

5. 20 ft-lbs, 0.3 feet

6. 390 N-m, 1 meter

In problems 7 – 9, find the length given values for the torque and the force.

7. 100 N-m, 0.1 Newtons

8. 100 N-m, 10 Newtons

9. 120 ft-lbs, 0.8 pounds

In problems 10–13, use this information: balancing bar (of negligible weight) has a length of 8 feet and a weight of 45 pounds placed on the left end.

10. If the fulcrum point of the balance bar is located 1 foot away from the left end, what weight must be placed on the right end of the bar?

11. If the fulcrum point of the balance bar is located 7 feet away from the left end, what weight must be placed on the right end of the bar?

12. If a 900 pound weight is placed on the right end of the bar, how far is the pivot point located from the right end? Round your answer to the nearest hundredth.

13. If a 90 pound weight is placed on the right end of the bar, how far is the pivot point located from the right end?

In problem 14, assume that momentum is conserved.

14. A truck traveling 55 miles per hour with a mass of 15,000 pounds, crashes into a stationary car of mass 2000 pounds. The car becomes mangled with the truck which continues to travel in the same direction as it was traveling prior to the collision. What is the speed of the truck and car following the collision? Round your answer to the nearest hundredth.

In problems 15–19, make use of the formula $E = IR$ where E is the voltage (in volts), I is the current (in amps), and R is the resistance (in ohms).

15. What is the voltage, when $I = 10$ amps and $R = 12$ ohms?

16. Write an equation for the current, "I", given the voltage "E" and the resistance "R". In other words, solve the equation $E = IR$ for the variable "I" and treat "E" and "R" as if they were constants.

17. Write an equation for the resistance "R", given the voltage "E" and the current "I". In other words, solve the equation $E = IR$ for the variable "R" and treat "E" and "I" as if they were constants.

18. What is the current when $E = 220$ volts and $R = 20$ amps?

19. What is the resistance when $E = 120$ volts and $I = 5$ amps?

In problems 20–22, use this formula to compute the degrees centigrade (C) given a temperature in degrees Fahrenheit (F):

$$C = 0.55556(F - 32)$$

20. Convert 212^{o}F to temperature in ^{o}C. Round your answer to the nearest hundredth.

21. Convert 0^{o}F to temperature in ^{o}C. Round your answer to the nearest tenth.

22. Rewrite the formula to obtain degrees Fahrenheit (F) given degrees centigrade (C). In other words, consider C to be a constant and solve the formula for the variable F:

In problems 23 – 30, use this formula (or a variation of it): $d = rt$ where d is the distance, r is the rate, and t is the time.

23. Consider the distance "*d*" and the rate "*r*" to be constants. Solve the distance formula for the variable time, t, in terms of distance and rate.

24. Consider the distance "*d*" and the time "*t*" to be constants. Solve the distance formula for the variable rate, r, in terms of distance and time.

25. If a jet travels an average of 660 mi/hr. for 7 hours. What is the distance the jet travels?

26. A truck driver drove an average of 55 mi./hr. for a total distance of 890 miles. What was the driving time?

27. A snail crawled 5.4 inches in 35 minutes. What was the average speed of the snail? Round your answer to the nearest hundredth.

28. A cheetah ran 0.5 mile in ¾ minute (or 0.0125 hours). What was its average speed of the cheetah in mi/hr?

29. Two cars leave the same point at the same time. One travels due north at an average speed of 40 mi/hr and the other travels due south at an average speed of 65 mi/hr. At what time will these cars be 1000 miles apart? Round your answer to the nearest tenth of an hour.

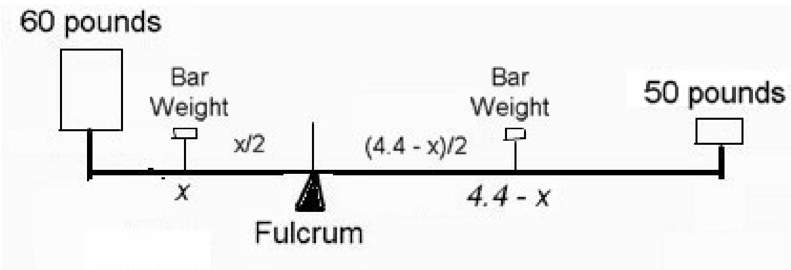
30. Two cars leave the same point. The first car travels north at an average speed of 55 mi/hr and the other car leaves 3 hours later and travels the same route as the first car at an average speed of 75 mi/hr. When will these cars meet? Round your answer to the nearest tenth of an hour.

3.4 Lever Arms with a Given Weight (Optional)

In section 3.3, you assumed that the weight of the lever arm or balancing bar was negligible (basically 0 pounds). Given a 4.4-foot long balancing bar (of negligible weight) with a 60-pound weight at the left end and a 50 pound weight at the right end. How far from the 60-pound weight will the fulcrum (or balancing) point be located? You call this unknown distance x. Then you can write an equation knowing that 60 pounds times the distance x to the fulcrum must equal 50 pounds times its distance to the fulcrum. Since the total length of the bar is 4.4 feet and the 60-pound weight is a distance x to the fulcrum, the remaining length of the bar from the fulcrum to the 50-pound weight must be $4.4 - x$, thus,

$$60x = 50(4.4 - x)$$

But now, assume the balancing bar is uniform along its entire length and weighs a total of 44 pounds. In the figure below, we have added the weight of the bar which can be computed based on length of each section on either side of the fulcrum. The balancing bar weight can be modeled as being applied at the center of each section on either side of the fulcrum.



The weight of the section of the bar that is x feet in length on the left side of the fulcrum is given by this expression:

$$\frac{x}{4.4}(44) \text{ or } 10x$$

The work contributed by the left section of bar is its weight times its distance from the fulcrum point, or

$$10x \bullet \frac{x}{2}$$

This work associated with the left section of the bar simplifies to $5x^2$.

The weight of the section of the bar that is $4.4 - x$ feet in length on the right side of the fulcrum is given by this expression

$$\frac{4.4-x}{4.4}(44) \text{ or } 44 - 10x$$

The work contributed by the right section of bar is its weight times its distance from the fulcrum point, or

$$(44 - 10x) \bullet \frac{4.4-x}{2}$$

This work associated with the left section of the bar simplifies to $5x^2 - 44x + 96.8$.

Now the sum of the work on the left side of the fulcrum (contributed by the left weight and left section of the bar) is equal to the sum of the work on the right side of the fulcrum (contributed by the right weight and right section of the bar):

$$60x + 5x^2 = 50(4.4 - x) + 5x^2 - 44x + 96.8$$

Expanding the right side of the equation using the distributive property, you have

$$60x + 5x^2 = 220 - 50x + 5x^2 - 44x + 96.8$$

Combining like terms on the right side of the equation you have

$$60x + 5x^2 = 316.8 - 94x + 5x^2$$

The $5x^2$ terms fortunately cancel on each side of the equation when we add $-5x^2$ to each side of the equation, thus simplifying the equation to

$$60x = 316.8 - 94x$$

Since we have a variable term on each side of the equation, you add $94x$ to each side of the equation, yielding

$$154x = 316.8$$

Finally, you perform the inverse operation and divide both sides of the equation by 154, yielding the solution $x = 2.057$ feet (rounded to the nearest thousandths).

Suppose we are given that W_1 is the weight on the left side of the balance bar, W_2 is the weight on the right side of the balance bar, W_{bar} is the weight of the bar, L is the length of the bar. As a challenge, let's find an equation that expresses the location of the fulcrum point relative to the left end in terms of these values. We find that the generalized formula is as follows:

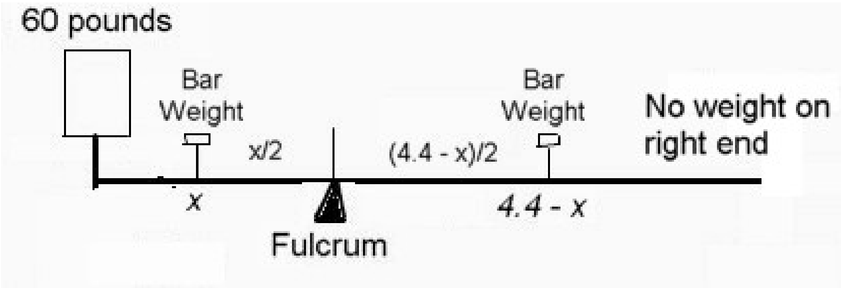
$$x = \frac{L(2W_2 + W_{bar})}{2(W_1 + W_2 + W_{bar})}$$

Consider the special case where there is no weight on the right end of the weighted balance bar. Thus, we can substitute $W_2 = 0$ into our equation and derive another specialized formula for the distance x of the fulcrum point relative to a single weight located on the left side of the balance,

$$x = \frac{L(W_{bar})}{2(W_1 + W_{bar})}$$

Also, as expected, please note that when $W_{\text{bar}} = 0$, which is the case of a bar of negligible weight, our generalized solution simplifies to the formula given in Section 3.3:

$$x = \frac{W_2 \bullet L}{W_1 + W_2}$$



Given the weight on the left side of the balance bar is $W_1 = 60$ pounds, the length of the balance bar is $L = 4.4$ feet, and the weight of the balance bar itself is $W_{bar} = 44$ pounds, then using the above specialized formula, the distance x to the fulcrum point is 0.931 feet rounded to the nearest thousandths. In this case the weight of the section of the balance bar to the right side of the fulcrum point exerts the same work as the weight and section of bar on the left side of the fulcrum point.

3.5 Scientific Notation

In some applications, engineers, physicists, and scientists encounter some rather large numbers. As an ordinary number, the speed of light, for example, is expressed as 300,000,000 m/sec (meters per second) or sometimes as 300 million or 0.3 billion m/sec. Both extremely large and extremely small numbers can be expressed in a convenient reduced form called scientific notation. In scientific notation, which is sometimes called exponential notation, the speed of light is 3.0×10^8 m/sec. A number expressed in scientific notation, will always be of the form $n \times 10^p$;

Where the first digit of "n" is greater than or equal to 1 and less than or equal to 9, and the exponent "p" is an integer number that designates a power of 10. To write 300,000,000 in scientific notation, you place a decimal point after the first digit (the leading 3), and usually show at least 1 additional digit after the decimal point. Then, drop the remaining zeroes and count the number of places you move the decimal point to the right, which in this example is 8.

In another source book, the speed of light is given as 299,792,458 m/sec. Again, to express the number in scientific notation, you place a decimal point after the first digit to obtain 2.99792458, and then determine the power of 10 by counting the number of places you move the decimal point, which is again 8. Thus, you have 2.99792458×10^8. Notice that when there are none (or perhaps only a few) ending zeros in an ordinary number, scientific notation does not yield much of a shortcut. Thus, 1.234 expressed in scientific notation is 1.234×10^0, where the power shown is 0 since the decimal point did not move.

Consider, for example, the earth's mass is 5,973,600,000,000,000,000,000,000 kg. To express this number in scientific notation, you place the decimal point after the 1st digit to obtain 5.9736. Since you moved the decimal point 24 places to the right, the number is 5.9736×10^{24} in scientific notation. Notice that 5.9736×10^{24} is also equivalent to 59.736×10^{23}, 597.36×10^{22}, 5973.6×10^{21}, and 59736×10^{20}; however, for the purposes of this instruction, you will always use what is referred to as the normalized result, 5.9736×10^{24}, where the first digit is between 1 and 9, inclusive, then followed by a decimal point and then any remaining digits.

Suppose you had the ordinary decimal number 0.01234 and wanted to express this in scientific notation. You first move the decimal point 2 places to the right so that the decimal point occurs following the first non-zero number ("1" in this case) and write 1.234×10^{-2}. Again, when there are few leading zeros in an ordinary decimal number, expressing it in scientific notation does not yield much of a shortcut.

Consider expressing the decimal number 0.1234 in scientific notation. It is a common error to write 0.1234×10^0, since you did not move the decimal point—but this would be incorrect since by definition, 0.1234 does not satisfy the criteria for a normalized result in which the first digit must not be zero. Thus, the correct normalized expression is 1.234×10^{-1}. Notice that since the decimal point was moved 1 place to the right, 1.234 is now multiplied by the power 10^{-1}.

Similarly, consider expressing a very small decimal number such as 0.00000003 in scientific notation. Again you move the decimal point after the 1st number (the 3), and count the number of places you moved the decimal point to obtain to determine the negative power of 10, which is –8 in this example. Thus, in scientific notation, you have the number 3.0×10^{-8}.

As another example, consider a proton's mass is 0.0000000000000000000000000016726 kg. Again, you place the decimal point after the first digit to obtain 1.6726 and count the number of places the decimal point is moved to the right, or 27. Thus, the proton's mass is 1.6726×10^{-27} kg.

A disk drive that contains 1,000,000 (or 1 million) bytes is said to have a capacity of 1 megabyte. In scientific notation, you could state the drive capacity is 1.0×10^6 bytes. Often, a power of 6, or 10^6 is referred to by the prefix "mega", thus 1.0×10^6 bytes is 1 megabyte. Other common powers of 10 that have been given prefixes are summarized in the table below. Thus, from the table, you see that in electronics, a capacitance of 4.7 microfarads (μF) is equivalent to 4.7×10^{-6} farads, since the prefix "micro" means times 10^{-6}. A 4.7 picofarad (pF) capacitor, likewise is equivalent to the rating 4.7×10^{-12} farads, or 0.0000000000047 farads when written as an ordinary number.

Interestingly, one of the smallest units of time is the yoctosecond, or one septillionth of a second. This is the amount of time taken for a quark (which is a subatomic particle in an atom) to emit a gluon!

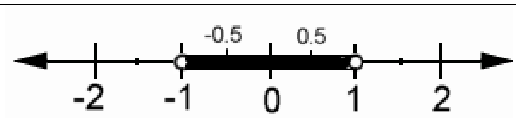
Common Prefixes corresponding to Powers of 10

Positive Powers of 10	Prefix	Negative Powers of 10	Prefix
10^{24}	yotta or septillion	10^{-3}	milli or one thousandth
10^{21}	zetta or sextillion	10^{-6}	micro or one millionth
10^{18}	exa or quintillion	10^{-9}	nano or one billionth
10^{15}	peta or quadrillion	10^{-12}	pico or one trillionth
10^{12}	tera or trillion	10^{-15}	femto or one quadrillionth
10^{9}	giga or billion	10^{-18}	atto or one quintillionth
10^{6}	mega or million	10^{-21}	zepto or one sextillionth
10^{3}	kilo or thousand	10^{-24}	yocto or one septillionth

It is also possible to express negative numbers in scientific notation. For example, the negative number, – 300,000,000, is expressed as -3.0×10^8 in scientific notation. Multiplication and division of numbers expressed in scientific notation follow the same rules of exponentiation discussed previously (see Table 1.5 and Table 3.5)

Summary of Converting Numbers to Scientific Notation

Type of Number to be converted to Scientific Notation	Example	Scientific Notation
Positive numbers ≥1: move decimal to the left and use a positive exponent.	1456.7	1.4567×10^3
Positive numbers <1: move decimal to the right and use a negative exponent.	0.0014567	1.4567×10^{-3}
Negative numbers ≤-1: move decimal to the left and use a positive exponent.	-1456.7	-1.4567×10^3
Negative numbers >-1: move decimal to the right and use a negative exponent.	-0.0014567	-1.4567×10^{-3}



Numbers between -1 and 1 on the number line are represented by using negative exponents when converting to scientific notation.

Addition and subtraction of numbers expressed in scientific notation generally requires an initial step of establishing the same powers of 10. Thus, $1.27 \times 10^6 - 3.2 \times 10^4$ must be first converted to this equivalent expression: $1.27 \times 10^6 - 0.032 \times 10^6$. Notice how 3.2×10^4 was changed from its normalized form to a form where the exponent is now the same as that given in the first number. In this particular case, we left the number that is associated with the higher power unchanged; the number that is associated with the lower power was changed so that the exponent on the power of 10 agreed with the higher power exponent. We compensated for moving the decimal point two places to the left (changing 3.2 to 0.032) by increasing the power of 10 by two (from 4 to 6) to obtain the same power of 10 associated with 1.27×10^6. Next we perform the subtraction by aligning the decimal points of the two numbers:

$$\begin{array}{r} 1.270 \\ \underline{-\,0.032} \\ 1.238 \end{array}$$

Finally, we must associate with this result the common power so that the final result is 1.238×10^6. Please note that alternatively, we could have adjusted 1.27×10^6 to 127×10^4 so that the power of 10 matched the second number, thus yielding the revised expression $127 \times 10^4 - 3.2 \times 10^4$. This yields 123.8×10^4 which when normalized yields the same result we previously obtained, 1.238×10^6.

Consider as another example $1.3 \times 10^6 - 1.1 \times 10^6$. Since both terms have the same power of 10, you first perform the subtraction (1.3 – 1.1) without having to adjust any of the terms, yielding the result 0.2; next we combine this result with the power to obtain the result 0.2×10^6. This result must be normalized so that the first digit is between 1 and 9 inclusive; thus, the final result is expressed as 2.0×10^5 (or simply 2×10^5). By moving the decimal point on 0.2 over 1 place to the right to yield 2.0 (or simply 2), we have effectively increased the result by a factor of 10. To compensate for this increase, we must subtract 1 from the exponent, thereby decreasing the power of 10 by one; thus, 10^6 becomes 10^5.

Table 3.5 Addition, Multiplication, Division, and Exponentiation Properties for Numbers Expressed in Scientific Notation

Property	**Example expression**	**Result**	**Normalized result***
$a \times 10^n + b \times 10^n = (a+b) \times 10^n$	$3.4 \times 10^5 + 8.2 \times 10^5$	11.6×10^5	1.16×10^6
$n \times 10^a \bullet m \times 10^b = n \bullet m \times 10^{a+b}$	$-3.1 \times 10^5 \bullet 4.3 \times 10^{-3}$	-13.33×10^2	-1.333×10^3
$\frac{n \times 10^a}{m \times 10^b} = \frac{n}{m} \times 10^{a-b}$	$\frac{1.8 \times 10^{-3}}{3.2 \times 10^4}$	0.5625×10^{-7}	5.625×10^{-8}
$(n \times 10^a)^b = n^b \times 10^{a \bullet b}$	$(-3.1 \times 10^{-5})^3$	-29.791×10^{-15}	-2.9791×10^{-14}

*When converting the result to the *normalized* result, for each place that the decimal point was moved to the left, add 1 to the exponent on the power 10 on the *normalized* result; for each place that the decimal point was moved to the right, subtract one from the exponent on the power 10 on the *normalized* result.

Exercise 3.5

In problems 1–10, express each ordinary number in normalized scientific notation.

1. 0.00034
2. 0.003
3. 0.00000015
4. 200
5. 260
6. –261
7. 261.1
8. 3,450,000
9. 0.86
10. -3,666,600,000

In problems 11–20, perform the indicated operation and express the result in normalized scientific notation.

11. $56.45 \text{ x } 10^5 \bullet 8.0 \text{ x } 10^4$
12. $1.03 \text{ x } 10^3 \div 2.06 \text{ x } 10^{-4}$
13. $8.4 \text{ x } 10^8 + 1.6 \text{ x } 10^8$
14. $9.3 \text{ x } 10^5 + 8.5 \text{ x } 10^4$
15. $9.3 \text{ x } 10^5 - 8.5 \text{ x } 10^4$
16. $-6.99 \text{ x } 10^7 - 6.99 \text{ x } 10^8$
17. $-7.23 \text{ x } 10^{-5} \bullet (-4.1 \text{ x } 10^{-4})$
18. $8.34 \text{ x } 10–3 \div (-2.0 \text{ x } 10^4)$
19. $3.042 \text{ x } 10^{-24} + 2.8 \text{ x } 10^{-26}$
20. $4.7 \text{ x } 10^{-8} - 8.8 \text{ x } 10^{-9}$

In problems 21–30, rewrite the number in normalized scientific notation.

21. $0.005 \text{ x } 10^{-6}$
22. $0.99 \text{ x } 10^{-12}$
23. $18.4 \text{ x } 10^{-8}$
24. $234.4 \text{ x } 10^6$
25. $0.000032 \text{ x } 10^4$
26. $18700.0 \text{ x } 10^9$
27. $1800.0 \text{ x } 10^{-3}$
28. $0.0006 \text{ x } 10^4$
29. $4433.22 \text{ x } 10^7$
30. $7766.55 \text{ x } 10^{-7}$

Answers to Chapter 3 Exercises

Exercise 3.1
1. $x = 4$; **2.** $x = -5$; **3.** $x = 8$; **4.** $x = 9$; **5.** $x = -6$; **6.** $x = -8$; **7.** $x = -7$; **8.** $x = 7$; **9.** $x = -5$; **10.** $x = 0$; **11.** x = 7; **12.** $x = -6$; **13.** $n = 12$; **14.** $e = 4$; **15.** $h = 15$; **16.** $x= 3.\overline{238095}$; **17.** $x = -1.6$; **18.** $w = 0.55$; **19.** $z = -3.55$; **20.** $n = 0.405$; **21.** $s = 40$; **22.** $p = -12$; **23.** $f = 6.27$; **24.** $g = -6.93$; **25.** $n = 0.5$ (or ½); **26.** $x = 5$; **27.** $x = 4.4$; **28.** $x = 12345670$; **29.** $y = 0.01234567$; **30.** no solution

Exercise 3.2
1. k = –3; **2.** $x = -8/11$; **3.** $t = -2$; **4.** $s = -2$; **5.** $f = 2$; **6.** $s = 6$; **7.** $v = -1$; **8.** $c = 3$; **9.** $x = -5$; **10.** $b = 3$; **11.** $w = 0.0013$; **12.** $e = 0$; **13.** z = all numbers; **14.** no solution; **15.** $q = -4$; **16.** $r = -2\frac{2}{3}$ (or $-2.\overline{6}$); **17.** $m = 1$; **18.** $d = -1$; **19.** $u = 10\frac{23}{25}$; **20.** $a = 6.4$; **21.** $x = 26.65$; **22.** $x = -5$; **23.** $x = 12$; **24.** $x = -0.89877$;**25.** $x = 1$; **26.** $x = -7/9$; **27.** $x = 4$; **28.** $x = 3/17$; **29.** $x=3$; **30.** $x = 20/37$

Exercise 3.3
1. 65.16 ft-lbs; **2.** 58.28 N-m; **3.** 0 ft-lbs; **4.** $133.\overline{3}$ (or 133⅓) lbs; **5.** $66.\overline{6}$ (or 66⅔) lbs; **6.** 390 N; **7.** 1000 m; **8.** 10 m; **9.** 150 ft; **10.** 6 3/7 (or $6.\overline{428571}$) lbs; **11.** 315 lbs; **12.** 0.38 ft; **13.** $2.\overline{6}$ (or 2⅔) ft; **14.** 48.53 mph; **15.** 120 volts; **16.** I = E/R; **17.** R = E/I; **18.** 11 amps; **19.** 24 ohms; **20.** 100.00°C; **21.** –17.8°C; **22.** $F = \frac{C+17.77792}{0.55556}$ or approximately F = 1.8C + 32; **23.** t = d/r; **24.** r = d/t; **25.** 4620 mi/hr; **26.** $16.\overline{18}$ (or 16 9/50) hr; **27.** 0.15 in/min; **28.** 40 mi./hr.; **29.** 9.5 hr; **30.** 11.25 (or 11¼) hr

Exercise 3.5
1. 3.4×10^{-4}; **2.** 3.0×10^{-3} or 3×10^{-3}; **3.** 1.5×10^{-7}; **4.** 2.0×10^{2} or 2×10^{2}; **5.** 2.6×10^{2}; **6.** -2.61×10^{2}; **7.** 2.611×10^{2}; **8.** 3.45×10^{6}; **9.** 8.6×10^{-1}; **10.** -3.6666×10^{9}; **11.** 4.516×10^{11}; **12.** 5.0×10^{6} or 5×10^{6}; **13.** 1.0×10^{9} or 1×10^{9}; **14.** 1.015×10^{6}; **15.** 8.45×10^{5}; **16.** -7.689×10^{8}; **17.** 2.9643×10^{-8}; **18.** -4.17×10^{-7}; **19.** 3.07×10^{-24}; **20.** 3.82×10^{-8}; **21.** 5.0×10^{-9} or 5×10^{-9}; **22.** 9.9×10^{-13}; **23.** 1.84×10^{-7}; **24.** 2.344×10^{8}; **25.** 3.2×10^{-1}; **26.** 1.87×10^{13}; **27.** 1.8×10^{0}; **28.** 6.0×10^{0} or 6×10^{0}; **29.** 4.43322×10^{10}; **30.** 7.76655×10^{-4}

Chapter 4: Expressions with Fractions and Ratios

4.1 Working with Fractions

You have two of your best friends over for a pre-algebra study group (at least that is what you have told your parents). Instead of working on pre-algebra problems and learning some math techniques from each other, you order a large pizza and the study turns into more of a party! The pizza arrives in a large box—pre-cut into eight slices. How can the pizza be divided equally among three people? Well, you divide the 8 pieces of pizza by 3 people to obtain 8/3 or $2\frac{2}{3}$ pieces per person. Did you remember learning that when you were younger? In this chapter we will review some operations with fractions.

First, here are a few essential definitions that are useful concerning fractions:

Table 4.1 Addition, Subtraction, multiplication, and division

Terminology	Definition	Example
fraction or **ratio**	One number divided by another number.	$\frac{2}{3}$
numerator	The top number of the fraction.	Given the fraction $\frac{2}{3}$, the numerator is "2".
denominator	The bottom number of the fraction.	Given the fraction $\frac{2}{3}$, the denominator is "3".
mixed number	A whole number, together with a fraction	$2\frac{2}{3}$
proper fraction	A fraction where the numerator is a number that is smaller than the denominator	$\frac{2}{3}$
improper fraction	A fraction where the numerator is the same or larger than the denominator	$\frac{3}{2}$

If you and your friends ate six slices of pizza out of the 8 total slices, how much of the pizza would have been eaten? $\frac{6}{8}$ is the answer; however, this can be further reduced to $\frac{3}{4}$ since both the numerator and denominator can be divided by 2 (or as you will learn later, 2 is called the greatest common factor of 6 and 8), thus we have,

$$\frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

Anytime you either divide both the numerator and denominator by the same value or multiply both the numerator and denominator by the same value, you have what is called an equivalent fraction. So, $\frac{3}{4}$ is equivalent to the fraction $\frac{6}{8}$, since $\frac{3 \bullet 2}{4 \bullet 2} = \frac{6}{8}$.

Since $\frac{3}{4}$ cannot be further simplified (as we will learn later, because the greatest common factor of 3 and 4 is 1), the fraction $\frac{3}{4}$ is said to be reduced to lowest terms.

Converting a Mixed Number to an Improper Fraction

Let's take a mixed number (which consists of a whole number, together with a fraction) and convert this to an improper fraction. This procedure often is helpful when adding, subtracting, multiplying, and dividing fractions. To convert $2\frac{3}{4}$ to an improper fraction, we start by multiplying the denominator (the "4") of the fraction times the whole number. So 4 times 2 is 8, (which corresponds to the fact that $\frac{8}{4} = 2$). To this result we add the numerator (the "3"), to obtain $8 + 3 = 11$ (or $\frac{8}{4} + \frac{3}{4} = \frac{11}{4}$) and then finally divide by the denominator (the "4"). Thus,

$$2\frac{3}{4} = \frac{11}{4}$$

Now, let's convert $15\frac{1}{2}$ to an improper fraction. Since 2 times 15 is 30, plus 1 is 31, we have

$$15\frac{1}{2} = \frac{31}{2}$$

Converting an Improper Fraction to a Mixed Number

We will now do the opposite procedure and convert an improper fraction to a mixed number. Again, let's use the improper fraction $\frac{11}{4}$. Since 11 divided by 4 is 2 with a remainder of 3, we can write

$$\frac{11}{4} = 2\frac{3}{4}$$

As another example, let's convert $\frac{31}{2}$ to a mixed number. As done in our previous example, 31 divided by 2 is 15 (with a remainder of 1), so we have

$$\frac{31}{2} = 15\frac{1}{2}$$

Now, there is the special case, such as $\frac{20}{4}$ where 20 divided by 4 is 5 with 0 remainder. Instead of writing $5\frac{0}{4}$, the convention is to simply write 5.

Exercise 4.1

In problems 1 – 10, express each mixed number as an improper fraction.

1. $1\frac{1}{2}$
2. $2\frac{1}{4}$
3. $4\frac{2}{5}$
4. $6\frac{5}{6}$
5. $25\frac{1}{4}$
6. $11\frac{3}{5}$
7. $12\frac{2}{3}$
8. $13\frac{2}{3}$
9. $10\frac{1}{8}$
10. $20\frac{3}{8}$

In problems 11 – 20, express each improper fraction as a mixed number.

11. $\frac{5}{4}$
12. $\frac{13}{10}$
13. $\frac{10}{3}$
14. $\frac{19}{6}$
15. 18/3
16. $\frac{182}{25}$
17. $\frac{29}{6}$
18. $\frac{27}{7}$
19. $\frac{195}{16}$
20. $\frac{17}{8}$

In problems 21 – 23, express each mixed number as an improper fraction.

21. What is the ratio of red candles indicated if 5 out of every 21 candies are red.

22. Of two pizzas that had 8 slices each, 5 slices were eaten. How many pizzas remained?

23. What is the ratio of Yankee fans of the total crowd (expressed as a fraction) if out of the crowd of 100,000 baseball fans, 78,000 are cheering for the Yankees. (Write your answer as a fraction in lowest terms.)

4.2 Factors and Prime Factorizations

Factors

Let's begin by considering an example. Take the number 12. Factors are simply numbers that when multiplied together equal 12. So the (whole number) factors of 12 are

1 x 12   2 x 6   3 x 4   4 x 3   6 x 2

In the language of math, we call these **"factors."** The factors of 12 are 1, 2, 3, 4, 6 and 12.

In other words, **all of the numbers that can be evenly divided into a particular number are called factors of that number.**

Another example: Say that you made brownies, and were trying to figure out how to cut them. You could leave the brownie as one humungous piece that was 20 inches long. Or you could decide to share them with someone else so that you would each have a piece that was 10 inches long. Or you could divide it evenly into four or five pieces.

So you could cut the brownies like this:

1 x 20   2 x 10   4 x 5

In the language of math, we would say that the factors of 20 are 1, 2, 4, 5, 10, and 20. We say that because those are the numbers that can be evenly divided into 20.

Prime and Composite Numbers

Sometimes there are numbers that don't have any other factor except one and themselves. These we call **prime numbers**. Prime numbers are integers that are greater than 1, that have as factors only 1 and the number itself. The prime numbers under 100 are as follows:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

A good example of a prime number is 41: you can ONLY divide 41 by 1 and itself. There are no other numbers that divide into 41 evenly.

Another good example of a prime number is 89. Eighty-nine is divisible by only 1 and 89. You cannot find any other divisors that will go into 89 evenly.

The number 2 is considered the first prime number. Again, it is divisible only by 1 and itself. Interestingly, 2 is the only even number that is prime—since other larger even numbers are divisible by 2 and therefore they fail to meet the definition of a prime number—which is a number evenly divisible by 1 and itself. 3 is the next prime number, followed by 5 and then 7. 9 is not a prime number because it is divisible by 3.

Numbers that are not prime, are called **composite (**com•POS'it**) numbers**.

A good example of a composite number is 20. While 20 is divisible by 1 and 20, these are not the only divisors to go evenly into 20. 20 is also divisible by 2, 4, and 10.

Another good example of a composite number is 50. You can certainly divide 50 by 1 and 50 to get to 50. But you can ALSO divide by 2, 5, 10, and 25.

To recap: if a given number is only divisible by 1 and the number itself, the number is called *prime*. If the number has any additional divisors, the number is called *composite*.

One of the largest prime numbers as of August 23, 2008 consists of 12,978,179 digits is $2^{43,112,609}$ minus 1 which was found by a computer.

Table 4.2 Prime and Composite Numbers

Prime Number	A prime number is a whole number that only has two factors which are itself and one. 2 is the first prime number
Composite Number	A composite number has factors in addition to one and itself. In other words, a composite number is not a prime number.
0 and 1	Considered neither prime nor composite.

Prime Factorization

Sometimes, we want to know all of the prime numbers that go into making a number. When a number is written as the product of all its prime factors, we call that **"prime factorization."**

To help us figure out the prime factorization of any number, we can use what is called a **"factor tree."**

So, for example, take the number 120. What are the factors of 120? We can choose any two factors we desire so let's start with 30 • 4 = 120.

So far, our factor tree looks like this:

```
 120
 / \
30  4
```

Next, we can factor 30 as 6•5 and 4 as 2•2, so our factor tree now looks like this:

```
   120
   / \
  30   4
 / \  / \
6  5  2  2
```

Finally, we can continue to factor the 6 as 2•3, but the factors at the other ends of the tree, 5, 2, and 2, are all prime and cannot be further factored. Thus, the final factor tree is shown below:

```
       120
        / \
     30   4
     / \   / \
   6  5  2  2
  / \
 2   3
```

So, the prime factorization of 120 = 2 • 3 • 5 • 2 • 2, where all of these factors are located at the ends of the factor tree.

Now, if we arrange the factors in order from lowest to highest, we have 120 = 2 • 2 • 2 • 3 • 5

We can further simplify the factors by using exponents: 2^3 • 3 • 5

It doesn't matter which factors of 120 were originally selected. Suppose you selected 60•2 = 120. As before, we keep factoring the terms at the end of each branch of the tree until we have only prime numbers that can no longer be further factored.

```
       120
        / \
     60   2
     / \
   20  3
   / \
 10  2
 / \
2   5
```

Again, we have 120 = 2 • 5 • 2 • 3 • 2, when we arrange the factors in order from lowers to highest, we have 120 = 2 • 2 • 3 • 4 • 5 = 2^3 • 3 • 5 which is identical to our previous result.

Exercise 4.2

In problems 1 – 10, write all the factors of the number in order from lowest to highest (using a comma as a separator):

1. 8
2. 53
3. 12
4. 144
5. 33
6. 27
7. 16
8. 12
9. 36
10. 60

In problems 11 – 20, designate each number as

A. prime
B. composite:

11. 7
12. 16
13. 21
14. 19
15. 121
16. 51
17. 84
18. 141
19. 20
20. 89

In problems 21 – 30, write the prime factorization of the number (use a factor tree):

21. 26
22. 58
23. 63
24. 85
25. 154
26. 202
27. 225
28. 160
29. 120
30. 125

4.3 Greatest Common Factor and Least Common Multiple

<u>Finding the Greatest Common Factor</u>

Suppose we are given 3 numbers: 60, 132, and 96 and desire to find the largest number that will divide evenly into each of these numbers. There are two different methods that can be used to find the solution to this problem.

<u>Method 1:</u> List the factors of each number. Identify the greatest number that is on every list.

Factors of 60: 1, 2, 3, 4, 5, 6, 10, **<u>12</u>**, 15, 20, 30, 60
Factors of 132: 1, 2, 3, 4, 6, 11, **<u>12</u>**, 22, 33, 44, 66, 132
Factors of 96: 1, 2, 3, 4, 6, 8, **<u>12</u>**, 16, 24, 32, 48. 96

So the greatest common factor would be 12 (shown underlined above).

<u>Method 2:</u> Write the prime factorization of each number. The GCF is the product of the common prime factors. Notice that the prime number 2 occurs twice in each of the three numbers.

$$60 = \boxed{2}\cdot\boxed{2}\cdot\boxed{3}\cdot 5$$
$$132 = \boxed{2}\cdot\boxed{2}\cdot\boxed{3}\cdot 11$$
$$96 = \boxed{2}\cdot\boxed{2}\cdot 2\cdot 2\cdot 2\cdot\boxed{3}$$

So we have determined that the greatest common factor (GCF) is 2 • 2 • 3 = 12.

<u>Greatest Common Factors of terms that include variables</u>

In algebra, you will be expected to find the greatest common factor with numbers that include variables. Don't panic. The strategies for finding the answers are very straightforward.

For example, consider finding the greatest common factor of $16x^2y$ and $18xy^2$. First, find the prime factorization for both numbers as well as the factors of the variables:

$$16x^2\,y = \underline{\mathbf{2}} \bullet 2 \bullet 2 \bullet 2 \bullet \underline{\boldsymbol{x}} \bullet x \bullet \underline{\boldsymbol{y}} \qquad\qquad 18xy^2 = \underline{\mathbf{2}} \bullet 3 \bullet 3 \bullet \underline{\boldsymbol{x}} \bullet \underline{\boldsymbol{y}} \bullet y$$

Then, figure out what BOTH terms have in common (shown underlined in bold).

The greatest common factor of $16x^2y$ and $18xy^2$ is $2xy$. Notice that when variables are included in the terms, the greatest common factor will contain each ***common*** variable with the smallest exponent among the common terms. Since the term $16x^2y$ has the variable x^2 and the term $18xy^2$ has the variable x (raised to the 1 power), x is chosen for use in the answer to the greatest common factor since its exponent (which is 1) is less than the exponent on x^2. Similarly, y was the variable with the smallest exponent, so it was chosen.

The greatest common factor of 10 and 51 is 1 since there are no prime factors common to both numbers (since 10 = 2•5 and 51 = 3•17).

<u>Least Common Multiple</u>

A **multiple** of a whole number is the product of the number and any nonzero whole number. So, remembering your times tables, the multiples of 5 are: 5, 10, 15, 20, 25, 30….
The multiples of 10 are: 10, 20, 30, 40, 50, 60…

As you can see, the multiples of 10 can also be obtained by adding 10 successively to each number.

A **common multiple** is a multiple that is shared by two or more numbers. For example:

Multiples of 4 are: 4, 8, **12**, 16, 20, **24**, 28…
Multiples of 6 are: 6, **12**, 18, **24**, 30, 36, 42…

The common multiples of 4 and 6 are therefore 12 and 24 (in the list shown).

The **least common multiple (LCM)** is simply the least of the common multiples of two or more numbers… In this case, the lowest common multiple of 4 and 6 is 12… because it is the lowest multiple that both numbers have in common. Another way to think about the least common multiple is to ask yourself this question: what is the smallest number that two (or more) selected numbers can divide into evenly? In the above example, 4 and 6 can both divide evenly into 12, thus 12 is the LCM.

Finding the Least Common Multiple

Let's find the least common multiple of these three numbers: 30, 56, and 60. It is time consuming to list the multiples of 30, 56, and 60, so we will go directly to a method that uses the prime factors. First, we list the prime factors of each number:

$30 = 2 \cdot 3 \cdot 5$
$56 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3 \cdot 7$
$60 = 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5$

Now, we simply take the product of each unique (or different) prime number having the highest exponent among the three terms. First we notice there are 4 different prime numbers among the numbers 30, 56, and 60: 2, 3, 5, and 7. So we must now compute the product of those primes with the highest exponents: 2^3, 3, 5, and 7 (the highest exponent for each of the primes 3, 5, and 7 is implied to be 1). Thus, we have the product

$$2^3 \cdot 3 \cdot 5 \cdot 7 = 840$$

This means that 840 is the smallest number that can be divided by 30, 56, and 60.

Let's do one more example by finding the LCM of 6 and 7. We will use two methods below:

Method 1: List the multiples of each number. Then find the least number that is on both lists:

Multiples of 6 are: 6, 12, 18, 24, 30, 36, **42**, 48…
Multiples of 7 are: 7, 14, 21, 28, 35, **42**, 49…

The lowest common multiple of 6 and 7 is 42.

Method 2: Find the prime factors of the numbers:

Prime Factors of 6 are: 2, and 3.
Prime Factors of 7 are: 7 (7 IS prime).

Now, we multiply all of the unique prime factors with the highest exponents. In this problem, each prime factor occurs only once, so we simply multiply the prime factors: $2 \cdot 3 \cdot 7 = 42$

The least common multiple of 6 and 7 is 42. Please note that the product of two numbers will always yield a common multiple, but it may not always produce the least common multiple.

An example where the product of two numbers fails to provide the least common multiple, consider the two numbers 3 and 9. The least common multiple is 9 not $3 \cdot 9 = 27$.

<u>Least Common Multiples with terms that include variables</u>

In algebra, you will be expected to find the least common multiple with numbers that include variables. Again, this is a straightforward process. For example, to find the least common multiple of $14x^4$ and $21x^2$ follow these steps:

First, perform the factorization of each term to obtain the prime factors of the coefficients (numbers):

$$14x^4 = 2 \cdot 7 \cdot x^4$$

$$21x^2 = 3 \cdot 7 \cdot x^2$$

As before, we multiply all of the unique prime factors with the highest exponents. In this problem, we have the unique prime numbers 2, 3, and 7 (all raised to the first power, so we have $2 \cdot 3 \cdot 7 = 42$. Finally we multiply by each unique variable (there is only 1 variable x in each term) that is of the highest power to obtain

$$42x^4.$$

The least common multiple is $42x^4$. Notice that when variables are included in the terms, the least common multiple contains each variable with the highest exponent found among each of the individual terms.

Let's try one final example**.** Find the least common multiple of $14x^4a$ and $21x^2b$. As before we first perform the prime factorization of the coefficient of each term:

$$14x^4a = 2 \cdot 7 \cdot x^4a$$

$$21x^2b = 3 \cdot 7 \cdot x^2b$$

Next, we multiply all of the unique prime factors with the highest exponents and then multiply by each unique variable with the highest power (x^4, a, and b) to obtain:

$$2 \cdot 3 \cdot 7 \cdot x^4ab = 42x^4ab$$

Recall that the LCM will always consist of each variable in any term that has the highest exponent.

Exercise 4.3

In problems 1 – 10, find the greatest common factor.

1. 21, 99
2. 76, 86
3. 54, 89
4. 12, 36
5. 32, 55
6. $16x$, $36x$
7. 64, 144
8. 120, 960
9. $18m^2$, 7m
10. $3x^2$, $15x^3$
11. $36x^2y^3z$, $42x^3y^2$
12. $18xy$, $21x^3y^3$, $24x^2y^2$
13. $28a$, $30b$, $32c$
14. 100, 150, 225
15. 100, 200, 300

In problems 11 – 20, find the least common multiple.

16. 55, 15
17. 42, 66
18. 3, 6, 12
19. 3, 8
20. 10, 12, 14
21. $5a^2$, $16a^3$
22. $25a$, $40a^2$
23. $14x^4$, $21x^2$
24. 18, 24
25. 10, 16
26. $8x^4y^3$, $10x^3y^2$, $12xyz$
27. $48x^2$, $54y^3$, $60z^4$
28. $18x$, $24x^2$, $36x^3$
29. 2, 4, 8
30. 50, 60, 70

4.4 Addition and Subtraction of Equivalent Fractions

Equivalent Fractions

You've heard before that you can't compare apples and oranges. The same is true in fractions – we can only add or subtract fractions when they have the same denominator. **Equivalent fractions** are fractions that have (or are made to have) the same denominator.

To help you understand better, think about trying to count the value of some Mexican and American coins that a traveler pulled out of his pocket. In order to figure the total amount of money, it would be easiest to first convert the Mexican coins into the equivalent American exchange values and THEN count all the money using only U.S. coin values.

It's the same with fractions. In order to add or subtract fractions, we need to first have them be equivalent, or of the same denominator.

As an example, consider this problem: $\frac{2}{3}+\frac{1}{6}$

These two fractions cannot be readily added since their denominators are different, thus they do not have a common denominator (or the same denominator). Note, however, that $\frac{2}{3}$ is equivalent to the fraction $\frac{4}{6}$. This is the case since we can multiply $\frac{2}{3}$ by $\frac{2}{2}$ to obtain $\frac{4}{6}$. Next, let's substitute $\frac{4}{6}$ which is the equivalent of $\frac{2}{3}$ into our addition problem and we have

$$\frac{4}{6}+\frac{1}{6}=\frac{5}{6}$$

Notice, that once all fractions have a common denominator, the numerators can simply be added (4+1) and the result has the same common denominator (6).

How to Make Equivalent Fractions

"Whatever you do to the numerator (or top of the fraction), you have to do the denominator (or bottom of the fraction)." This procedure works because essentially you are multiplying the fraction by a factor of 1 which does not change the value of the fraction.

Suppose we want to add these two fractions:

$$\frac{7}{10}+\frac{17}{20}$$

These fractions cannot be added because they do not have a common (or same) denominator. BUT note that if we multiply the denominator of $\frac{7}{10}$ by 2, we obtain 20--or the same denominator as the other fraction ($\frac{17}{20}$). If we multiply the denominator of $\frac{7}{10}$ by a factor of 2, then we must also multiply the numerator by that same factor of 2 so that we do not change the value of the fraction. Thus, $\frac{7}{10}$ is equivalent to $\frac{14}{20}$ since, $\frac{7}{10}\bullet\frac{2}{2}=\frac{14}{20}$.

Now we can add the two fractions, since both $\frac{14}{20}$ and $\frac{17}{20}$ have a common denominator. Please also note that 20 is the least common multiple of 10 and 20.

Here's a hint: If you are having difficulty finding a common denominator, find the least common multiple (LCM) of both numbers, then multiply the numerator and the denominator of both numbers by the appropriate factor such that the denominator is equal to the LCM. This will always yield the least common denominator that is useful for adding or subtracting the given terms.

Adding and Subtracting Equivalent Fractions

Now that you know how to create or change fractions so that they become equivalent, you can begin to add and subtract them. Keep in mind, when adding OR subtracting fractions, the fractions must have a common denominator and the numerator is the only number that is affected. The common denominator will stay the same:

$$\frac{3}{10}+\frac{2}{10}=\frac{5}{10}$$

$$\frac{17}{20}-\frac{14}{20}=\frac{3}{20}$$

$$\frac{15}{30}+\frac{20}{30}=\frac{35}{30}\text{ (which could be also written } 1\frac{5}{30} \text{ and simplified to } 1\frac{1}{6}\text{)}$$

Simplest Form

Keep in mind whenever you are working with fractions, to always put your answers in simplest form. What do we mean by this? A fraction is in simplest form when the numerator and denominator can no longer be further divided by the same whole number.

To write a fraction in simplest form, divide the numerator and the denominator by their greatest common factor (GCF). For example:

$$\frac{5}{20}+\frac{10}{20}=\frac{15}{20}$$

However, $\frac{15}{20}$ is not in simplest form. Since both 15 and 20 are divisible by 5 (which is the GCF of 15 and 20), we can simplify the fraction as follows:

$$\frac{15}{20}\frac{\div 5}{\div 5}=\frac{3}{4}$$

As another example, consider the following:

$$\frac{2}{9}+\frac{1}{9}=\frac{3}{9}$$

However, $\frac{3}{9}$ is not in simplest form. Since both 3 and 9 are divisible by 3 (and 3 is the GCF of 3 and 9), we simplify the fraction as follows:

$$\frac{3 \div 3}{9 \div 3} = \frac{1}{3}$$

Now, let's look at a problem where the result is an improper fraction (i.e., the numerator is larger than the denominator):

$$\frac{7}{8} + \frac{5}{8} = \frac{12}{8}$$

We must now divide the 12 by 8 to get 1 with a remainder of 4 or 1 4/8, but 4/8 can be reduced to ½, so we have

$$\frac{12}{8} = 1\frac{4}{8} = 1\frac{1}{2}$$

Unless otherwise stated, always reduce your answers to simplest form.

Simplest Form with Variables

To put fractions that contain variables in simplest form, simply factor the numerator (using prime factors of the coefficient and factors of the variables) and similarly factor the denominator. Then divide out common factors. For example, let's reduce the following fraction to simplest form:

$$\frac{12xyz^3}{15y^2}$$

Notice the prime factors of the coefficient (or numeric value) and factors of each of the variables in numerator and denominator:

$$\frac{2 \bullet 2 \bullet \mathbf{3} \bullet x \bullet \mathbf{y} \bullet z \bullet z \bullet z}{\mathbf{3} \bullet 5 \bullet \mathbf{y} \bullet y}$$

The bold factors shown in the numerator, cancel with those shown in the denominator, thus leaving us with the remaining terms as shown in the reduced fraction in simplest form:

$$\frac{4xz^3}{5y}$$

Please note, that there is a shortcut to reducing the variables in a fraction. First, locate any variable that exists in both the numerator and denominator. Using the above example,

$$\frac{12xyz^3}{15y^2}$$

only the variable y is common to both the numerator and denominator. Thus, we can readily simplify $\frac{y}{y^2}$ by recalling the fact that when the same base (in this case the variable y) exists in the numerator and denominator, the exponents subtract (see Section 1.5) as follows:

$$\frac{y}{y^2} = \frac{y^1}{y^2}\; y^{1-2} = y^{-1} = \frac{1}{y}.$$

Thus, the variables in the fraction can be readily reduce to $\frac{xz^3}{y}$.

Exercise 4.4

In problems 1 – 5, find the lowest common denominator (or least common multiple, LCM, of the denominators) and convert each fraction to an equivalent fraction using the LCM.

1. $\frac{12}{16}$ $\frac{2}{8}$

2. $\frac{8}{14}$ $\frac{3}{7}$

3. $\frac{5}{15}$ $\frac{4}{9}$

4. $\frac{8}{30}$ $\frac{3}{5}$

5. $\frac{6}{10}$ $\frac{5}{8}$

In problems 6 – 10, perform the indicated addition and write the answer in simplest form.

6. $\frac{12}{16} + \frac{2}{8}$

7. $\frac{8}{14} + \frac{3}{7}$

8. $\frac{5}{15} + \frac{4}{9}$

9. $\frac{8}{30} + \frac{3}{5}$

10. $\frac{6}{10} + \frac{5}{8}$

In problems 11 – 15, perform the indicated subtraction and write the answer in simplest form.

11. $\frac{12}{16} - \frac{2}{8}$

12. $\frac{8}{14} - \frac{3}{7}$

13. $\frac{5}{15} - \frac{4}{9}$

14. $\frac{8}{30} - \frac{3}{5}$

15. $\frac{6}{10} - \frac{5}{8}$

In problems 16 – 20, write the fraction or mixed number in simplest form.

16. $\frac{2}{8}$

17. $\frac{33}{6}$

18. $5\frac{9}{12}$

19. $\frac{5}{20}$

20. $\frac{12}{4}$

4.5 Solving Algebra Using Sums and Differences

<u>Common Denominators</u>

Suppose we have fractions that we want to add and they all have a common denominator. Then we simply add the numerators and place the result over the common denominator. As an example, consider the following:

$$\frac{2}{a}+\frac{1}{a}+\frac{5}{a}+\frac{11}{a}=\frac{17}{a}$$

Subtraction works much the same way. Remember, in regular math, when you subtract fractions with common denominators, you just subtract the numerators and leave the common denominator as is (unless you're simplifying). Consider the following subtraction problem

$$\frac{19}{a}-\frac{2}{a}-\frac{5}{a}-\frac{11}{a}=\frac{1}{a}$$

<u>Different Numeric Denominators</u>

What if the denominators are different? You simply find a common denominator, then make equivalent fractions that all have the same denominator as you learned previously. Here is an example, let's add these fractions that have different denominators.

$$\frac{3}{24}+\frac{5}{12}+\frac{1}{6}$$

First, we find the least common multiple (LCM) which is 24 and then rewrite $\frac{5}{12}$ and $\frac{1}{6}$ as equivalent fractions with a denominator of 24:

$$\frac{5}{12}\bullet\frac{2}{2}=\frac{10}{24} \quad \text{and} \quad \frac{1}{6}\bullet\frac{4}{4}=\frac{4}{24}$$

Now we have all the terms with the same common denominator (or least common denominator), so we simply sum all the numerators:

$$\frac{3}{24}+\frac{10}{24}+\frac{4}{24}=\frac{17}{24}$$

Now, let's consider another similar example, except this problem contains some variables in the numerator.

$$\frac{3a}{24}+\frac{5a}{12}+\frac{a}{6}$$

We first need to determine the least common denominator of 24, 12, and 6. If we list the multiples of each term:

24: **<u>24</u>**, 48, 72, 96, …
12: 12, **<u>24</u>**, 36, 48, 60, …
6: 6, 12, 18, **<u>24</u>**, 30, 36, 42, …

We quickly note that 24 is the least common multiple of all three numbers. So that is going to be our least common denominator.

Alternatively, we could have used our knowledge of prime factors to determine the least common multiple. First we perform the prime factorization for each number:

$24 = 2 \bullet 2 \bullet 2 \bullet 3 = 2^3 \bullet 3$
$12 = 2 \bullet 2 \bullet 3 = 2^2 \bullet 3$
$6 = 2 \bullet 3$

Next, we take the product of each unique prime number with the highest exponent:

$$2^3 \bullet 3 = 24$$

Since the first fraction already has 24 as its denominator, we leave that term unchanged.

However, we must rewrite $\frac{5a}{12}$ and $\frac{a}{6}$ as equivalent fractions, both with a denominator of 24:

$$\frac{5a}{12} \bullet \frac{2}{2} = \frac{10a}{24} \quad \text{and} \quad \frac{a}{6} \bullet \frac{4}{4} = \frac{4a}{24}$$

Now we have all the terms with the same common denominator, so we simply sum all the numerators:

$$\frac{3a}{24} + \frac{10a}{24} + \frac{4a}{24} = \frac{17a}{24}$$

Next, consider this example carefully. Here, the variables are not the same (that is, they are not the same letter) so that the terms cannot be combined

$$\frac{3a}{24} + \frac{5b}{12} + \frac{c}{6}$$

We next find equivalent fractions so that we have a common denominator:

$$\frac{3a}{24} + \frac{10b}{24} + \frac{4c}{24}$$

Finally we add the numerators to obtain the result:

$$\frac{3a+10b+4c}{24}$$

The above fraction is in simplest form since the different terms in the numerator cannot be combined.

Different Algebraic Denominators

Now let's add fractions that have different variables in the denominator, such as

$$\frac{11}{a} + \frac{7}{b}$$

First we need to find a common denominator. One way you can do so is by simply multiplying the two denominators together (in general, this will not always produce the *least* common multiple—but that is okay; it will always provide a common denominator). The least common denominator of *a* and *b* in this case also turns out to be the product of the two denominators, or *ab*.

Next, looking at the 1st fraction, $\frac{11}{a}$, we must ask ourselves, by what factor must we multiply the denominator (a) to obtain *ab*? Well, this is not a difficult question, since given the denominator a, we must simply multiply it by the factor *b* to yield the result *ab*. Thus, we have determined that we must multiply the 1st fraction by $\frac{b}{b}$ to yield an equivalent fraction that has *ab* as the denominator:

$$\frac{11}{a} \bullet \frac{b}{b} = \frac{11b}{ab}$$

Similarly, for the 2nd fraction, $\frac{7}{b}$, in our addition problem above, we must ask ourselves, by what factor must we multiply the denominator (*b*) to obtain *ab*? Again, the answer is obvious; we must multiply by $\frac{a}{a}$ to yield an equivalent fraction that has *ab* as the denominator:

$$\frac{7}{\mathrm{b}} \bullet \frac{a}{a} = \frac{7a}{ab}$$

So, now that we have created equivalent factors for all terms, we can rewrite the original equation as

$$\frac{11b}{ab} + \frac{7a}{ab} = \frac{7a+11b}{ab}$$

Exercise 4.5

In problems 1 – 5, perform the indicated addition or subtraction of fractions and reduce your answer to lowest terms.

1. $\frac{1}{3}+\frac{1}{3}$

2. $\frac{3}{10}+\frac{1}{10}$

3. $\frac{7}{16}+\frac{5}{16}$

4. $\frac{15}{y}+\frac{11}{y}$

5. $\frac{20}{rs}-\frac{14}{rs}$

In problems 6 – 10, perform the indicated addition or subtraction of fractions having unlike numeric denominators.

6. $\frac{2}{5}+\frac{2}{3}$

7. $\frac{5}{8}+\frac{1}{6}$

8. $\frac{x}{6}-\frac{y}{4}$

9. $\frac{2a}{3}-\frac{b}{4}$

10. $\frac{5n}{6}-\frac{1}{3}$

In problems 11 – 15, perform the indicated addition or subtraction of fractions and state answer in simplest form.

11. $\frac{1}{b}+\frac{2}{a}$

12. $\frac{2}{a}-\frac{3}{b}$

13. $\frac{x}{1}-\frac{3}{y}$

14. $\frac{a}{2}-\frac{3}{b}$

15. $\frac{s}{1}+\frac{s}{r}$

4.6 Multiplying Fractions

Multiplying Fractions

Multiplying (and dividing) fractions is actually easier than adding or subtracting fractions—since you do not have to find a common denominator. Consider this multiplication problem

$$\frac{1}{2} \bullet \frac{3}{4}$$

To multiply fractions, all you need to do is multiply the numerators together, and then multiply the denominators together. So we have,

$$\frac{1 \bullet 3}{2 \bullet 4} = \frac{3}{8}$$

Simple—isn't it!

Multiplying Mixed Numbers

If we are multiplying factors that are mixed numbers, it is easiest to first convert the mixed numbers to improper fractions. Consider this problem:

$$1\frac{1}{2} \bullet \frac{1}{2}$$

We convert $1\frac{1}{2}$ to the improper fraction $\frac{3}{2}$, so that we now have

$$\frac{3}{2} \bullet \frac{1}{2} = \frac{3}{4}$$

Consider another example where we are multiplying factors that are both mixed numbers:

$$3\frac{1}{3} \bullet 2\frac{1}{2}$$

Again, we convert both mixed numbers to improper fractions and perform the multiplication:

$$\frac{10}{3} \bullet \frac{5}{2} = \frac{10 \bullet 5}{3 \bullet 2} = \frac{50}{6} = 8\frac{1}{3}$$

Multiplying Fractions that consist of Variables in the numerator and denominator

We treat fractions with variables no differently than if we were working with constants—the numerators are multiplied together, and the denominators are multiplied together as shown in this example:

$$\frac{m}{x} \bullet \frac{c}{d} = \frac{mc}{xd} \text{ or } \frac{cm}{dx} \text{ (if we list the variables in alphabetical order)}$$

As another example consider this problem:

$$\frac{\mathrm{m}}{\mathrm{n}} \bullet \frac{1}{2}$$

Multiplying the numerators together and the denominators, we obtain the answer:

$$\frac{\mathrm{m}}{\mathrm{n}} \bullet \frac{1}{2} = \frac{m \bullet 1}{n \bullet 2} = \frac{m}{2n}$$

The Method of Cancelling

A little shortcut for how to multiply fractions is what is called **"cancelling."** Watch how it works. Let's begin with the multiplication of two fractions. We note that 5 is the greatest common factor of 5 in the numerator and 10 in the denominator. So we divide 5 into both the numerator and denominator to obtain:

$$\frac{7}{{}_{2}\cancel{10}} \bullet \frac{\cancel{5}^{1}}{4}$$

Now we have simplified the problem through cross-cancelling to this:

$$\frac{7 \bullet 1}{2 \bullet 4} = \frac{7}{8}$$

By pulling out the common factors (in this case, "5") the expression is much easier to simplify and reduce to lowest terms.

The same is true when variables are present in the fractions. Consider this multiplication problem:

$$\frac{m}{3} \bullet \frac{9}{n}$$

We now cross-cancel in a manner similar to the previous problem and we have

$$\frac{\mathrm{m}}{{}_{1}\cancel{3}} \bullet \frac{\cancel{9}^{3}}{\mathrm{n}}$$

So that our final answer is

$$\frac{3m}{n}$$

Exercise 4.6

In problems 1 – 5 perform the indicated multiplication of the fractions.

1. $\frac{5}{12} \bullet \frac{3}{2}$
2. $\frac{7}{10} \bullet \frac{5}{4}$
3. $\frac{6}{5} \bullet \frac{7}{8}$
4. $\frac{11}{12} \bullet 3$
5. $\frac{1}{2} \bullet \frac{1}{2}$

In problems 6 – 10 perform the multiplication of the mixed fractions.

6. $3\frac{3}{4} \bullet 2\frac{2}{3}$
7. $7\frac{1}{10} \bullet 3\frac{1}{3}$
8. $10\frac{2}{8} \bullet 3\frac{4}{6}$
9. $11\frac{1}{2} \bullet 6\frac{1}{9}$
10. $5\frac{3}{8} \bullet 2\frac{6}{9}$

In problems 11 – 15 perform the indicated multiplication of fractions with variables.

11. $\frac{r}{s} \bullet \frac{r}{s}$
12. $\frac{m}{3} \bullet \frac{9}{n}$
13. $\frac{5}{j} \bullet \frac{k}{10}$
14. $\frac{2r}{s} \bullet \frac{8}{2r}$
15. $\frac{7}{d} \bullet \frac{3d}{3}$

4.7 Solving Algebra Using Products of Fractions

Review

$$\frac{1}{2}\bullet\frac{3}{4}=\frac{1\bullet 3}{2\bullet 4}=\frac{3}{8}$$

Or here is another example:

$$\frac{5}{8}\bullet\frac{4}{6}=\frac{5\bullet 4}{8\bullet 6}=\frac{20}{48}=\frac{5}{12}$$

Is there a shortcut?

But of course everybody wonders if there's an easier way to do things. Everyone likes a shortcut. Maybe the building inspector has a lot of work to do or maybe he wants to get to his lunch break faster but he still needs to do the job correctly.

A little shortcut for how to multiply fractions is called **'cancelling'**. It's called 'cancelling' because you can cancel numbers out if they have a common factor. Then the numbers are easier to work with because they are smaller (you were introduced to cancelling in 4.6, but this is a great reminder!).

Have a look at the example of 'cancelling' below. See how the 10 and the 5 have the number 5 in common. That's a common factor. The 5 obviously has one 5 in it. And the 10 has two 5's in it. So you just cross out the 5 and make it a 1. And you cross out the 10 and make it a 2.

See how this works:

$$\frac{7}{{}_{2}\cancel{10}}\bullet\frac{\cancel{5}^{\,1}}{4}$$

So now we have simplified the problem to

$$\frac{7\bullet 1}{2\bullet 4}=\frac{7}{8}$$

Here is another problem that involves multiplying fractions that contain variables in the numerators and denominators. Since there are no terms in the numerator that are factors or multiples of terms in the denominator, we cannot perform any cancelling on this problem:

$$\frac{u}{x}\bullet\frac{v}{y}=\frac{uv}{xy}$$

Here are two more examples where simplifying first by cancelling is not possible:

$$\frac{n}{m}\bullet\frac{2}{p}=\frac{2n}{mp}\quad\text{and}\quad\frac{6}{f}\bullet\frac{4a}{b}=\frac{24a}{bf}$$

(notice that the denominator *bf* was rearranged so as to have the variables in alphabetical order).

In this next example, even though the problem looks more complicated than those we have previously considered, we can cross-cancel the numbers that are factors of each other as well as those variables that are the same:

$$\frac{24y^2x^3}{8a^3b} \div \frac{30y^3x^2}{32a^2b^3}$$

First, we must convert this problem to a multiplication problem by inverting the fraction that follows the division sign:

$$\frac{24y^2x^3}{8a^3b} \bullet \frac{32a^2b^3}{30y^3x^2}$$

Next, we can recognize that 6 is the greatest common factor of both 24 in the numerator and 30 in the denominator; also 8 is the greatest common factor of both 32 in the numerator and 8 in the denominator. We can also cross-cancel the y^2 in the numerator with the y^3 in the denominator, leaving just y in the denominator. When we cross-cancel the x^3 in the numerator with the x^2 in the denominator, we are left with just x in the numerator. Similarly, we can cancel the a^2 in the numerator with a^3 in the denominator, leaving a in the numerator; and finally, we can cancel the b^3 in the numerator with the b in the denominator, leaving b^2 in the numerator. Another way to visualize these cancellations is to expand out the variables that have exponents and express them as shown below. So, after performing these cancellations, we now have reduced the entire problem to this:

$$\frac{24\cancel{y}\cancel{y}\cancel{x}\cancel{x}x}{8\cancel{a}\cancel{a}a\cancel{b}} \bullet \frac{32\cancel{a}\cancel{a}\cancel{b}bb}{30\cancel{y}\cancel{y}y\cancel{x}\cancel{x}} = \frac{4x}{1a} \bullet \frac{4b^2}{5y}$$

Now let's complete the multiplication. Since $4x \bullet 4b^2 = 16b^2x$ (notice I have re-arranged the variables so that they appear in alphabetical order, although $16xb^2$ is also correct) and $1a \bullet 5y = 5ay$, we now have the final reduced answer.

$$\frac{16b^2x}{5ay}$$

When there are variables involved in the numerator or denominator, mathematicians usually do not go to the further step of expressing the improper fraction, $\frac{16}{5}$, as a mixed number $3\frac{1}{5}$. So the above answer is considered simplified enough and is the final answer. Although technically correct, *never* write

$$3\frac{1}{5}\frac{b^2x}{ay}$$

Likewise, it is unconventional to write a term in pre-algebra (or algebra) using a mixed fraction as a coefficient, for example: $4\frac{1}{8}x$**.** Usually when the coefficient of a variable is a mixed number, we write the coefficient as an improper fraction so instead, we would write $\frac{33x}{8}$ or $\frac{33}{8}x$. Additionally, if you cancel the x's in the numerator and denominator of a fraction such as $\frac{x}{2x^2}$ then all terms in the numerator are gone thus yielding a numerator equal to "1", so that this yields the simplified fraction $\frac{1}{2x}$.

Exercise 4.7

In problems 1 – 15, perform the indicated multiplication.

1. $\frac{6}{q} \bullet \frac{p}{6}$

2. $\frac{2u}{y} \bullet \frac{v}{2y}$

3. $\frac{20m}{3} \bullet \frac{9}{4n}$

4. $\frac{6}{5} \bullet \frac{5u}{v}$

5. $\frac{2r}{s} \bullet \frac{8}{3r}$

6. $\frac{2}{9e} \bullet \frac{3d}{4}$

7. $\frac{4}{9} \bullet \frac{3x}{y}$

8. $\frac{6}{j} \bullet \frac{k}{10}$

9. $\frac{4v}{3} \bullet \frac{1}{8}$

10. $12 \bullet \frac{5x}{3}$

11. $\frac{8}{j} \bullet 3$

12. $\frac{2a}{b} \bullet \frac{10}{n}$

13. $\frac{5t}{2} \bullet \frac{7e}{p}$

14. $\frac{6}{15n} \bullet \frac{2p}{7}$

15. $\frac{x}{5} \bullet \frac{1}{x}$

4.8 Multiplying and Dividing Fractions

Dividing by a fraction is the same as multiplying by the reciprocal or inverse.

The block around Sammy's house is exactly $\frac{1}{3}$ mile. Sammy knows she rode her bike for exactly 4 miles today (according to an app on her phone). She wants to know then how many laps she rode exactly.

Given there is 1/3 mile/lap and she rode a distance of 4 miles, we must divide the distance by the distance per lap. Just looking at the dimensions we have

laps = distance in miles ÷ distance in miles/lap

Next, we change the divide sign to a multiplication sign and invert the fraction (or divisor), thus

laps = distance in miles • laps/distance in miles

Notice that "distance in miles" cancels in the numerator and denominator, leaving the units as laps—which is what we desire. Many times such "dimensional analysis" will help us determine whether it is appropriate to multiple or divide two numbers to get the desired answer. So, now let's perform the division:

$$4 \div \frac{1}{3}$$

First, we change the division to a multiplication problem and invert the fraction:

$$\frac{4}{1} \bullet \frac{3}{1}$$

Thus, Sammy rode her bike 12 laps around the block.

Take a look at another example: Michelle and Andrea are track and field stars, and are comparing the tracks they use to train on. Michelle's track is $\frac{9}{10}$ of a mile long, and Andrea's is $\frac{3}{5}$ of a mile long. Michelle wants to know how many of Andrea's tracks would fit into hers, that is, by what factor is Michelle's track longer than Andrea's track. The problems is

$$\frac{9}{10} \div \frac{3}{5} = \frac{9}{10} \bullet \frac{5}{3} = \frac{45}{30} = 1\tfrac{1}{2}$$

If we perform some preliminary cross-cancelling, since 3 in the denominator is a factor of 9 in the numerator and 5 in the numerator is a factor of 10 in the denominator, we have

$$\frac{{}^{3}\cancel{9}}{{}_{2}\cancel{10}} \cdot \frac{\cancel{5}^{1}}{\cancel{3}_{1}}$$

This simplifies the problem to

$$\frac{3 \bullet 1}{2 \bullet 1} = \frac{3}{2} = 1\frac{1}{2}$$

Therefore, Michelle's track could fit $1\frac{1}{2}$ of Andrea's tracks; that is, Michelle's track is $1\frac{1}{2}$ times longer than Andrea's track.

Dividing Mixed Numbers

Justin's favorite hiking trail is $6\frac{1}{2}$ miles long, and takes $2\frac{3}{4}$ hours to complete. How many miles is Justin covering, on average, per hour?
Since we are looking for an answer with the units of miles per hour, we must divide the given miles by the given hours:

$$6\frac{1}{2} \div 2\frac{3}{4}$$

First, we convert each mixed number to an improper fraction and rewrite the problem:

$$\frac{13}{2} \div \frac{11}{4}$$

Next, we change the division of a fraction to multiplication of the inverse (or reciprocal):

$$\frac{13}{2} \bullet \frac{4}{11}$$

Noting that we can cross-cancel the 2 with the 4, we can simplify the problem to

$$\frac{13}{1} \bullet \frac{2}{11} = \frac{26}{11} = 2\frac{4}{11}$$

Thus, Justin's speed averages $2\frac{4}{11}$ miles/hour.

Please note that when performing a division of fractions, you should not perform any cross cancelling until you have converted the problem to a multiplication of fractions. For example,

$\frac{1}{10} \div \frac{5}{4}$ must first be changed to $\frac{1}{10} \cdot \frac{4}{5}$ before attempting to simplify by cross cancelling.

Dividing Fractions with Variables

The procedures for dividing fractions that contain variables are really not different from the procedures given in the previous section. Again, we change the division problem to a multiplication problem and invert the fraction that is the divisor. Here is one more example:

$$26 \div \frac{x}{3} = 26 \bullet \frac{3}{x} = \frac{26 \bullet 3}{x} = \frac{78}{x}$$

Exercise 4.8

In problems 1 – 6, perform the indicated division of fractions and express your answer in lowest terms.

1. $\frac{5}{12} \div \frac{3}{2}$

2. $\frac{7}{10} \div \frac{5}{4}$

3. $\frac{6}{5} \div \frac{7}{8}$

4. $\frac{11}{12} \div 3$

5. $\frac{1}{2} \div \frac{1}{2}$

In problems 6 – 10, perform the indicated division of mixed numbers and express your answer is lowest terms.

6. $3\frac{3}{4} \div 2\frac{2}{3}$

7. $7\frac{1}{10} \div 3\frac{1}{3}$

8. $10\frac{2}{8} \div 3\frac{4}{6}$

9. $11\frac{1}{2} \div 6\frac{1}{9}$

10. $5\frac{3}{8} \div 2\frac{6}{9}$

In problems 11 – 15, perform the indicated division of fractions.

11. $\frac{r}{s} \div \frac{r}{s}$

12. $\frac{m}{3} \div \frac{9}{n}$

13. $\frac{5}{j} \div \frac{k}{10}$

14. $\frac{2r}{s} \div \frac{8}{2r}$

15. $\frac{7}{d} \div \frac{3d}{3}$

4.9 Solving Algebra Using Quotients of Fractions

Here are some more examples of division problems that utilize fractions that contain variables:

$$\frac{1}{r} \div \frac{s}{t} = \frac{1}{r} \bullet \frac{t}{s} = \frac{t}{rs}$$

$$\frac{r}{2} \div \frac{5t}{2} = \frac{r}{2} \bullet \frac{2}{5t} = \frac{2r}{10t} = \frac{r}{5t}$$

Please note that only the fraction that follows the division sign is inverted.

Let's consider still another example where we are dividing by a term that is not a fraction. Notice that we effectively consider f to be the same as $\frac{f}{1}$, so we are able to still apply the rule of inverting the divisor as shown in these steps:

$$\frac{6d}{e} \div f = \frac{6d}{e} \bullet \frac{1}{f} = \frac{6d}{ef}$$

Summary of Addition, Subtraction, Multiplication, and Division of Fractions

Operation involving Fractions	**Solution**
Adding fractions with common denominators	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
Subtracting fractions with common denominators	$\frac{a}{c} \quad \frac{b}{c} = \frac{a-b}{c}$
Adding fractions with different denominators	$\frac{a}{m} + \frac{b}{n} = \frac{an+bn}{mn}$
Subtracting fractions with different denominators	$\frac{a}{m} - \frac{b}{n} = \frac{an-bn}{mn}$
Multiplying fractions	$\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$ $a \bullet \frac{b}{c} = \frac{ab}{c}$
Dividing fractions*	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \bullet \frac{d}{c} = \frac{ad}{bc}$ $a \div \frac{b}{c} = a \bullet \frac{c}{b} = \frac{ac}{b}$

***Note:** You may not cross-cancel fractions that are being divided. Instead, you must first rewrite the division as a product (by multiplying by the inverse of the 2nd fraction) and then, if possible, do cross-cancelling.

Exercise 4.9

In problems 1 – 15, perform the indicated division.

1. $\frac{6}{q} \div \frac{p}{6}$

2. $\frac{2u}{y} \div \frac{v}{2y}$

3. $\frac{m}{3} \div \frac{9}{n}$

4. $\frac{6}{5} \div \frac{5u}{v}$

5. $\frac{2r}{s} \div \frac{8}{2r}$

6. $\frac{2}{9e} \div \frac{3d}{4}$

7. $\frac{4}{9} \div \frac{3x}{y}$

8. $\frac{6}{j} \div \frac{k}{10}$

9. $\frac{4v}{3} \div \frac{1}{8}$

10. $12 \div \frac{5x}{3}$

11. $\frac{8}{j} \div 3$

12. $\frac{2a}{b} \div \frac{10}{n}$

13. $\frac{5t}{2} \div \frac{7e}{p}$

14. $\frac{6}{15n} \div \frac{2p}{7}$

15. $\frac{x}{5} \div \frac{1}{x}$

Answers to Chapter 4 Exercises

Exercise 4.1

1. $\frac{3}{2}$; **2.** $\frac{9}{4}$; **3.** $\frac{22}{5}$; **4.** $\frac{41}{6}$; **5.** $\frac{101}{4}$; **6.** $\frac{58}{5}$; **7.** $\frac{38}{3}$; **8.** $\frac{41}{3}$; **9.** $\frac{81}{8}$; **10.** $\frac{163}{3}$; **11.** $1\frac{1}{4}$; **12.** $1\frac{3}{10}$; **13.** $3\frac{1}{3}$; **14.** $3\frac{1}{6}$; **15.** 6; **16.** $7\frac{7}{25}$; **17.** $4\frac{5}{6}$; **18.** $3\frac{6}{7}$; **19.** $12\frac{3}{16}$; **20.** $2\frac{1}{8}$; **21.** $\frac{5}{21}$; **22.** $\frac{11}{8}$; **23.** $\frac{39}{50}$

Exercise 4.2

1. 1,2,4,8; **2.** 1,53; **3.** 1,2,3,4,6,12; **4.** 1,2,3,4,6,8,9,12,16,18,24,36,48,72,144; **5.** 1,3,11,33; **6.** 1,3,9,27; **7.** 1,2,4,8,16; **8.** 1,2,3,4,6,12; **9.** 1,2,3,4,6,9,12,18,36; **10.** 1,2,3,4,5,6,10,12,15,20,30,60; **11.** A; **12.** B; **13.** A; **14.** A; **15.** B; **16.** B; **17.** B; **18.** B; **19.** B; **20.** A; **21.** 2•13; **22.** 2•29; **23.** 3^2•7; **24.** 5•17; **25.** 2•7•11; **26.** 2•101; **27.** 3^2•5^2; **28.** 2^5•5; **29.** 2^3•3•5; **30.** 5^3

Exercise 4.3

1. 3; **2.** 2; **3.** 1; **4.** 12; **5.** 1; **6.** $4x$; **7.** 16; **8.** 120; **9.** m; **10.** $3x^2$; **11.** $6x^2y^2$; **12.** $3xy$; **13.** 2; **14.** 25; **15.** 100; **16.** 165; **17.** 462; **18.** 12; **19.** 24; **20.** 420; **21.** $80a^3$; **22.** $200a^2$; **23.** $42x^4$; **24.** 72; **25.** 80; **26.** $120x^4y^3z$; **27.** $2160x^2y^3z^4$; **28.** $72x^3$; **29.** 8; **30.** 2100

Exercise 4.4

1. $\frac{12}{16}, \frac{4}{16}$; **2.** $\frac{8}{14}, \frac{6}{14}$; **3.** $\frac{15}{45}, \frac{20}{45}$; **4.** $\frac{8}{30}, \frac{18}{30}$; **5.** $\frac{24}{40}, \frac{25}{40}$; **6.** 1; **7.** 1; **8.** $\frac{7}{9}$; **9.** $\frac{13}{15}$; **10.** $1\frac{9}{40}$; **11.** $\frac{1}{2}$; **12.** $\frac{1}{7}$; **13.** $-\frac{1}{9}$; **14.** $-\frac{1}{3}$; **15.** $-\frac{1}{40}$; **16.** $\frac{1}{4}$; **17.** $5\frac{1}{2}$; **18.** $5\frac{3}{4}$; **19.** $\frac{1}{4}$; **20.** 3

Exercise 4.5

1. $\frac{2}{3}$; **2.** $\frac{2}{5}$; **3.** $\frac{3}{4}$; **4.** $\frac{26}{y}$; **5.** $\frac{6}{rs}$; **6.** $1\frac{1}{15}$; **7.** $\frac{19}{24}$; **8.** $\frac{2x-3y}{12}$; **9.** $\frac{8a-3b}{12}$; **10.** $\frac{5n-2}{6}$; **11.** $\frac{a+2b}{ab}$; **12.** $\frac{2b-3a}{ab}$; **13.** $\frac{xy-3}{y}$; **14.** $\frac{ab-6}{2b}$; **15.** $\frac{rs+s}{r}$

Exercise 4.6

1. $\frac{5}{8}$; **2.** $\frac{7}{8}$; **3.** $1\frac{1}{20}$; **4.** $2\frac{3}{4}$; **5.** $\frac{1}{4}$; **6.** 10; **7.** $23\frac{2}{3}$; **8.** $37\frac{7}{12}$; **9.** $70\frac{5}{18}$; **10.** $14\frac{1}{3}$; **11.** $\frac{r^2}{s^2}$; **12.** $\frac{3m}{n}$ **13.** $\frac{k}{2j}$; **14.** $8/s$; **15.** 7

Exercise 4.7

1. $\frac{p}{q}$ **2.** $\frac{uv}{y^2}$; **3.** $\frac{15m}{n}$; **4.** $\frac{6u}{v}$; **5.** $\frac{16}{3s}$; **6.** $\frac{d}{6e}$; **7.** $\frac{4x}{3y}$; **8.** $\frac{3k}{5j}$; **9.** $\frac{v}{6}$; **10.** $20x$; **11.** $\frac{24}{j}$; **12.** $\frac{20a}{bn}$; **13.** $\frac{35et}{2p}$; **14.** $\frac{4p}{35n}$; **15.** $\frac{1}{5}$

Exercise 4.8

1. $\frac{5}{18}$; **2.** $\frac{14}{25}$; **3.** $1\frac{13}{35}$; **4.** $\frac{11}{36}$; **5.** 1; **6.** $1\frac{13}{32}$; **7.** $2\frac{13}{100}$; **8.** $2\frac{35}{44}$; **9.** $1\frac{97}{110}$; **10.** $2\frac{1}{64}$; **11.** 1; **12.** $\frac{mn}{27}$; **13.** $\frac{50}{jk}$; **14.** $\frac{r^2}{2s}$; **15.** $\frac{7}{d^2}$

Exercise 4.9

1. $\frac{36}{pq}$; **2.** $\frac{4u}{v}$; **3.** $\frac{mn}{27}$; **4.** $\frac{6v}{25u}$; **5.** $\frac{r^2}{2s}$; **6.** $\frac{8}{27de}$; **7.** $\frac{4y}{27x}$; **8.** $\frac{60}{jk}$; **9.** $\frac{32v}{3}$; **10.** $\frac{36}{5x}$; **11.** $\frac{8}{3j}$; **12.** $\frac{an}{5b}$; **13.** $\frac{5pt}{14e}$; **14.** $\frac{21}{15np}$; **15.** $\frac{x^2}{5}$

Chapter 5: Rational Numbers, Expressions, and Equations

5.1 Rational Numbers

<u>Rational Numbers – An Introduction</u>

A rational number is any number that can be written or expressed as the quotient (or fraction) of any two integers (or whole numbers). The denominator of such a fraction cannot be zero.

Any whole number can be called a rational number, since it can be expressed as a fraction with 1 as the denominator.

Here are some examples:

Rational numbers:

$\frac{3}{4}$ $\quad -\frac{5}{6}$ $\quad \frac{18}{11}$ $\quad -\frac{9}{5}$ $\quad 2\frac{1}{3}$ $\quad -1\frac{2}{3}$ $\quad 14$ $\quad .29$ $\quad 0.26$ $\quad -0.798156$ $\quad \sqrt{81}$ $\quad \sqrt{6.25}$

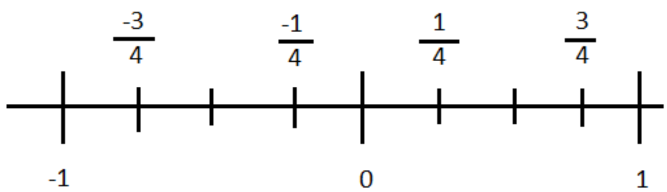
Note: $\sqrt{81}$ is rational since $\sqrt{81} = 9$. Also, $\sqrt{6.25}$ is rational since $\sqrt{6.25} = 2.5$. These square roots are of numbers that yield a whole number or non-repeating, terminating decimal number (discussed in Section 5.2).

NOT rational numbers:

$\frac{8}{0}$ $\quad 5\frac{3}{0}$ $\quad \pi$ ($\pi = 3.1415926535\ldots$) $\quad e$ ($e = 2.718281828459\ldots$) $\quad \sqrt{7}$ (approx. $2.64575131106\ldots$)

<u>Number Lines</u>

Number lines are another way to visualize rational numbers.



<u>Comparing Rational Numbers</u>

With integers, it is easy to observe that 17 is larger than 11 just by comparing the values of the numbers.

But comparing rational numbers in fraction form will often require rewriting the fractions with a common denominator.

<u>Comparing rational numbers with *same* denominator:</u>

When the denominators are identical, simply compare the numerators to establish which fraction is larger or smaller:

$\frac{5}{6}$ is greater than $\frac{3}{6}$ $\frac{2}{10}$ is less than $\frac{7}{10}$

Comparing rational numbers with *different* denominators:

- This task is often easier if you first change the rational numbers (fractions) so that they have the same denominator by forming equivalent fractions. Then you simply compare the numerators as discussed above.

Suppose we want to determine which is larger\? $\frac{2}{3}$ or $\frac{5}{6}$

Since the fractions that we are considering do not have a common denominator, and 6 is the least common multiple of the denominators 3 and 6, we can multiply $\frac{2}{3}$ by $\frac{2}{2}$ to obtain the equivalent fraction $\frac{4}{6}$.

Now that the denominators of both fractions are the same, we can compare $\frac{4}{6}$ with $\frac{5}{6}$ and it is obvious that $\frac{4}{6}$ is less than $\frac{5}{6}$.

Exercise 5.1

Are the following rational numbers, yes or no?

1. $\frac{5}{9}$

2. $\frac{2}{3}$

3. $\sqrt{7}$

4. e^2

5. 9

Which of the following rational numbers are *larger*? If the numbers are equal, then write equal.

6. $\frac{5}{4}$ or $\frac{7}{6}$

7. $\frac{1}{4}$ or $\frac{9}{36}$

8. $\frac{1}{9}$ or $\frac{3}{13}$

9. 3 or $\frac{51}{17}$

10. $\frac{9}{11}$ or $\frac{7}{8}$

Which of the following rational numbers are *smaller*? If the numbers are equal, then write equal.

11. 2.3 or $2\frac{1}{3}$

12. 4.5 or $\frac{90}{20}$

13. $\frac{1}{3}$ or 0.3333533

14. 7.2 or $\frac{15}{2}$

15. $\frac{5}{13}$ or $\frac{9}{17}$

Solve each word problem for the requested information. Often you can use *dimensional analysis* to help solve these word problems (see p. 104).

16. Each text message costs $0.27. How much would 89 messages cost?

17. It takes you 16 minutes to download a 900 Mbyte movie with your laptop using your home network. But you know that the speed in the library hotspot is 1 Mbyte per second. Which place would you choose to download your files the fastest? How long would the 900 Mbyte movie take to download in the library?

18. You are about to choose a car and find two used cars in excellent condition: one is a 2002 model and the other is a 2005. Each vehicle is priced the same at $6000.00. When you ask for the car the salesman, he tells you that the 2002 vehicle has a fuel consumption rate of 12 liters of gas/100 km while the 2005 car has fuel consumption rate of 0.188 liters/km. Which vehicle would save more money in gas?

5.2 Writing Rational Numbers

Writing a Whole Numbers as Rational Numbers

How do we write a whole number as a rational number (or fraction)? We simply divide by 1. For example, if we have the numbers 5, 11 and 654, then to convert these to rational numbers we have

$$\frac{5}{1} \qquad \frac{11}{1} \qquad \frac{654}{1}$$

Writing a Rational Number in Decimal Form

To write a rational number such as $\frac{6}{10}$ in decimal form, divide the numerator (top number) by the denominator (bottom number). This operation is easily performed with a calculator by keying in the numerator **6**, pressing the ÷ (divide) button, and then keying in the denominator **10** and pressing the = (equal) button; otherwise you must simply resort to doing the operation by hand using long division:

$$10\overset{\quad 0.6}{\overline{\smash{)}6.0}}$$

Terminating Versus Repeating Decimals

There are two kinds of decimals – terminating and repeating. Terminating decimals are those that occur when you change a rational number into a decimal and the string of numbers contained in the decimal answer STOPS. The following decimals are terminating:

0.25 0.333 0.56 0.89 0.1124

However, at other times the string of numbers in the decimal answer *doesn't* stop. We call this type of decimal a repeating decimal. For example:

0.2333333… 0.7777… 0.89999… 0.565656... 0.13241324132413 24...

So, to summarize: if the numbers in a decimal end at some point, we call that kind of decimal terminating; however, if a number or sequence of numbers in a decimal continue to repeat on into infinity, we call that a repeating decimal.

Writing Terminating Decimals as Fractions

So what if you wanted to convert a decimal number back into a fraction? The procedure varies somewhat depending upon whether we are converting a terminating decimal or a repeating decimal. First, let's consider the converting of a terminating decimal to a fraction—since this is very straightforward. Use the place of the last digit to determine the denominator of the fraction. For example, given the decimal number 0.9, since 9 is the last digit of the decimal and it is in the tenths place, the denominator is 10. Thus, we have the fraction $\frac{9}{10}$.

Given the decimal number 0.98, since 8 is the last digit of the decimal and it is in the hundredths place, the denominator is 100. Thus, we have the fraction $\frac{98}{100}$ which can be further reduced to

$$\frac{49}{50}.$$

Here is one more example using a whole number and fraction combination.

$$3.05 = 3\frac{5}{100} = 3\frac{1}{20}$$

Notice that when there is a whole numbers with the decimal, you simply keep the whole number and then follow the above steps to convert the decimal portion to a fraction. Don't forget to simplify.

Writing Repeating Decimals as Fractions

To convert a repeating decimal to a fraction takes an additional step that involves some pre-algebra. To learn how to write a repeating decimal as a fraction, let's consider converting $0.4\overline{5}$ (or 0.455555555….)

First, we set the variable x equal to the repeating decimal, so let

$$x = 0.4555555555...$$

Next, because there is only a single digit (the "5") that repeats, we multiply both sides of the above equation by 10, so that we obtain

$$10x = 4.5555555555...$$

Now, here is where a "trick" comes into play. We next do something that we have never done before—we actually subtract the two equations as follows:

$$\begin{array}{rr} & 10x = 4.5555555555... \\ - & x = 0.4555555555... \\ \hline & 9x = \ 4.1 \end{array}$$

Finally, we solve for x by dividing each side of the equation by 9, so that we have

$$x = \frac{4.1}{9}$$

To obtain a whole number in the numerator, let's now multiply both numerator and denominator by 10 to form the equivalent fraction

$$\frac{41}{90}$$

Thus, we have accomplished our goal and have determined that $0.4\overline{5} = \frac{41}{90}$

Let’s try one more problem where we convert a repeating decimal to a fraction. This time, we will convert $0.\overline{31}$ (or 0.31313131...)

As we have done previously, we first set x equal to the repeating decimal:

$$x = 0.31313131...$$

Next, since two digits are repeating, we multiply both sides of the equation by 100 to obtain

$$100x = 31.31313131...$$

Next, we subtract the two equations to obtain:

$$\begin{array}{rr} & 100x = 31.31313131... \\ - & \underline{x = \ \ 0.31313131...} \\ & 99x = 31 \end{array}$$

Finally, since x is multiplied by 9, we divide both sides of the equation by 9 to obtain

$$x = \frac{31}{99}$$

Thus, we have determined that $0.\overline{31} = \frac{31}{99}$

Let’s consider one more example: convert $13.\overline{31}$ (which is a whole number with a repeating decimal) to a fraction. Well, an easy way to think about this is to consider that we are going to end up with 13 plus a fraction that is equivalent to the decimal portion $0.\overline{31}$. Following the same procedure given above, we determined that $0.\overline{31} = \frac{31}{99}$, therefore $13.\overline{31} = 13 + \frac{31}{99}$. Adding the whole number to the fraction we obtain the result $13\frac{31}{99}$

Comparing Fractions Using Decimals

When comparing two or more numbers to determine which number is the greatest or smallest, you can change the fractions (or rational numbers) to decimal form as an alternative to trying to convert the fractions so that they all have a common denominator. Let’s do a sample problem. Which is greater: 79/86 or 92/100. The decimal equivalent of 79/86 = 0.918604651163...; whereas, the decimal equivalent of 92/100 = 0.92, therefore we see that 92/100 is larger since the “2” in the hundredths place of 0.92 is greater than the “1” in the hundredths place of 0.9186... .

Exercise 5.2

Write the following rational numbers as decimals:

1. $\frac{5}{9}$

2. $\frac{2}{7}$

3. $\frac{3}{11}$

The following rational numbers, are
A. terminating decimals
B repeating decimals

4. $\frac{1}{4}$

5. $\frac{2}{3}$

6. $\frac{4}{9}$

7. $\frac{12}{16}$

Write the following terminating decimals as fractions:

8. 0.85

9. 0.72

10. 0.56

11. 0.68

12. 0.25

13. 0.125

14. 0.126

15. 0.5113

Write the following repeating decimals as fractions:

16. $0.\overline{714}$

17. $0.\overline{81}$

18. $1.\overline{3}$

19. $0.4\overline{83}$ (note: only the "83" is repeating)

20. $0.\overline{1}$

5.3 Solving Rational Equations

Solving Equations Containing Rational Solutions

We have explored the concept of solving basic algebraic equations back in Chapters 1 to 3. Let's review how to solve a two-step algebraic equation as follows:

$$8x - 7 = 0$$

First, we note that the variable appears on the left-side of the equation; however, there is the constant -7 that we must eliminate in order to isolate the variable x. So the first step is to do the opposite and add 7 to each side of the equation:

$$8x - 7 + 7 = 0 + 7$$

or

$$8x = 7$$

Next, since x is multiplied by 8, we do the inverse operation and divide both sides of the equation by 8:

$$\frac{8x}{8} = \frac{7}{8}$$

Simplifying we have

$$x = \frac{7}{8}$$

By way of review, let's do one more sample problem and solve

$$2 = -11x + 7$$

Since the variable x is on the right side of the equation, we note there is a +7 that must be eliminated. So the first step is to do the opposite and subtract 7 from each side of the equation:

$$2 - 7 = -11x + 7 - 7$$

or

$$-5 = -11x$$

Now, since the variable is multiplied -11, we do the inverse operation and divide each side of the equation by -11

$$\frac{-5}{-11} = \frac{-11x}{-11}$$

Simplifying (see Table 1.4 on page 15), since we have a negative term in both the numerator and denominator, we have a positive result, $x = \frac{5}{11}$ (since $\frac{-5}{-11} = \frac{-1 \bullet 5}{-1 \bullet 11}$ and canceling the -1 in both the numerator and denominator yields $\frac{5}{11}$).

Some More Examples

Now let´s solve some word problems where we can apply the procedures and lessons we have learned.

1. Melissa has a total \$14, which is exactly four times more than the money that Tanya has, plus another \$9. How much money does Tanya have?

   In order to solve this problem remember the steps to solve word problems that we learned in section 1.12

Extract the key information that is given:

- Melissa has \$14.
- Melissa has 4 times more than what Tanya has plus \$9.

What are we trying to find:

- The amount that Tanya has.

Assign variables to any quantities that are unknown:

Since we don´t know how much money Tanya has, let´s assign the variable t to this unknown:

- Let t = the amount of money (in dollars) that Tanya has

Now, let's write an equation with the unknown t and solve for t:

$$4t + 9 = 14$$

Solving this equation, we find $t = 1\frac{1}{4} = 1.25$. So in answer the original question, we have determined that Tanya has \$1.25 (notice we have used a dollar sign which shows the proper units of the answer). Typically an answer to a word problem will have two parts, the numeric (or number) answer along with the proper units. Sometimes problems are missed because the numeric portion is correct, but the units are wrong. Had we written 1.25¢, this answer would have been incorrect since the units are dollars and not cents. We know the units are dollars since we added 9 to $4t$ in the initial equation—and the 9 represented \$9. We also used 14 in the initial equation which again represented \$14. Had we used 900 instead of 9, to represent 900¢ and set the expression equal to 1400¢, we would have solved for t which represented the unknown number of cents that Tanya had. Using units of cents in our equation, we obtain $t = 125$, and 125¢ = \$1.25, which is equivalent to the same answer that we previously obtained using units of dollars.

Let's now try another example.

Sally is trying to make some pretty boxes of chocolate to give her associates at the office. She originally picks enough chocolate bars to have a total of 7 boxes, but only ends up having enough chocolates to make a total of 3 complete boxes, because 15 of her chocolate bars melted when she left them in the car one warm afternoon. How many chocolate bars did Sally place in each box?

Extract the information that is given:

- Originally she bought enough chocolate bars to fill 7 boxes with an unknown amount of chocolate.
- 15 chocolate bars melted.
- At the end she had enough chocolate for 3 boxes.

What are we trying to find:

- The unknown amount of chocolate in each box.

Assign variables to any unknown quantities:

- Let k be the number of chocolate bars placed in each box.

Now let's write an equation in terms of the unknown k and solve for k:

$$7k - 15 = 3k$$

To solve the equation we first want to group all the variables on one side of the equation. To accomplish this, notice that if we subtract $3k$ from each side of the equation, we will no longer have the variable k on the right side of the equation:

$$7k - 15 \mathbf{- 3k} = 3k \mathbf{- 3k}$$

When we simplify by combining like terms, we obtain $4k - 15 = 0$. Next, to isolate the variable k, we must add 15 to each side of the equation, thus we have

$$4k - 15 \mathbf{+ 15} = 0 \mathbf{+ 15}$$

Again, simplifying the equation, we have $4k = 15$. Since k is multiplied by 4, we do the inverse and divide both sides of the equation by 4 to obtain

$$k = \frac{15}{4} = 3\frac{3}{4}$$

Thus, Sally placed 3¾ chocolate bars in each box. [Note: It is interesting that Sally must have had to break some of the chocolate bars to get 3/4th of a piece!]

Let's consider one more problem that requires use of a special formula when tasks are performed in parallel but at two different speeds.

James always takes two hours to clean his room; whereas, his sister Anna only takes an hour to clean her room. Both rooms are the same size. If they work together, how long will it take for them to finish cleaning both rooms?

Extract the information that is given:

- James can clean 1 room in 2 hours.
- Anna can clean 1 room in 1 hour.

Now we can reason this way. While James is cleaning his room and Anna is cleaning her room, they can work independently on their own rooms for 1 hour. After 1 hour Anna's room will be clean, but James' room will be only ½ finished. We know that by himself, James would require 1 hour more to finish cleaning the remaining half of his room; however, if Anna worked by herself, she could finish cleaning the remaining half of James' room in ½ hour (or 30 minutes).

So, now the real question is this, if James and Anna work together (in other words in parallel) to complete the remaining ½ of James room, how much time will they complete the cleaning in? Let's let *t* be the time it takes them to clean ½ of James room working together, then we have this interesting formula for a parallel process:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{t}$$

Where *p* is the time it takes for one process to complete a given job and *q* is the time it takes for another process to complete the same job independently. Then when both processes work together, that job is completed in time *t*.

So, let's let *p* be the time it takes James to clean ½ his room = 60 minutes (1 hours)

Let's let *q* be the time it takes Anna to clean ½ of James's room = 30 minutes (1/2 hour)
Then we have the time, *t (in minutes)*, to finish cleaning the room with both working together as

$$\frac{1}{60} + \frac{1}{30} = \frac{1}{t}$$

To simplify this equation, we can multiply both sides of the equation by the common denominator 60*t*, to obtain

$$\mathbf{60t}\left(\frac{1}{60} + \frac{1}{30}\right) = \frac{1}{t} \bullet \mathbf{60t}$$

Using the distributive property on the left side of the equation and simply multiplying on the right side of the equation, all the denominators in the original equation are eliminated and we have

$$t + 2t = 60$$

Now, we combine like terms to obtain $3t = 60$ and divide both sides of the equation by 3 to obtain $t = 20$. Thus, the time it takes to clean both rooms is the 1 hour that they each worked on their own rooms independently, plus another 20 minutes that it took both of them to complete the last half of James' room, so 1 hour and 20 minutes = 80 minutes or $1\frac{1}{3}$ hours.

Another way to approach this problem is to consider that it would have taken James 4 hours (240 minutes) total to clean both his room and Anna's—if James worked alone. It would take Anna just 2 hours (120 minutes) to clean both rooms—if she worked alone. So, together if they both worked to clean both rooms (in what we call a parallel process), we can use the equation:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{t}$$

Let's let p and q be the time for James and Anna to do the cleaning, respectively. Then we have,

$$\frac{1}{240} + \frac{1}{120} = \frac{1}{t}$$

Multiplying both sides of the equation by the common denominator, $240t$, we have

$$t + 2t = 240$$

Simplifying, we have $3t = 240$, or $t = 80$ minutes = 1 hour and 20 minutes = $1\frac{1}{3}$ hours as before. Oftentimes, there is more than one correct strategy that can be used to solve any given problem as demonstrated above.

Let's do still another example. Suppose Martha can cook 2 potatoes in 1 hour and Fred can cook 5 potatoes in 1 hour. How long will it take Marth and Fred working together to cook 20 potatoes?

Well, the trick to this problem is in the method. First, let's determine how long it takes Martha to individually cook 20 potatoes. At the rate of 2 potatoes/hr, it will take her 10 hours to cook the 20 potatoes alone. Again, we repeat the analysis with Fred. At the rate of 5 potatoes/hr, it will take him 4 hours to cook the 20 potatoes alone. Now that we have determined the lengths of time to cook 20 potatoes individually by Martha (10 hours) and Fred (4 hours), we can use the parallel formula as follows:

$$\frac{1}{Martha's\,Time} + \frac{1}{Fred's\,Time} = \frac{1}{Combined\,Time}$$

Substituting the individual times for Martha and Fred to cook the 20 potatoes, we have

$$\frac{1}{10\,hrs} + \frac{1}{4\,hrs} = \frac{1}{t_{combined}}$$

To simplify and remove all of the denominators, we can multiply by the least common denominator $20t$. ***Note:*** The author quickly determined the necessary multiplier to eliminate all denominators by quickly noting (1) that 20 is the least common multiple of 10 and 4, and (2) he had to multiply by t to eliminate the $t_{combined}$. Thus, we now have $2t + 5t = 20$. Solving for t, we have

$$7t = 20 \text{ or } t = \frac{20}{7} = 2\,\frac{6}{7}\text{ hr}$$

Since $\frac{6}{7}$ of an hour is equivalent to $\frac{6}{7}$ hr • 60 $\frac{minutes}{hr} = \frac{360}{7}$ min. = 51 $\frac{3}{7}$ min, the answer 2 $\frac{6}{7}$ hr is equivalent to 2 hr 51 $\frac{3}{7}$ min. Converting $\frac{3}{7}$ min. to seconds by multiplying by the conversion factor $\frac{60\,sec.}{1\,min.}$, we could go one step further and also write the equivalent answer as

$$2\text{ hr }51\text{ min }25\,\frac{5}{7}\text{ sec}$$

Exercise 5.3

Solve the following equations for the given variable and state the answer is simplest form.

1. $17a + 2 = 8a + 6$

2. $12z = 5z + 4$

3. $16m - 7 = 7$

4. $s = 14 - 9s$

5. $40v + 16 = 30v + 27$

6. $6w = 10 + 3w$

7. $2n - 1 = 1 - 3n$

8. $k = 23 - 4k$

9. $8t - 10 = 12$

10. $5x - 12 = -9x$

Solve the following word problems, using x as the variable for the requested unknown:

11. Charles ate 8 caramel squares which is 7 times what he usually eats minus 6. How many caramel squares does he usually eat?

12. Martha has some milk she wants to refrigerate. She knows that she needs 10 bottles to store all the milk she has, and so far she has filled 2 bottles and has 10 liters left. How many liters fit in one bottle?

13. Karla knows that to prepare a certain cake she is going to need 7 times the amount of sugar she needs to prepare a muffin. If she knows that that the cake requires 4 lb of sugar minus 2 times the amount of sugar required to prepare a muffin, how much sugar does she need to prepare a muffin?

14. John and Kyle are twins and their sister's name is Anne. John says that his age is 5 times that of Anne's age plus 5 years. Kyle says that his age is 3 times that of Anne's age plus 8 years. What is Anne's age?

15. Michael is in the lab and he is going to run an experiment to process 5 samples; he is going to use 3 grams of the reactive minus the amount he uses to process one sample. How much reactive does he need to process one sample?

5.4 Simplifying Expressions by Combining Like Terms

Like Terms

You remember from section 1.2 that terms are the components of expressions in an equation, and these terms can be either constants or variables.

In math the phase ***like terms*** refers to terms whose variables are the same. So, for example:

$7x$ and $3x$ $12y$ and $15y$ $2n$ and $17n$ $-14r$ and $9r$

…are all examples of pairs of like terms because they do have the same variables.

The following would NOT be examples of like terms:

$8t$ and $9s$ $4r$ and $3x$ $10m$ and $13p$ $15h$ and $18e$

…because they do NOT have the same variables.

Combining Like Terms

In math, and especially in algebra, when we "combine like terms" in an equation, we make the problem simpler or easier to solve. In the following examples, the equations on each line are the same; however the equation on the right has combined like terms:

$7x – 3x = 2$ simplifies to $4x = 2$

$12a + 15a = 10$ simplifies to $27a = 10$

$2b – 17b = 5$ simplifies to $-15b = 5$

$-14k + 9k = 12$ simplifies to $-5k = 12$

It is easier to solve or work with the equations on the right than the equations on the left.

How to Combine Like Terms

- Simply perform the math on the constants or numerical values, and combine those variables that are the same. Consider, for example the expression $18m + 6m + 4 + 3$

First add the constants (4+3). Next, using the distributive property by noticing that the term $18m$ and $6m$ each contain the same common variable, m, we have

$$(18 + 6)m + 7.$$

This further simplifies to

$$24m + 7.$$

Let's consider another example where the like terms are in different locations throughout the equation:

$$5a + 2 - 3a = 4$$

We still combine the terms that are alike ($5a - 3a$), to obtain

$$2a + 2 = 4$$

Here is another example. Consider the equation:

$$\mathbf{17\boldsymbol{b}} - 5 + \mathbf{5\boldsymbol{b}} = 10$$

Using the distributive property and noting that ***b*** is common to two of the terms, we can write,

$$(17 + 5)\boldsymbol{b} - 5 = 10$$

This simplifies to,

$$22b - 5 = 10$$

You can also combine like terms with rational expressions (but must adhere to the various rules of adding and subtracting fractions):

$$\frac{\mathbf{3}}{\mathbf{4}}\boldsymbol{m} + 6 - \frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{m} = 5$$

$$\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{m} + 6 = 5$$

Since ¾ - ¼ = ½, we simplify the equation to $\frac{m}{2} + 6 = 5$. Please note that $\frac{m}{2}$ is the same as the expression *m/2*.

You can also combine like terms with rational expressions that do NOT have common denominators:

$$\frac{\mathbf{7}}{\mathbf{12}}\boldsymbol{t} - 6 + \frac{\mathbf{2}}{\mathbf{6}}\mathbf{t} = 10$$

Using the distributive property we have

$$\left(\frac{\mathbf{7}}{\mathbf{12}} + \frac{\mathbf{2}}{\mathbf{6}}\right)t - 6 = 10$$

which simplifies to $\left(\frac{7}{12} + \frac{4}{12}\right)t - 6 = 10$ or

$$\frac{11}{12}t - 6 = 10$$

Exercise 5.4

Combine like terms in the following equations:

1. $5s^2 + 7 + 2s^2 + 3$

2. $7x + 2a + 3x$

3. $3ab + 4a + 6ab$

4. $2xy^2 - 3xy + 7xy^2$

5. $8v + 2 - \frac{6}{7}v$

6. $6z + 3yz + 16yz$

7. $8x - 9x^2 + 3x^2$

8. $6 + 9d + 7$

9. $13m + 2mn - 5m$

10. $27 + 14k + 3k$

11. $\frac{\mathrm{m}}{2} + 3 + \frac{3\mathrm{m}}{7}$

12. $\frac{4}{5}x^2 - y + 2x^2$

13. $\frac{8n}{3} - 5 + 2n$

14. $27s^3 + 3t + 4s^3$

15. $15l + 4j - \frac{3}{11}l$

5.5 Solving Equations by Combining Like Terms

Examples of Equations

Jenelle and Sasha went shopping. Jenelle bought 9 tee-shirts and Sasha bought 2. Sasha though also bought a \$23 pair of shoes. The girls spent \$98 altogether. What was the cost of each tee-shirt?

Extract the information that is given:

- Jenelle bought 9 tee-shirts.
- Sasha bought 2 tee-shirts and spent \$23 in shoes.
- Jenelle and Sasha spent \$98 total.

What are we trying to find?

- The cost of a single tee-shirt.

Assign variables to any unknown quantities:

- Let t be the cost of each tee-shirt.

Now, let's write an equation in terms of the unknown t and then solve for t.

$$\mathbf{9t + 2t} + 23 = 98$$

Next, we combine like terms,

$$\mathbf{11t} + 23 = 98$$

We now isolate the variable by subtracting 23 from each side of the equation:

$$11t + 23 \mathbf{- 23} = 98 \mathbf{- 23}$$

Simplifying, we have,

$$11t = 75$$

Next, since t is multiplied by 11, we do the inverse and divide both sides of the equation by 11,

$$\frac{11t}{11} = \frac{75}{11}$$

Finally, we have the solution,

$$t = \frac{75}{11} \text{ or } 6\frac{9}{11} \text{ or approximately } 6.818$$

Thus, the cost of a single tee-shirt is \$6.82 (rounded to the nearest cent).

Let's consider another example. Sam and Dave are going to the fair. Sam has 7 tickets plus \$5 cash. Dave has 3 tickets, plus \$2 cash. When Sam and Dave's tickets and cash are combined, the total adds up to \$17 that they can use towards the rides. How much is each ticket worth?

Extract the information that is given:

- Sam has 7 tickets plus $5 cash.
- Dave has 3 tickets plus $2 cash.
- All the tickets plus cash adds up to $17.

What are we trying to find?

- The value of each ticket.

Assign variables to any unknown quantities:

- Let c be the cost of a ticket.

Now, let's write an equation in terms of the unknown c and solve for c:

$$\underline{7c} + 5 + \underline{3c} + 2 = 17$$

Combining like terms we have

$$\underline{10c} + 7 = 17$$

We now isolate the variable by first subtracting 7 from each side of the equation,

$$10c + 7 - 7 = 17 - 7$$

After simplifying, we have

$$10c = 10$$

Next, since c is multiplied by 10, we do the inverse and divide each side of the equation by 10,

$$\frac{10c}{10} = \frac{10}{10}$$

or $c = 1$. Thus, the price of each ticket is $1.00

Let's consider one more example problem.

CD's were on sale at the mall. Both Suzie and Jamie each bought a CD that was half-off the regular price. Together they paid a total of $14. How much was each CD before the half-off sale?

Extract the information that is given:

- Suzie and Jamie each bought a CD with 50% discount.
- The total amount spent was $14.

Assign variables to any unknown quantities:

Let c represent the regular price of one CD.

Now we can write the initial equation:

$$\frac{1}{2}c + \frac{1}{2}c = 14$$

Next, we combine like terms

$$c = 14$$

Thus, the regular price of each CD was \$14.00.

Consider another example that contains fractions:

$$\frac{x}{5} + 5x + \frac{11}{3}x = 22$$

Let's first get rid of the fractions by multiplying both sides of the equation by the least common multiple of 5 and 3 which is 15. Thus we have

$$15(\frac{x}{5} + 5x + \frac{11}{3}x) = 15\bullet(22)$$

Using the distributive property, we have

$$15\bullet\frac{x}{5} + 15\bullet5x + 15\bullet(\frac{11}{3}x) = 330$$

Further simplifying, we have

$$3x + 75x + 55x = 330$$

Combining like terms, we have

$$133x = 330$$

Finally, dividing both sides of the equation by 133, we have $x = 2\frac{64}{133}$

Notice that the answer was given as a mixed fraction since attempting to represent the number in decimal format as 2.48, or 2.482, or 2.482030075 are only approximate answers.

Alternatively, we could have solved the initial problem by converting all the whole numbers and fractions that contain the variable *x,* to equivalent fractions with a common denominator of 15. Thus,

$\frac{x}{5} + 5x + \frac{11}{3}x = 22$ is equivalent to $\frac{3x}{15} + \frac{75x}{15} + \frac{55x}{15} = 22$, which is equivalent to $\frac{3x+75x+55x}{15} = 22$

This can be further reduced to $\frac{133x}{15} = 22$. Now, multiplying both sides of the equation by the reciprocal factor, $\frac{15}{133}$, we have $x = 22\bullet\frac{15}{133}$, or $x = \frac{330}{133}$. Thus,

$x = 2\frac{64}{133}$ which is the same as the previous result.

Exercise 5.5

Solve the following equations by combining like terms:

1. $5t + 18 - 18t = 15$

2. $6a + \frac{2}{3} - \frac{a}{4} = 10$

3. $\frac{s}{2} + 6s - 2s = 5$

4. $6m + 4 + 17m = 27$

5. $2n - 16 - 4n = 32$

6. $7k + \frac{2k}{3} - \frac{k}{7} = 18$

7. $7x + 12 - 14x = 19$

8. $3ty + 7ty - 5t = 15t$

9. $11z - \frac{7}{9} - 7z = \frac{130}{18}$

10. $9r + 33 - 17r = 12$

Solve the following word problems for the requested value or values.

11. Tom wanted to know his mom's age. His mom told him that her age is $\frac{7}{3}$ the age of his sister Mary. His mom's age and his sister's age total 50 years. What is the age of Tom's sister and Tom's mom?

12. Jake has been driving at a certain speed on his way to the next town. On the way back he drives 50% faster than on the way there. If the average speed of the entire trip was 100 km/hr, how fast was he going on his way to the town?

13. Georgia went to the movies and bought a combo food item that cost $\frac{13}{4}$ times the cost of the ticket. If she spent a total of $12.75 for the ticket and combo, what was the cost of the ticket?

14. Arthur wants to buy a new cell phone, he knows that besides what his father is going to give him this week he has 5 times that amount plus $100 saved. If he knows that all together it is going to be enough to buy the $250 cell phone he wants, how much does his dad give him every week?

15. The teacher notices that the number of boys exceeds the amount of girls in her class by three. If in the class there are 31 students, how many girls are in the classroom?

5.6 Solving Equations with a Variable on Each Side

Sometimes algebra problems will have variables on each side of the equation. When this occurs we must first perform a procedure whereby we rewrite the equation with the variables all on one side of the equation as demonstrated below.

When the Variable is on Both Sides

Consider this example where the variable occurs in one term on the left side of the equation and one term on the right side of the equation:

$$2p + 25 = 3p + 10$$

Now, since $2p$ is on the left side, we can eliminate this term by subtracting $2p$ from each side of the equation:

$$2p - \mathbf{2p} + 25 = 3p - \mathbf{2p} + 10$$

Notice that the resulting equation now has the variable only appearing on the right side of the equation:

$$25 = p + 10$$

We can solve this equation as we usually do by isolating the variable. In this case we subtract 10 from each side of the equation $25 - \mathbf{10} = p + 10 - \mathbf{10}$
This gives us the solution, $p = 15$.

Alternatively, given the same equation that we started with above, $2p + 25 = 3p + 10$

Noticing that $3p$ appears on the right side of the equation, we could have subtracted $3p$ from both sides of the equation as follows:

$$2p - \mathbf{3p} + 25 = 3p - \mathbf{3p} + 10$$

The resulting equation has the variable p only on the left side of the equation: $-p + 25 = 10$

We isolate the variable p by first subtracting 25 from each side of the equation to obtain

$$-p = -15 \quad \text{(which is equivalent to } -1p = -15\text{)}$$

Finally, we divide both sides by -1 (since p is multiplied by -1), to obtain the same answer as before, $p = 15$

Let's consider another example in the form of a word problem.

Juan had two coupons to use at the restaurant and then paid an extra \$11. Dave had 3 coupons to use at the restaurant and then had to pay an extra \$7. If both of their meals were the same original cost, how much were their coupons worth?

Based on the given information, let's assign the variable c to represent the value of each coupon,

$$2c + 11 = 3c + 7$$

We eliminate the variable on the left side of the equation by subtracting $2c$ from each side,

$$2c - \mathbf{2c} + 11 = 3c - \mathbf{2c} + 7$$

Finally, with the variable on one side of the equation, w solve for c,

$$11 = c + 7$$

Solving for c, we obtain $c = 4$. Thus, the coupons were worth \$4 each.

Here is another example.

Jimmy and Todd are athletes who are training for an upcoming marathon. Jimmy ran twelve times around the track and then ran an addition 8 kilometers. Todd ran only four times around the track, but then ran an additional 24 kilometers. If they both ran the same distance, what is the length of the track that they ran on?

Let's assign the variable k to be the length of the track. Then we can write the equation,

$$12k + 8 = 4k + 24$$

We next get the variables all on one side of the equation by subtracting $4k$ from each side of the equation:

$$12k - \mathbf{4k} + 8 = 4k - \mathbf{4k} + 24$$

Simplifying, we have

$$8k + 8 = 24.$$

To isolate the variable k, we must first subtract 8 from each side of the equation,

$$8k + 8 - \mathbf{8} = 24 - \mathbf{8}$$

Simplifying, we have

$$8k = 16.$$

Finally, we divide both sides by 8 to obtain the solution, $k = 2$. Thus, the track that Jimmy and Todd both ran on is 2 km.

On rare occasions, an equation may have an extraneous or unnecessary variable, such as in this example:

$$4xy + 5y = 3xy$$

Please notice, however, that the variable, y, is common to each and every term. Therefore, if we divide both sides of the equation by y, all of the y's will cancel, yielding the new equation:

$$4x + 5 = 3x$$

Now, we can solve for x to obtain $x = -5$. In the original equation, notice that it does not matter what value is assigned to y, since the variable y simply cancels out in every case, but x must be equal to -5 for the original equation to be true.

Exercise 5.6

Solve the following equations when the variable is on both sides:

1. $\frac{3x}{2} - 2 = 18 + \frac{17x}{2}$

2. $at - 2t = 14at + 4t$

3. $4m + 21 = -5m + 30$

4. $9s + 72 = 21s - 12$

5. $n + 18 = \frac{n}{3} - 4$

6. $y - 16 = 18y + 3$

7. $14r - \frac{12}{5} = \frac{2r}{5} + 10$

8. $10g - 11 = 14g + 21$

9. $5cx + 3x = -35cx + 83x$

10. $-13t - 26 = t + 2$

11. Mario realized that yesterday he had \$10 more than today, and he will have the same amount as yesterday in three days when he will have twice the amount he has today minus \$2. How much money does he have today?

12. Terry is half the age of his father right now, but in ten years he will be $\frac{3}{5}$ the age of his dad. How old are Terry and his father?

13. John is 4 inches taller than Laura, and Laura is 1 foot and 4 inches taller than her cousin. If John and Laura´s height together is 2.5 times the height of Laura´s cousin, how tall is Laura?

14. Mark bought 3 identical shirts and one \$10 tie. His friend spent \$100 on gift items and bought 2 t-shirts for himself that cost half the price Mark had to pay for his shirts. If both of them spent the same amount of money, what is the price Mark paid for each shirt?

15. Anne went to the candy shop and bought 7 chocolate bars and spent \$15 on other items, her friend bought 2 chocolate bars and spent an additional \$20 on other items. If both of them spent the same amount, what is the cost of a chocolate bar?

5.7 Using Multiplicative Inverses to Solve Equations

Equations with Only One-Step

So far you have learned how to solve equations by combining like terms and isolating the variables on either side of the equation by applying the inverse operation. In the last section we have been focused on adding and subtracting operations. However you will find equations where, in order to solve them, you will have to use the multiplicative inverse of the coefficient (the constant that appears multiplied by the variable).

How can you find the multiplicative inverse of a number? A number multiplied by its inverse (called the multiplicative inverse) always yields a result of 1. Let´s consider some examples:

Number	Multiplicative inverse	Product
5	$\frac{1}{5}$	$5 \bullet \frac{1}{5} = 1$
$\frac{1}{8}$	8	$\frac{1}{8} \bullet 8 = 1$
0.8	$\frac{1}{0.8}$	$0.8 \bullet \frac{1}{0.8} = 1$
$\frac{4}{5}$	$\frac{5}{4}$	$\frac{4}{5} \bullet \frac{5}{4} = 1$

Sometimes when solving an equation, you will be able to do so just by multiplying both sides of the equation by the multiplicative inverse of the coefficient of the variable. In other words, what we are talking about is multiplying both sides of the equation by the multiplicative inverse to solve equations where either on the right or left side of the equation we have a term such as $\frac{3x}{5}$ (which is equivalent to $\frac{3}{5}x$) or $\frac{x}{3}$ (which is equivalent to $\frac{1}{3}x$). Consider the following example equations that can be solved in one-step using the multiplicative inverse.

Sally's invested money generates \$0.25 per week in interest. How many months will it take for her to accumulate \$12 in interest?

Let's assign x to the number of weeks needed to accumulate \$12.00. The equation is

$$0.25x = 12$$

Multiplying both sides of the equation by the inverse we have

$$0.25x \bullet \frac{1}{0.25} = 12 \bullet \frac{1}{0.25}$$

Simplifying, we have $x = 48$. Thus, it will take 48 months to accumulate \$12.00 in interest.

We could have done this problem by first converting the decimal number into a rational number:

$$0.25 = \frac{25}{100} = \frac{1}{4}$$

Then we could write the equation,

$$\frac{1}{4}x = 12$$

As before, we now multiply both sides of the equation by the multiplicative inverse to obtain

$$\frac{4}{1} \bullet \frac{1}{4}x = \frac{4}{1} \bullet 12$$

Simplifying, we have $x = 48$ as before.

Equations with Two-Steps

The process to solve two-step equations is demonstrated by the following example:

$$\frac{3}{5}t + \frac{1}{10} = \frac{4}{20}$$

We first isolate the variable by subtracting 1/10 from each side of the equation,

$$\frac{3}{5}t + \frac{1}{10} - \frac{1}{10} = \frac{4}{20} - \frac{1}{10}$$

Simplifying both sides of the equation, we have

$$\frac{3}{5}t = \frac{4}{20} - \frac{2}{20}$$

or

$$\frac{3}{5}t = \frac{2}{20}$$

Now, multiplying by the inverse we have,

$$\frac{5}{3} \bullet \frac{3}{5}t = \frac{5}{3} \bullet \frac{2}{20}$$

This yields,

$$t = \frac{10}{60} = \frac{1}{6}$$

Exercise 5.7

Solve the following one-step equations for the specified variable and write your answer in simplest form.

1. $\frac{a}{6} = \frac{7}{3}$

2. $2t = \frac{9}{2}$

3. $8r = 96$

4. $16m = 4$

5. $28s = 3\frac{2}{7}$

6. $\frac{2}{7}d = 42$

7. $\frac{9}{4}v = 81$

8. $\frac{3}{7}x = \frac{5}{14}$

9. Mark owes \$300 to the bank; if he is paying \$30 per month, how many weeks is it going to take him to pay his debt? (Consider that each month has 4 weeks).

10. A newspaper prints 1025 newspapers using one huge paper roll, how many paper rolls are needed daily if they print around 5125 newspapers?

Solve the following two-step equations for the specified variable and write your answer in simplest form.

11. $6y + 14 = 18$

12. $\frac{m}{2} - \frac{5}{7} = \frac{18}{4}$

13. $\frac{a}{9} + \frac{2}{3} = 11$

14. $\frac{3}{4}L - 12 = 10$

15. $8g + \frac{2}{7} = \frac{15}{2}$

16. $\frac{r}{20} + 10 = 32$

17. $\frac{3}{7}a - \frac{4}{9} = \frac{17}{2}$

18. $28z - \frac{1}{4} = \frac{222}{8}$

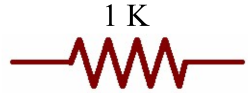
19. Sean currently owes \$50 to his friend, which is $\frac{4}{3}$ times what his friend lent him minus \$30. How much money did Sean ask to borrow from his friend originally?

20. Andy´s mom asked him to buy 4 toothpastes and one \$10 shampoo bottle. If the total amount he spent was \$22, how much does it cost for the toothpaste?

5.8 Application of Pre-Algebra to Electronics (Optional)

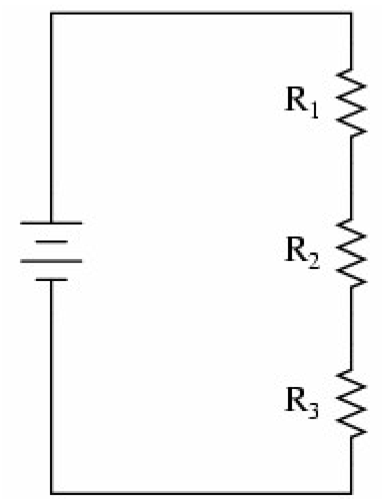
In electronics, resistors are one of several components that appear on electronic circuit boards. Low-wattage resistors are often made out of carbon and have three colored bands that indicate the value of the resistance which is measured in units of ohms. A fourth color-band is used to indicate the tolerance to which the resistor has been manufactured (gold ±5%, silver ±10%, red ±2%, brown ±1%). Here is how a typical resistor looks:

The above resistor is rated at 1,000-ohms (with a 10% tolerance due to the silver 4th band). In electronics, the Greek capital letter omega (Ω) is used to designate ohms. Thus, the above photo shows a 1,000 Ω resistor (sometimes abbreviated as 1-KΩ). On an electronic schematic diagram, the above resistor would be shown as follows:

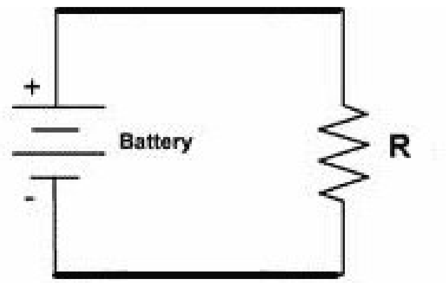


Computing Series Resistance

To compute the total resistance of several resistors connected in series, you simply add the value of each resistor. As an example, consider the following circuit showing a battery connected to three resistors connected in series:



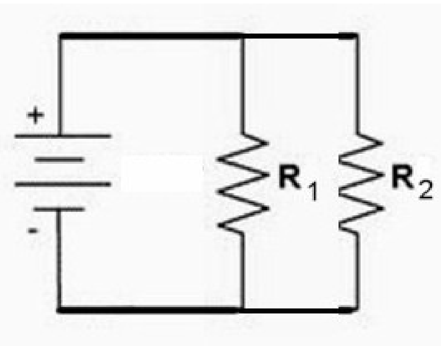
If R_1 = 1,000 Ω, R_2 = 4,700 Ω, and R_3 = 2,200 Ω, then the above circuit can be modeled as one resistor with a resistance that is the sum of the three resistances. Since $R_1 + R_2 + R_3$ = 7,900 Ω, we can simplify the circuit to



where the resulting total resistance, R = 7,900 Ω (or 7.9k Ω).

Computing Parallel Resistance

Shown below is a schematic showing two resistors that are connected in parallel to a battery voltage source.



To find the equivalent resistance of two resistors in parallel, we must use the parallel formula that we used back in Section 5.3:

$$\frac{1}{\mathrm{R1}} + \frac{1}{\mathrm{R2}} = \frac{1}{R}$$

In the above equation, the parallel resistance, R, can be computed if the values of R_1 and R_2 are known. Previously, we have used the formula using the variables p, q, and t.

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{t}$$

In fact, an astute mathematician or electronics engineer who had to solve parallel resistance problems over and over again, would opt to derive a more direct but equivalent formula than always having to resort to the above formula.

Notice, we can start to simplify the parallel process equation (and solve for t) by eliminating all the denominators. This is accomplished by first multiplying both sides of the equation by the common denominator, pqt, giving us

$$pqt\left(\frac{1}{p} + \frac{1}{q}\right) = \left(\frac{1}{t}\right)pqt$$

Simplifying, we have

$$qt + pt = pq$$

Next, we notice the like terms having the common variable t on the left side of the equation. So using the distributive property, we can write $(p + q)t = pq$. Now, we divide both sides by $(p + q)$ to obtain

$$t = \frac{pq}{p+q}$$

So, let's try a practical problem—since often students will say, "When will we ever use that in our lifetime?" Suppose you have two 8-ohm speakers that you wire in parallel to connect to an audio amplifier. What is the resulting resistance (in ohms) of the two speakers when connected in parallel? [Note: For purposes of this problem, we are considering the speaker's impedance (which is defined as voltage divided by current and which represents a measure of both magnitude and phase) as simple resistance.]

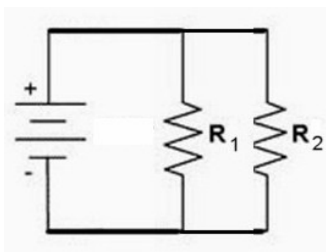
We could apply the parallel formula (which works for resistance) and then solve for t, where p and q are both each equal to 8:

$$\frac{1}{8} + \frac{1}{8} = \frac{1}{t}$$

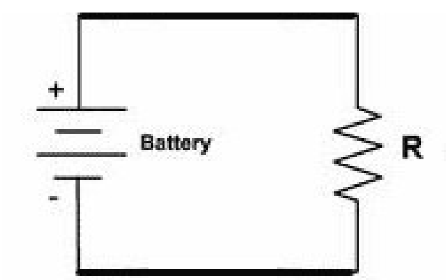
Or instead, we could use the formula that we just derived:

$$t = \frac{pq}{p+q} = \frac{8 \bullet 8}{8+8} = \frac{64}{16} = 4$$

Thus, two 8-ohm speakers wired in parallel produce the equivalent of a 4 Ω load. Thus, we could represent this situation as two parallel resistors, R_1 and R_2 both equal to 8 Ω as follows



However, we have determined that the two resistors in parallel can be modeled as a single resistance, R = 4 Ω as shown below:



Now, if we connected a 16-ohm speaker and a 4-ohm speaker in parallel, we could easily compute the resulting resistance as follows:

$$t = \frac{pq}{p+q} = \frac{16 \bullet 4}{16+4} = \frac{64}{20} = 3.2$$

Thus, the resulting "load" (the two speakers connected in parallel) to be attached to the amplifier would have a resistance of 3.2-ohms.

Now, one final challenge problem. Suppose we desire to connect three speakers (or 3 resistors) in parallel, call them p, q, and r. We could use the parallel formula above first with the two resistors p and q, and obtain their equivalent resistance in parallel; then, use that answer and the remaining resistance r in the parallel formula again to obtain the resulting resistance of all three in parallel.

Another strategy would be to use the parallel resistance formula for three different resistors:

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{1}{t}$$

Again, if we are astute mathematicians or electronic engineers and will be performing often this calculation of the combined resistance of three resistors wired in parallel, we could solve for t by first multiplying by the common denominator $pqrt$, to eliminate all of the denominators and obtain:

$$qrt + prt + pqt = pqr$$

Now, solving for t, we have

$$(\mathrm{qr} + \mathrm{pr} + \mathrm{pq})t = \mathrm{pqr}$$

or

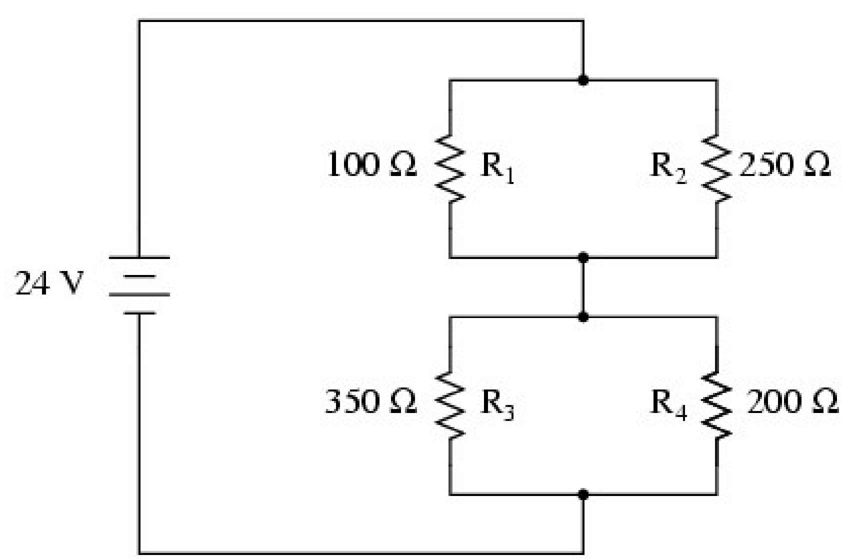
$$t = \frac{pqr}{qr+pr+pq}$$

Suppose we have 3-speakers (or resistors) with a rating of p = 16-ohms, q = 8-ohms, and r = 8 ohms in parallel. The two 8-ohm speakers in parallel are equivalent to a resistance of 4-ohms. So, now we have a 4-ohm resistance in parallel with a 16-ohm resistance which yields a combined equivalent resistance of 3.2-ohms. Let's check this with the formula we derived above:

$$t = \frac{16\bullet 8\bullet 8}{8\bullet 8+16\bullet 8+16\bullet 8} = 3\frac{1}{5} = 3.2 \text{ ohms}$$

Computing Resistance in Series-Parallel Circuits

Using the principles we discussed, you should be able to compute the equivalent resistance of this combination parallel-series circuit:



The parallel combination of R_1 and R_2 is equivalent to a resistance of about 71.2 Ω. The parallel combination of R_3 and R_4 is equivalent to a resistance of about 127.3 Ω. Now we simply add the two previously computed resistances that are in series to obtain 71.2 + 127.3 = 198.5 Ω, which is the total equivalent resistance of R_1, R_2, R_3, and R_4 combined.

Since ohms law states $V = IR$, where V is voltage (in volts), I is current (in amps), and R is resistance (in ohms), we can solve this equation for I by dividing both sides of the equation by R, thus we obtain

$$I = V/R$$

Since we know the voltage is 24-volts, and the equivalent resistance is 198.5, we can determine that the resistor circuitry is drawing $I = 24/198.5 = 0.121$ amps.

Since Power, P (in watts), is given by the formula $P = I^2R$, we can calculate the power consumed by the circuitry is $P = (0.121^2)(198.5) = 2.91$ watts. In college level courses you may learn how to model components such as diodes, transistors, triacs, silicon-controlled rectifiers and more. Using mesh and nodal analysis, which is beyond the scope of this course, it is possible to determine the current and voltage anywhere in a circuit. Using still more advanced mathematics, Fourier Series and Laplace Transforms, it is possible to analyze exponentially decaying sinusoidal waveforms that are characteristic outputs from circuits that contain components such as inductors and capacitors.

Answers to Chapter 5 Exercises

Exercise 5.1

1. yes; **2.** yes; **3.** no; **4.** no; **5.** yes; **6.** 5/4; **7.** equal; **8.** 3/13; **9.** equal; **10.** 7/8; **11.** 2.3; **12.** equal; **13.** 1/3; **14.** 7.2; **15.** 5/13; **16.** \$24.03; **17.** library; first convert library download rate: $\frac{1\,sec}{1\,Mbyte} = \frac{60\,sec}{60\,Mbytes} = \frac{1\,minute}{60\,Mbytes}$, download time: $\frac{1\,min}{60\,Mbytes}$ • 900 Mbytes = 15 minutes; **18.** 2002 vehicle

Exercise 5.2

1. $0.\overline{5}$; **2.** $0.\overline{285714}$; **3.** $0.\overline{27}$; **4.** A; **5.** B; **6.** B; **7.** A; **8.** 17/20; **9.** 18/25; **10.** 14/25; **11.** 17/25; **12.** 1/4; **13.** 1/8; **14.** 63/500; **15.** 5113/10000; **16.** 238/333; **17.** 9/11; **18.** 4/3 or 1 1/3; **19.** 479/990; **20.** 1/9

Exercise 5.3

1. 4/9; **2.** 4/7; **3.** 7/8; **4.** 1 2/5; **5.** 1 1/10; **6.** 3 1/3; **7.** 2/5; **8.** 4 3/5; **9.** 2 3/4; **10.** 6/7; **11.** $7x - 6 = 8$; $x = 2$ caramel squares; **12.** $10x = 2x + 10$; $x = 1¼$ liters; **13.** $7x = 4 - 2x$; $x = 4/9$ lb; **14.** $5x + 5 = 3x + 8$; $x = 1½$ years old; **15.** $5x = 3 - x$; $x = ½$ gram of reactive

Exercise 5.4

1. $7s^2 + 10$; **2.** $10x + 2a$; **3.** $9ab + 4a$; **4.** $9xy^2 - 3xy$; **5.** $50v/7 + 2$; **6.** $6z + 19yz$; **7.** $8x - 6x^2$; **8.** 13 + 9d; **9.** $8m + 2mn$; **10.** $27 + 17k$; **11.** $13m/14 + 3$; **12.** $14x^2/5 - y$; **13.** $14n/3 - 5$; **14.** $31s^3 + 3t$; **15.** $162l/11 + 4j$

Exercise 5.5

1. 3/13; **2.** 1 43/69; **3.** 1 1/9; **4.** 1; **5.** -24; **6.** 2 31/79; **7.** -1; **8.** $10ty = 20t$, $10y = 20$, $y = 2$; **9.** 2; **10.** 2 5/8; **11.** Let x = Mary's age, then mom's age is $\frac{7}{3}x$; mom's age plus Mary's age is 50 or $\frac{7}{3}x + x = 50$; $x = 15$ years old; mom's age = 35 years old; **12.** Let x = speed into town; speed is 50% (or $\frac{50}{100} = 0.5$) faster going back--which is his speed into town plus 50% faster or $x + 0.5x = 1.5x$; the average speed is the sum of the speed to and from town divided by two: $\frac{x+1.5x}{2} = 100$; $x = 80$ km/hr; **13.** Let x = ticket cost; combo food item = $\frac{13}{4}$; $\frac{13}{4}x + x = 12.75$; $x = \$3.00$; **14.** Let x = dad's contribution; $x + 5x + 100 = 250$; $x = \$25$; **15.** Let x = no. of girls; $x + x + 3 = 31$; $x = 14$ girls

Exercise 5.6

1. -2 6/7; **2.** -6/13; **3.** 1; **4.** 7; **5.** -33; **6.** -1 2/17; **7.** 31/34; **8.** -8; **9.** 2; **10.** -2; **11.** Let x = money that Mario has today; $x + 10 = 2x - 2$; $x = \$12$; **12.** Let x = Father's age; Terry's age = $x/2$; in 10 years, Terry's age will be 3/5 his Father's age: $(x/2 + 10) = (3/5)(x + 10)$; $x = 40$ yrs. old; Terry´s age, 20 yrs. old; **13.** Let x = Laura's height (in ft); John's height = $x + 1/3$ (ft); Cousin's height = $x - 4/3$ (feet); $x + 4/12 + x = 2.5(x - 4/3)$; $x = 7\ 1/3$ ft or 7' 4"; **14.** Let x = Mark's shirt cost; $3x + 10 = 2 \bullet (x/2) + 100$; $x = \$45$; **15.** Let x = chocolate bar cost; $7x + 15 = 2x + 20$, $x = \$1.00$

Exercise 5.7

1.14; **2.** 2¼; **3.** 12; **4.** 1/4; **5.** 23/196; **6.** 147; **7.** 36; **8.** 5/6; **9.** Let x = no. of weeks; payment = \$30/4 per week; $30x/4 = 300$; $x = 40$ weeks; **10.** Let x = no. of rolls; $1025x = 5125$; $x = 5$ rolls; **11.** 2/3; **12.** 10 3/7; **13.** 93; **14.** 29 1/3; **15.** 101/112; **16.** 440; **17.** 20 47/54; **18.** 1; **19.** Let x = loaned money; $50 = 4x/3 - 30$; $x = \$60$; **20.** Let x = cost of toothpaste; $4x + 10 = 22$; $x = \$3.00$

Chapter 6. Polynomials

6.1 Monomials and Polynomials

By now, you have noticed that any given algebraic equation consists of expressions comprised of one or more terms. These terms can be either constants or variables (see Section 1.8 of this book). An algebraic expression that consists of just one term is called a **monomial**, while an expression that consists of two or more terms is called a **polynomial**.

In this chapter you will learn how to add, subtract, multiply and divide polynomials. You will also learn how to factor a polynomial.

Since polynomials are comprised of two or more monomials, let's first identify what a monomial is. **Monomials** are single terms that are present in algebraic expressions. Examples of monomials include the following:

$$-4a \qquad 15t \qquad 7ab \qquad \frac{-15m}{k} \qquad 14x^2 \qquad 8 \qquad x$$

Remember, a **term** can be either a single number or a variable, or the product or quotient of variables and their coefficients. In a polynomial, two or more monomials will be added or subtracted. Each monomial term is separated from an adjacent monomial term by a + or – sign, depending upon whether the term is added or subtracted within the expression.

Earlier in this book, we learned how to work with monomials in equations (see Sections 2.2 and 5.7); therefore, let's move on to working with polynomials.

A polynomial is a sum (or difference) of monomials. Polynomials most often are presented as either binomials (which contain two monomial terms) or trinomials (which contain three monomial terms). See 3 examples of monomials, binomials, and trinomials in the table below.

Monomial (1 term)	Binomial (2 terms)	Trinomial (3 terms)
$-7ab$	$14x + 3$	$10x^2 + 14x - 6$
$15a$	$x^2 + 5$	$29x^2 - 15x + 8$
9	$17x^2 - 6$	$9x^2 + 26x - 12$

To identify polynomials it is important to look for the following:

- Each term within the expression must be added or subtracted to another term.
- The terms can be either positive or negative. The terms can consist of variables raised to some power (for example, x^2, x^3, x^4, etc.)

Degree of a Polynomial

The degree of a polynomial is the highest degree assigned to any individual term. For example, given the polynomial (or trinomial): $10x^2 + 14x - 6$, let's evaluate the degree of each term—or, in other words, the highest *exponent* when a term consists of only a single variable.

Term	Degree
$10x^2$	2
$14x$	1
6 (which is equivalent to $6x^0$)	0*

*The degree of any constant is always zero (0). Thus, -1.3, 3.14, 5, 5^8, etc. all have a degree of 0 since there is no variable associated with the value.

The first term has the highest exponent (which is 2), so the degree of $10x^2 + 14x - 6$ is 2. Let's consider another example and find the degree of this polynomial: $a + a^3b^2 - c^4$. When two or more variables are present in a term, the degree is the sum of that term's exponents.

Term	Degree
a	1
a^3b^2	5
c^4	4

The greatest degree of the terms is 5 (because the sum of the exponents of the term a^3b^2 is 5). So the degree of $a + a^3b^2 - c^4$ is 5. Terms are usually listed in order by degree so that the terms should be typed as $a^3b^2 - c^4 + a$.

Now, suppose we had an expression such as $x^2 - 1/x^3$. At first glance you might suggest that the degree is 3, since 3 is the highest exponent found on the variable x in the 2nd term. But notice carefully, that x in that term is located in the denominator. Thus, we could write the equivalent expression as $x^2 - x^{-3}$. The highest exponent in this expression is the 2 associated with the first term, x^2. Therefore, the degree of $x^2 - 1/x^3$ is 2 (and not 3). Let's consider one more unusual example, $a + b^2/c^2$. Here again, we can rewrite this expression as $a + b^2c^{-2}$. The degree of the first term, a, is 1; however, the degree of the second term, b^2c^{-2}, is the sum of the exponents 2 and -2 which is zero. Thus, the degree of $a + b^2/c^2$ is actually 1 (and not some other number).

Standard Form

It is easier to work with polynomials, especially in terms of simplifying the addition or subtraction of them, if they are in written in standard form.

The **standard form** of a polynomial is when the expression is simplified (with all like terms combined) and the terms are arranged so the degree of each term decreases or stays the same from left to right. These two different polynomials are shown in standard form:

$$r^4 - 4r^3 + 7r^2 - 2r + 5$$

$$x^5 - 8x^3 + 7x^2 - x$$

Notice that the expressions have been arranged so that the terms with the largest degree (that is, the largest exponents) are first, followed by the terms with successively smaller degrees.

Evaluating Polynomials

The following equation involves a polynomial which represents the braking distance *B* (in feet) that is needed to stop a car; where *v* is the car's speed per hour and *r* is the reaction time of the driver:

$$B = 1.47vr + 0.05v^2$$

If a car was going 80 miles an hour and the driver took two seconds to react, how long would it take the car to stop? Substituting these values for *v* and *r*, we have

$$1.47(80)(2) + 0.05(80)^2 = 235.2 + 320 = 555.2$$

Thus, it would take the car approximately 555.2 feet to stop.

Exercise 6.1

State whether the following are polynomials.

1. $ma + t$
2. $x + 2$
3. $q_1q_2/2$
4. $x \bullet x$
5. $np + 5$

State whether the following are
A. monomials B. binomials C. trinomials

6. $2a + t$
7. $np + p^2 - 5$
8. d^2
9. $8 - 15z$
10. $4acx$

Find the degree of the following polynomials:

11. $a^3 - b$
12. $7x + at^4 + x^2$
13. $qp + 3$
14. $x + y - 7m$
15. $n - k$

Write the following polynomials in standard form:

16. $2 + m^2 - 2m$
17. $3bz^4 - 6 + 2az^5 + 5z^4 - z^3$
18. $-8d - d^4 + 5d^2$
19. $6dr^2 + 7r^3 - 10 - br$
20. $-8 + 10j^2 + 3j$

Evaluate the following polynomials:

21. Anne bought 3 jackets and 6 belts, and her dad gave her \$100 for her expenses. If each jacket cost \$40 and each belt \$15, what is Anne's out-of-pocket expense?

22. Mary is printing her own business cards on a rectangular piece of paper. She can fit the same number of cards in the length and in the width of the piece of paper. The cost of each printed card is \$0.60. She is saving \$20 from the total amount she would have to pay a professional printer. How much was the professional printer charging if Mary can fit 10 cards along the length of the piece of paper?

23. According to Newton´s first law of motion the final velocity, v, is equal to the initial velocity, u, plus the product of the acceleration, a, and the time, t. Thus, $v = u + at$. If the initial velocity of a car is 40 km/hr and the acceleration rate is 960 km/hr^2, what will be the car´s velocity (in units of km/hr) after 5 minutes?

6.2 Adding and Subtracting Polynomials

As you are well aware, adding and subtracting are two of the most basic operations you have learned. In the past, when solving equations with variables on each side, you have been able to rearrange all variables to be on one side of the equation by adding or subtracting all the monomials on a selected side of the equation. To simplify an equation, you have combined like variables by adding or subtracting terms with the same variable and exponents (See section 5.4).

To add or subtract polynomials, the first step is to put both polynomials in their standard form. In the following example, we have accomplished that step:

$$(-17x^3 + 8x^2 - 20x + 5) + (20x^3 - 4x^2 + 32x - 3)$$

Next, we can arrange the like terms in columns and add as usual:

$$\begin{array}{rrrrr} & -17x^3 & +8x^2 & -20x & +5 \\ + & \underline{20x^3} & \underline{-4x^2} & \underline{+32x} & \underline{-3} \\ & 3x^3 & +4x^2 & +12x & +2 \end{array}$$

Thus, the result is $3x^3 + 4x^2 + 12x + 2$

The process is similar when subtracting polynomials. First, both polynomials must be arranged in their standard form, which we have accomplished in this example,

$$(14z^3 - 12z^2 + 15z - 3) - (7z^3 + 3z^2 - 10z + 5)$$

Since the entire second polynomial is subtracted, the sign of each term in the second polynomial can be changed so that all positive terms become negative terms and all negative terms become positive terms. Thus, we have

$$\begin{array}{rrrrr} & 14z^3 & -12z^2 & +15z & -3 \\ + & \underline{-7z^3} & \underline{-\ \ 3z^2} & \underline{+10z} & \underline{-5} \\ & 7z^3 & -15z^2 & +25z & -8 \end{array}$$

Thus, the result is $7z^3 - 15z^2 + 25z - 8$.

Another strategy is to add or subtract polynomials horizontally by combining like terms. Consider the addition of these two polynomials: $(4x^2 - 7x + 15) + (10x^2 + 29x - 8)$
First, we identify the like terms (the same color is used on like terms):

$$(\underline{4x^2} - 7x + \mathbf{15}) + (\underline{10x^2} + 29x - \mathbf{8})$$

Next, let's group the like terms together: $\underline{4x^2} + \underline{10x^2} - 7x + 29x + \mathbf{15} - \mathbf{8}$ and then add (or subtract):

$$\underline{14x^2} + 22x + \mathbf{7}$$

Let's consider one more example of subtracting polynomials:

$$(9x^2 - 18x + 2) - (-5x^2 + 8x - 6)$$

First, we identify like terms:

$$(\underline{9x^2} - 18x \mathbf{+ 2}) - (\underline{-5x^2} + 8x \mathbf{- 6})$$

Now, let's simply the expression evaluating – ($\underline{-5x^2}$ + 8x **– 6**), so that we now have

$$\underline{9x^2} - 18x \mathbf{+ 2} + \underline{5x^2} - 8x \mathbf{+ 6}$$

Next, group the like terms and add or subtract as necessary:

$$\underline{9x^2} + \underline{5x^2} - 18x - 8x \mathbf{+ 2 + 6}$$

Combining like terms, we have $\underline{14x^2}$ – 26x **+ 8**

Exercise 6.2

Perform the indicated operation and simplify these polynomial expressions, writing the answers in standard form (with the highest degree terms listed first).

1. $(x^3 + 3x^2 - 2) + (2x^4 - 2x^3 - 2x^2 + 15)$
2. $(3m^4 - 7m^3 + 4m + 14) + (m^4 + 15m^3 + 4m^2 - 2m - 16)$
3. $(7z^3 - 3z^2 + 2z + 15) + (3z^3 + z^2 - 9z - 28)$
4. $(4k^4 + 2k^2 - 15) + (6k^4 - 3k^3 + 7)$
5. $(5p^5 - 4p^4 + 3p - 12) + (7p^4 + 3p + 15)$
6. $(14a^4 + 5a^3 - 2a^2 + 16) + (7a^4 - 11a^3 + 5a^2 + 16)$
7. $(6b^3 - 7b^2 + 13b - 7) + (5b^3 + 10b^2 - 8b + 22)$
8. $(9r^3 + 4r - 24) + (5r^4 - 3r^3 + 17r - 14)$
9. $(17q^4 - 5q + 52) + (9q^4 + 18q - 15)$
10. $(15c^3 + 18c^2 - 7c + 20) + (10c^3 - 7c^2 + 7c - 12)$
11. $(x^3 + 3x^2 - 2) - (2x^4 - 2x^3 - 2x^2 + 15)$
12. $(3m^4 - 7m^3 + 4m + 14) - (m^4 + 15m^3 + 4m^2 - 2m - 16)$
13. $(7z^3 - 3z^2 + 2z + 15) - (3z^3 + z^2 - 9z - 28)$
14. $(4k^4 + 2k^2 - 15) - (6k^4 - 3k^3 + 7)$
15. $(5p^5 - 4p^4 + 3p - 12) - (7p^4 + 3p + 15)$
16. $(14a^4 + 5a^3 - 2a^2 + 16) - (7a^4 - 11a^3 + 5a^2 + 16)$
17. $(6b^3 - 7b^2 + 13b - 7) - (5b^3 + 10b^2 - 8b + 22)$
18. $(9r^3 + 4r - 24) - (5r^4 - 3r^3 + 17r - 14)$
19. $(17q^4 - 5q + 52) - (9q^4 + 18q - 15)$
20. $(15c^3 + 18c^2 - 7c + 20) - (10c^3 - 7c^2 + 7c - 12)$

6.3 Multiplying Monomials and Polynomials

Multiplying a Monomial and a Binomial

To multiply a monomial by a binomial, simply use the distributive property. Let's consider the following product:

$$4g(10g + 2) \text{ or equivalently, } (4g)(10g + 2)$$

Using the distributive property, we have

$$4g(10g) + 4g(2)$$

After performing the indicated multiplication associated with each term, this simplifies to

$$40g^2 + 8g$$

Multiplying a Monomial by a Trinomial

The distributive property can also be applied when computing the product of a monomial and a trinomial or any other polynomial. Consider this example,

$$9n(6n^2 – 7n + 10) \text{ or equivalently, } (9n)\ (6n^2 – 7n + 10)$$

Using the distributive property and multiply $9n$ by each of the three other terms, we have

$$9n(6n^2) – 9n(7n) + 9n(10) = 54n^3 – 63n^2 + 90n$$

In case you have a negative monomial, always remember to check the sign of the product of every term. Let's consider this product:

$$-10x^2(18x^2 – 5x + 7)$$

Using the distributive property, we have

$$-10x^2(18x^2) + 10x^2(5x) – 10x^2(7)$$

After performing the indicated multiplication associated with each term, this simplifies to

$$-180x^4 + 50x^3 – 70x^2$$

Exercise 6.3

Perform the indicated multiplication, writing the answer in standard form (with the highest degree terms listed first).

1. $(2a)(6a^2 - 3ab)$
2. $(7x)(2xy + x^5)$
3. $(pq)(7p^2 - 9q^3)$
4. $(-9x^2)(-3x - x^3)$
5. $(5t)(-6at + t^2v)$
6. $(-8d)(60k - 3d)$
7. $(at)(ap + 2t)$
8. $(10j^2)(14j + 7)$
9. $(-y)(-y^3 + 14y)$
10. $(8c^2)(bc + 13c)$
11. $(8y^2)(6y^2 - 15y - 7)$
12. $(7x^2)(8ax - 6x - 3a)$
13. $(bt)(5t^2 - 7b + 11)$
14. $(2j)(16j^5 - 8j^2 + 6j)$
15. $(5v^2)(24 + 8v - 9v^3)$
16. $(13u^3)(6x + 13ux - 5u^2)$
17. $(14j^3)(21 - 17j + 8j^2)$
18. $(7n^2)(5m - mn + 16m^2)$
19. $(6m)(lm + 4m^2 - 23)$
20. $(15k)(14k^3 - 5t + k)$

6.4 Multiplying Binomials

Now that you have mastered the multiplying of monomials, the next step is to multiply expressions with two terms, in this case binomials.

Multiplying Binomials Using the Distributive Property

As you learned previously the distributive property is used to derive the product of a monomial and binomials or trinomials. The distributive property is also used to derive the product of a binomial and another binomial or polynomial. Let's consider the example of the product of two binomials as follows:

$$(15x + 8)(22x + 4)$$

In general, this is of the form $(a + b)(c + d) = ac + ad + bc + bd$. A special mnemonic has been developed to describe this result: **FOIL**, or **F**irst + **O**uter + **I**nner + **L**ast which terms are defined below:

- First: Multiply the first terms of both binomials—ac
- Outer: Multiply the outer terms of both binomials, the first term of the first binomial and the last term of the last binomial—ad
- Inner: Multiply the inner term of both binomials, the last term of the first binomial and the first term of the last binomial—bc
- Last: Multiply the last term of both binomials—bd

Thus, applying this general distributive property to the example product of two binomials above, we have,

$$15x(22x) + 15x(4) + 8(22x) + 8\,(4)$$

where we have made the following substitutions: $a = 15x$, $b = 8$, $c = 22x$, and $d = 4$. Next we perform the multiplications to obtain,

$$330x^2 + 60x + 176x + 32$$

Finally, we simplify by combining like terms (the $60x$ and the $176x$), to obtain the final result,

$$330x^2 + 236x + 32$$

Let's do one more example together, this time using some negative terms:

$$(-11x + 6)(5x - 18)$$

Using the distributive property (FOIL method), we have,

$$-11x(5x) + 11x(18) + 6(5x) - 6(18)$$

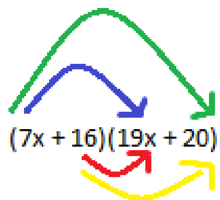
Notice that $-11x$ times -18 was written as the term $+11x(18)$, since a negative term times another negative term is a positive term. Likewise, since $+6$ times -18 is negative, we wrote the term as $-6(18)$. Finally, we perform the multiplications indicated, and combine like terms:

$$-55x^2 + 198x + 30x - 108 = -55x^2 + 228 - 108$$

This procedure is so very fundamental, let's do one more example demonstrating the product of two binomial expressions and representing each step by a different color. Consider

$$(7x + 16)(19x + 20)$$

Now, using the distributive property or more specifically the FOIL method, we are going to perform the sum of the following four multiplications: **F**irst (shown in blue), **O**uter (shown in green), **I**nner (shown in red) and **L**ast (shown in yellow):



$$\mathbf{(7x)(19x)} + \mathbf{(7x)(20)} + \mathbf{(16)(19x)} + (16)(20)$$

Performing the indicated multiplications and then combining like terms we have,

$$133x^2 + 140x + 304x + 320 = 133x^2 + 444x + 320$$

Following the above steps will allow you to multiply binomials successfully. A common mistake that a student will often make is to forget the FOIL method and simply perform $(7x)(19x) + (16)(20)$, which yields only the first and last terms—producing an incomplete result. Of course, now that you have been warned about this, you will be certain to always obtain all four terms.

Table 6.4 Rules for multiplication of monomials and binomials

Product of	Product	Result
monomial • monomial	$a \cdot b$	ab
monomial • binomial	$a(b + c)$	$ab + ac$
binomial • binomial	$(a + b)(c + d)$	$a(c + d) + b(c + d)$ or $ac + ad + bc + bd$

Exercise 6.4

Perform the indicated multiplication of the binomials, giving the answer in standard form.

1. $(x - 6)(2x - 3)$
2. $(6a - 3)(7a - 15)$
3. $(z + 4)\ (-2z + 9)$
4. $(t + 9)(-4t - 11)$
5. $(3k - 21)(-k + 8)$
6. $(7m - 6)(4m - 2)$
7. $(n^2 - 5)(n^2 + 2)$
8. $(y + 6)(-9y + 10)$
9. $(a^2 - 5)(11a^2 - 2)$
10. $(t^3 - 2)(-2t^3 + 7)$
11. $(-8d + 2)(5 - d)$
12. $(9c + 11)(7c - 5)$
13. $(q - 2)(6 - 3q)$
14. $(k - 2)(8k - 7)$
15. $(7 - 9p)(p + 14)$
16. $(x + 9)(x - 17)$
17. $(y - 11)(3y - 2)$
18. $(3w + 14)(w - 10)$
19. $(-b^4 + 3b^2)(8b^4 - 5b^2)$
20. $(-4n^4 - 2n^2)(-6n^4 - 3n^2)$

6.5 Dividing Polynomials

Dividing a Polynomial by a Monomial

Let's first consider how to divide a polynomial by a monomial. In this type of division, the only thing necessary is to divide each term of the polynomial by the monomial. This follows the basic rule in fractions, whereby such a fraction can be rewritten as the sum of fractions consisting of each numerator term divided by the term in the denominator:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Consider the following example word problem:

Mary Anne bought 6 identical shirts at the store, and spent an additional \$28 for a new pair of jeans. She had a coupon that gave her half off of her total purchase. If she ended up paying a total of \$48, what was the original cost of each shirt (prior to the half off coupon)?

Let x = the original cost of each shirt. Prior to her use of the coupon, Mary Anne spent $6x + 28$ for the six shirts and pair of jeans. However, since she had a coupon that gave her half off, we must divide the total cost of her purchase by 2, thus we have the equation:

$$\frac{6x + 28}{2} = 48$$

This can be simplified by dividing each term on the left side of the equation by 2, so that we have

$$\frac{6x}{2} + \frac{28}{2} = 48$$

Performing the division, we have

$$3x + 14 = 48$$

To solve for x, we must subtract 14 from each side of the equation:

$$3x = 48 - 14$$

Next, we divide by 3, to obtain

$$x = \frac{34}{3} = 11\frac{1}{3}$$

Thus, the original price for each shirt was \$11.33.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial. The following property of exponents is useful when a polynomial is divided by a variable raised to some power:

$$\frac{x^a}{x^b} = x^{a-b}$$

Let's apply this property using the following example:

$$\frac{20x^3 - 80x^2 + 16x}{4x}$$

First, we divide each term in the numerator by the term in the denominator,

$$\frac{20x^3}{4x} - \frac{80x^2}{4x} + \frac{16x}{4x}$$

Then using the property of exponents (or simply cancelling like terms), we can reduce the expression to,

$$5x^2 - 2x + 4$$

Let's try one more example problem:

$$\frac{80x^3 + 20x^2 - 5x + 10}{10x}$$

First, we divide each term in the numerator by the monomial in the denominator:

$$\frac{80x^3}{10x} + \frac{20x^2}{10x} - \frac{5x}{10x} + \frac{10}{10x}$$

Next, we perform the division of each term:

$$8x^2 + 2x - \frac{1}{2} + \frac{1}{x}$$

Exercise 6.5

Perform the division, writing the answer in standard form.

1. $(6x^2 - 3x + 9)/3$
2. $(4a^6 - 9a^3 + a^2)/a^2$
3. $(9z^4 + 5z^2 + 1)/z^3$
4. $(12t^3 + 8t^2 + 6t)/(2t)$
5. $(15m^6 - 25m^3 - 30m)/(5m)$
6. $(28c^5 - 49c^3 + 77c^7)/(7c^2)$
7. $(54n^4 - 36n^2 - 45)/(9n^2)$
8. $(121 + 77k^5 - 22k^4)/(11k)$
9. $(96r^2 - 156 + 84r^3)/(12r^2)$
10. $(21b^3 - 105b^2 - 42b^4)/(3b)$
11. $(72p^3 - 90p^2 + 126)/(18p)$
12. $(28 - 70y^3 + 238y^2)/(14y)$
13. $(78x + 26x^2 - 182)/26$
14. $(208 - 169q^3 + 65q^6)/(13q)$
15. $(6w^9 - w^6 - 4w^3)/w^3$

6.6 Factoring Trinomials in the Standard Form

Multiplying Binomials

Recall that earlier you learned how to use the FOIL method to multiply two binomial factors such as the following:

$$(x + 5)(x + 7)$$

If you conclude that $(x + 5)(x + 7) = x^2 + 35$, then you are INCORRECT—so please re-read and study the previous section 6.5 before continuing.

Using the FOIL method, we have the first term as x^2, outer term as $7x$, inner term as $5x$, and last term as 35, thus we can write

$$(x + 5)(x + 7) = x^2 + \mathbf{7x} + \mathbf{5x} + 35$$

Combining the like terms $7x$ and $5x$, we have $x^2 + \mathbf{12x} + 35$. Please note that the reverse is also true:

$$x^2 + 12x + 35 = (x + 5)(x + 7)$$

Thus, if we were given the expression $x^2 + 12x + 35$, we could conclude that the equivalent expression in terms of factors would be $(x + 5)(x + 7)$. Using the FOIL method it is easy to multiply two factors and then find the equivalent polynomial such that $(x + 5)(x + 7) = x^2 + 12x + 35$. It is somewhat more challenging to do the reverse procedure, that is, given the polynomial $x^2 + 12x + 35$, find the factors. Let's learn how we can factor such trinomials (polynomials with three terms).

How to Factor Trinomials

The key to factoring is to observe very carefully how the terms of the factor contribute toward the resulting polynomial. Please notice in the last example how we derived the fact that

$$(x + 5)(x + 7) = x^2 + 12x + 35$$

Notice these individual steps that were actually involved:

- $x \bullet x = x^2$ — The first terms of each factor multiply together to become the first term of the trinomial.
- $5 \bullet 7 = 35$ — The last terms of each factor multiply together to become the last term of the trinomial.
- $5 + 7 = 12$ — The sum of the last terms of each factor yield the middle term of the trinomial.

Using these observations, let's now attempt to factor this polynomial:

$$x^2 + 12x + 20$$

Now, based on our previous experience with factors, we would expect that this polynomial can be expressed as the product of two binomials, so that we know the basic form of the answer:

$$x^2 + 12x + 20 = (\_\ \_\ \_)(\_\ \_\ \_)$$

Since the only way to obtain x^2 is to have x times x, we know that the first term of each binomial factor must be x. At this particular time, we have deduced the incomplete factors are

$$(x\ \_\ \_)(x\ \_\ \_)$$

To complete the factors that yield the remaining part of the polynomial, $12x + 20$, we must now find the proper combination of constants *such that when the constants are multiplied together we obtain the constant 20; however, when the constants are added together, we obtain 12*—which is the coefficient of the middle term $12x$.

Here is a table showing all the possible factors of 20. We have included a column for the product of the two factors as well as the sum of the two factors (which must yield the coefficient of the middle term (12).

1st Factor	2nd Factor	1st Factor • 2nd Factor	1st Factor + 2nd Factor
1	20	20	21
2	**10**	**20**	**12**
4	5	20	9
-1	-20	20	-21
-2	-10	20	-12
-4	-5	20	-9

Out of these six different combinations of factors, we could quickly eliminate the last three negative factors (-1 • -20, -2 • -10, and -4 • -5) by noticing that since both the coefficient of the middle term ($12x$) and the constant 20 are positive, this requires that both factors be positive. Notice that only the factors 2 and 10 add to give 12 (the middle term of the trinomial) but multiply together to get 20 (the last term of the trinomial). Thus, we have finally determined

$$x^2 + 12x + 20 = (x + 2)(x + 10)$$

or equivalently, we can write the factors in the reverse order,

$$x^2 + 12x + 20 = (x + 10)(x + 2)$$

So the steps for factorizing a trinomial may involve use of these steps:

- Factor the first term
- Find all the pairs of factors of the last term
- Be careful to check the sign of each factor of the first and last terms
- Use the process of elimination

Let's consider another example in which the middle term is negative,

$$x^2 - 10x + 21$$

As we did previously, we start by writing the general form of the factors:

$$x^2 - 10x + 21 = (\_\ \_\ \_)(\_\ \_\ \_)$$

Next, we note that the only way to obtain x^2 is to write the factors as

$$x^2 - 10x + 21 = (x\ \_\ \_)(x\ \_\ \_)$$

To complete the factoring, we must find all the factors of 21 as shown in the table:

1st Factor	2nd Factor	1st Factor • 2nd Factor	1st Factor + 2nd Factor
1	21	21	22
3	7	21	10
-1	-21	21	-22
-3	**-7**	**21**	**-10**

Notice that because the constant 21 is positive, and we have a negative middle term (-10*x*), the only way that we can obtain these results is by choosing BOTH the 1st and 2nd factors as negative. Of the 4 possibilities for factors shown in the table, only -3 and -7 multiply to yield 21 and add to yield -10. Thus, the polynomial is factored as follows:

$$x^2 - 10x + 21 = (x - 3)(x - 7)$$

Let's consider another example where the constant term in the trinomial is negative:

$$x^2 + 10x - 24$$

We see that some of the factors of -24 are given in the table below

1st Factor	2nd Factor	1st Factor • 2nd Factor	1st Factor + 2nd Factor
1	-24	-24	-23
-1	24	-24	-23
2	-12	-24	-10
-2	**12**	**-24**	**10**

There is no need to make a complete list of the factors, since we can stop once we find the two factors that work, -2 and 12. Thus, we can factor the polynomial as follows:

$$x^2 + 10x - 24 = (x - 2)(x + 12)$$

When the third or constant term of the polynomial is negative, this always indicates that one of the numbers in the factored binomial will be negative.

Let's try one more factoring problem where we have a negative sign on the 1st term of the polynomial. Consider

$$-x^2 - 13x - 12$$

We note that the only way to obtain $-x^2$ is to start by writing,

$$(x\ \_\ \_)(-x\ \_\ \_)$$

since $x \bullet (-x) = -x^2$. Equivalently, we could have written the factors in the reverse order,

$$(-x\ \_\ \_)(x\ \_\ \_)$$

Now because we have a $-x$ as a factor, this is going to somewhat complicate our selection of the other factors. If we started trying factors, we might think that either -1 and 12, or 12 and -1 are good to begin with since when multiplied together they yield the constant that we have in the polynomial (-12). So let's try -1 and 12, so that we have $(x - 1)(-x + 12)$. We can use the FOIL method and see that this equates to $-x^2 + 13x - 12$, which does not equal our original polynomial $-x^2 - 13x - 12$. The reason for this is that we have $-x$ as one of the terms in the factors. However, if we experiment with the signs of the factors 1 and 12, we finally see that this combination of +1 and -12 works:

$$-x^2 - 13x - 12 = (-x - 12)(x + 1)$$

This sort of trial and error approach with the factors is also useful if we have a more complicated polynomial that has coefficients on the x^2 term, such as

$$-8x^2 - 18x - 10$$

We know that to obtain the first $-8x^2$ term, we have these 4 possible factors

$(-x\ \_\ \_)(8x\ \_\ \_)$
$(x\ \_\ \_)(-8x\ \_\ \_)$
$(-2x\ \_\ \_)(4x\ \_\ \_)$ **[We will see that this factor works later]**
$(2x\ \_\ \_)(-4x\ \_\ \_)$

Now, in addition to those 4 factors shown above, to obtain the 3rd term of the polynomial, -10, we have these 4 possible factors.

$(? + 1)(? - 10)$
$(? - 1)(? + 10)$
$(? + 2)(? - 5)$
$(? - 2)(? + 5)$ **[We will see that this factor works later]**

So, now begins the process of trial and error. We must keep trying up to 16 different possible combinations until we find the combination of factors that work. Finally we see that the factors 1 (or -1) and 10 (or -10) do not work with the factors $-x$ and $8x$: $(-x + 1)(8x - 10)$, $(-x - 1)(8x + 10)$, $(-x - 10)(8x + 1)$, $(-x + 1)(8x - 10)$; also they do not work with the factors x and $-8x$: $(x + 1)(-8x - 10)$, $(x - 1)(-8x + 10)$, $(x - 10)(-8x + 1)$, $(x + 1)(-8x - 10)$. Also, the factors 2 (or -2) and 5 (or -5) do not work with either the factors $-x$ and $8x$ or the factors x and $-8x$: $(-x + 2)(8x - 5)$, $(-x - 2)(8x + 5)$, $(-x - 5)(8x + 2)$, $(-x + 5)(8x - 2)$, $(x + 2)(-8x - 5)$, $(x - 2)(-8x + 5)$, $(x - 5)(-8x + 2)$, $(x + 5)(-8x - 2)$. The one combination of factors (highlighted above) that is found (through trial and error) to yield the original polynomial is this:

$$-8x^2 - 18x - 10 = (-2x - 2)(4x + 5)$$

Later (in Chapter 11, Section 11.3), we will learn the quadratic formula which will provide a way to solve polynomials without resorting to factoring—which in some cases can be time-consuming and difficult (especially if the coefficients and constant are not whole numbers).

Also, consider that it is not possible to factor all polynomials. $x^2 + 9$ is one such example of a polynomial that cannot be factored (at least not through use of real numbers).

If we try factors such as $(x + 3)(x + 3)$, we get $x^2 + 6x + 9$. And if we try $(x - 3)(x - 3)$, we get $x^2 - 6x + 9$. So, no matter what we try, we get an unwanted middle term of either $6x$ or $-6x$. If we resort to trying $(x + 3)(x - 3)$, we get $x^2 - 9$. While there is no middle term here, the operator between the terms is incorrect. However, did you know that using higher mathematics it is indeed possible to factor $x^2 + 9$? If you want to impress your friends, you can show them that

$$x^2 + 9 = (x + 3i)(x - 3i);$$

where i is called an imaginary number and is defined in higher mathematics as $i = \sqrt{-1}$. Yes, I know you were told and perhaps learned that it is "impossible" to take the square root of a negative number; but, in this instance it is convenient to use this definition. These factors, called **complex expressions**, are suitable factors since $3i \bullet (-3i) = -9i^2 = -9 \bullet (-1) = 9$. You will learn more about this if you pursue college courses beyond the level of calculus. **Complex numbers** are used in various fields today, including signal processing, control theory, electromagnetism, fluid dynamics, quantum mechanics, cartography, and vibration analysis. Once I saw in a catalog a tee-shirt with the slogan, "$\sqrt{-1}$ love math!"--meaning "I love math!"

Well, we got off on a "tangent" there—but wasn't it fun? While $x^2 + 9$ cannot be factored (using real numbers), $x^2 - 9$ can be factored as $(x +3)(x - 3)$, since this yields (using the FOIL method—see pp. 151-152):

$$x^2 - 3x + 3x - 9$$

which simplifies to

$$x^2 + 0x - 9 \quad \text{or} \quad x^2 - 9$$

where the coefficient of the coefficient of x (the middle term) is 0. Also, in many practical cases, the factors of a quadratic expression will not be whole numbers as is exemplified in the case of factoring

$$x^2 - 5 = (x +\sqrt{5})(x - \sqrt{5}).$$

Note: Please remember that, in the general case, $(a + b)(c + d)$, using the distributive property, yields these four terms:

$$ac + ad + bc + bd$$

Only when we have $(a + b)(a - b)$ do we finally obtain the two terms shown:

$$a^2 - ab + ab - b^2 = a^2 - b^2$$

In other words, the lesson here is to never multiply two general binomial terms, such as

$$(a + b)(a + b)$$

and simplify this expression to

$$a^2 + b^2$$

THIS RESULT IS INCORRECT! The correct answer is $a^2 + ab + ab + b^2$ or

$$a^2 + 2ab + b^2$$

Exercise 6.6

Factor the following trinomials:

1. $x^2 + 13x + 36$
2. $x^2 + 9x + 14$
3. $x^2 + 15x + 44$
4. $x^2 + 11x + 30$
5. $x^2 + 4x + 3$
6. $x^2 - 11x + 28$
7. $x^2 - 11x + 18$
8. $x^2 - 15x + 36$
9. $x^2 - 8x + 15$
10. $x^2 - 7x + 6$
11. $x^2 + 10x - 11$
12. $x^2 + 8x - 20$
13. $x^2 + 5x - 36$
14. $x^2 + 5x - 24$
15. $x^2 + 11x - 26$
16. $x^2 - 7x - 44$
17. $x^2 - 36x - 117$
18. $x^2 - 7x - 60$
19. $x^2 - 7x - 18$
20. $x^2 - x - 42$

6.7 Solving Equations Containing Polynomials

Factorization or factoring is a useful tool when solving an equation that consists of a polynomial. An equation that is comprised of a second degree polynomial is also called a quadratic equation. In the future we will consider other methods to solve quadratic equations (using the quadratic formula). For the present, let's consider an example of how to solve a quadratic equation though use of factoring.

Consider the trinomial in the following equation:

$$x^2 + 10x + 10 = -11$$

To solve for x, the first step is to write the equation in the standard form $ax^2 + bx + c = 0$. So, we must add 11 to each side of the equation to obtain the standard form,

$x^2 + 10x + 21 = 0$ (where $a = 1$ since the coefficient of x^2 is implied to be 1; $b = 10$; and $c = 21$)

Next, we can mentally note that 3 • 7 = 21 and 3 + 7 = 10, so that we can instantly factor the expression on the left side of the equation

$$(x + 3)\ (x + 7) = 0$$

Also, mentally, we should have been able to quickly discard other possible factors for 21, such as 1 • 21, since the addition (or subtraction) of these factors does not yield the desired coefficient of the middle term (10).

Now, since the two factors, $(x + 3)$ and $(x + 7)$ are equal to zero, if any one of these factors is zero, then the product will automatically be zero. So the next step is to set each factor equal to zero and solve for x. Thus we have

$$x + 3 = 0 \qquad \text{and} \qquad x + 7 = 0$$

Solving each equation, we find that $x = -3$ or $x = -7$ will satisfy the original equation.

To verify that our answers (-3 and -7) are correct we substitute each answer into original standard equation. First we check $x = -3$,

$$\begin{gathered} x^2 + 10x + 21 \\ (-3)^2 + 10(-3) + 21 \\ 9 + (-30) + 21 = 0 \\ 9 - 30 + 21 = 0 \\ 0 = 0 \end{gathered}$$

Next, we check $x = -7$,

$$\begin{gathered} x^2 + 10x + 21 \\ (-7)^2 + 10(-7) + 21 \\ 49 + (-70) + 21 = 0 \\ 49 - 70 + 21 = 0 \\ 0 = 0 \end{gathered}$$

Since 0 = 0 is a true statement in both cases, we have confirmed our answers are correct.

Exercise 6.7

Solve the following quadratic equations by first factoring.

1. $x^2 - 5x - 36 = 0$
2. $x^2 + 18x + 65 = 0$
3. $x^2 - 16x + 48 = 0$
4. $x^2 - 5x + 6 = 0$
5. $x^2 - 15x + 54 = 0$
6. $x^2 - x - 90 = 0$
7. $x^2 - 15x + 26 = 0$
8. $x^2 - 2x - 8 = 0$
9. $x^2 - 7x - 60 = 0$
10. $x^2 + 6x + 9 = 0$
11. $x^2 - 11x + 18 = 0$
12. $x^2 + 20x + 96 = 0$
13. $x^2 + 4x - 77 = 0$
14. $x^2 - 8x + 12 = 0$
15. $x^2 - 19x + 78 = 0$

Solve these more challenging equations.

16. $x^2 - 9 = 0$
17. $3x^2 - 3 = 0$ (Look for a number that is common to each of the terms and use the distributive property first. Then factor the remaining polynomial.)
18. $3x^2 - 12x + 9 = 0$ (Look for a number that is common to each of the terms and use the distributive property first. Then factor the remaining polynomial.)
19. $x^2 + 12x + 36 = 0$
20. $8x^2 - 8x - 16 = 0$ (Look for a number that is common to each of the terms and use the distributive property first. Then factor the remaining polynomial.)

Answers to Chapter 6 Exercises

Exercise 6.1

1. yes; **2.** yes; **3.** no; **4.** no; **5.** yes; **6.** b; **7.** c; **8.** a; **9.** b; **10.** a; **11.** 3; **12.** 5; **13.** 2; **14.** 1; **15.** 1; **16.** $m^2 - 2m + 2$; **17.** $2az^5 + 3bz^4 + 5z^4 - z^3 - 6$; **18.** $-d^4 + 5d^2 - 8d$; **19.** $6dr^2 + 7r^3 - br - 10$ or $7r^3 + 6dr^2 - br - 10$; **20.** $10j^2 + 3j - 8$; **21.** \$110.00; **22.** \$80.00; **23.** 120 km/hr (you must convert 5 minutes to 1/12 hr in order to have the proper units for the answer)

Exercise 6.2

1. $2x^4 - x^3 + x^2 + 13$; **2.** $4m^4 + 8m^3 + 4m^2 + 2m - 2$; **3.** $10z^3 - 2z^2 - 7z - 13$; **4.** $10k^4 - 3k^3 + 2k^2 - 8$; **5.** $5p^5 + 3p^4 + 6p + 3$; **6.** $21a^4 - 6a^3 + 3a^2 + 32$; **7.** $11b^3 + 3b^2 + 5b + 15$; **8.** $5r^4 + 6r^3 + 21r - 38$; **9.** $26q^4 + 13q + 37$; **10.** $25c^3 + 11c^2 + 8$; **11.** $-2x^4 + 3x^3 + 5x^2 - 17$; **12.** $2m^4 - 22m^3 - 4m^2 + 6m + 30$; **13.** $4z^3 - 4z^2 + 11z + 43$; **14.** $-2k^4 + 3k^3 + 2k^2 - 22$; **15.** $5p^5 - 11p^4 - 27$; **16.** $7a^4 + 16a^3 - 7a^2$; **17.** $b^3 - 17b^2 + 21b - 29$; **18.** $-5r^4 + 12r^3 - 13r - 10$; **19.** $8q^4 - 23q + 67$; **20.** $5c^3 + 23c^2 - 14c + 32$

Exercise 6.3

1. $12a^3 - 6a^2b$ or $-6a^2b + 12a^3$; **2.** $7x^6 + 14x^2y$; **3.** $-9pq^4 + 7p^3q$; **4.** $9x^5 + 27x^3$; **5.** $5t^3v - 30at^2$; **6.** $-480dk + 24d^2$ or $24d^2 - 480dk$; **7.** $a^2pt + 2at^2$; **8.** $140j^3 + 70j^2$; **9.** $y^4 - 14y^2$; **10.** $8bc^3 + 104c^3$; **11.** $48y^4 - 120y^3 - 56y^2$; **12.** $56ax^3 - 42x^3 - 21ax^2$; **13.** $5bt^3 - 7b^2t + 11bt$; **14.** $32j^6 - 16j^3 + 12j^2$; **15.** $-45v^5 + 40v^3 + 120v^2$; **16.** $169u^4x - 65u^5 + 78u^3x$ or $-65u^5 + 169u^4x + 78u^3x$; **17.** $112j^5 - 238j^4 + 294j^3$; **18.** $112m^2n^2 - 7mn^3 + 35mn^2$ or $-7mn^3 + 112m^2n^2 + 35mn^2$; **19.** $24m^3 + 6lm^2 - 138m$; **20.** $210k^4 - 75kt + 15k^2$ or $210k^4 + 15k^2 - 75kt$

Exercise 6.4

1. $2x^2 - 15x + 18$; **2.** $42a^2 - 111a + 45$; **3.** $-2z^2 + z + 36$; **4.** $-4t^2 - 47t - 99$; **5.** $-3k^2 + 45k - 168$; **6.** $28m^2 - 36m + 12$; **7.** $n^4 - 3n^2 - 10$; **8.** $-9y^2 - 44y + 60$; **9.** $11a^4 - 57a^2 + 10$; **10.** $-2t^6 + 11t^3 - 14$; **11.** $8d^2 - 42d + 10$; **12.** $63c^2 + 32c - 55$; **13.** $-3q^2 + 12q - 12$; **14.** $8k^2 - 23k + 14$; **15.** $-9p^2 - 119p + 98$; **16.** $x^2 - 8x - 153$; **17.** $3y^2 - 35y + 22$; **18.** $3w^2 - 16w - 140$; **19.** $-8b^8 + 29b^6 - 15b^2$; **20.** $24n^8 + 24n^6 + 6n^4$

Exercise 6.5

1. $2x^2 - x + 3$; **2.** $4a^4 - 9a + 1$; **3.** $9z + 5/z + 1/z^3$ (or $9z + 5z^{-1} + z^{-3}$); **4.** $6t^2 + 4t + 3$; **5.** $3m^5 - 5m^2 - 6$; **6.** $11c^5 + 4c^3 - 7c$; **7.** $6n^2 - 4 - 5/n^2$; **8.** $7k^4 - 2k^3 + 11/k$; **9.** $7r + 8 - 13/r^2$; **10.** $-14b^3 + 7b^2 - 35b$; **11.** $4p^2 - 5p + 7/p$; **12.** $-5y^2 + 17y + 2/y$; **13.** $x^2 + 3x - 7$; **14.** $5q^5 - 13q^2 + 16/q$; **15.** $6w^6 - w^3 - 4$

Exercise 6.6

1. $(x + 4)(x + 9)$; **2.** $(x + 2)(x + 7)$; **3.** $(x + 4)(x + 11)$; **4.** $(x + 5)(x + 6)$; **5.** $(x + 3)(x + 1)$; **6.** $(x - 4)(x - 7)$; **7.** $(x - 2)(x - 9)$; **8.** $(x - 3)(x - 12)$; **9.** $(x - 3)(x - 5)$; **10.** $(x - 1)(x - 6)$; **11.** $(x - 1)(x + 11)$; **12.** $(x - 2)(x + 10)$; **13.** $(x - 4)(x + 9)$; **14.** $(x - 3)(x + 8)$; **15.** $(x - 2)(x + 13)$; **16.** $(x + 4)(x - 11)$; **17.** $(x + 3)(x - 29)$; **18.** $(x + 5)(x - 12)$; **19.** $(x + 2)(x - 9)$; **20.** $(x + 6)(x - 7)$

Exercise 6.7

1. -4, 9; **2.** -13, -5; **3.** 4, 12; **4.** 2, 3; **5.** 6, 9; **6.** -9, 10; **7.** 2, 13; **8.** -2, 4; **9.** -5, 12; **10.** -3; **11.** 2, 9; **12.** -12, -8; **13.** -11, 7; **14.** 2,6; **15.** 6, 13; **16.** -3, 3; **17.** -1, 1; **18.** 1, 3; **19.** -6; **20.** -1, 2

Chapter 7: Linear Equations and Graphing

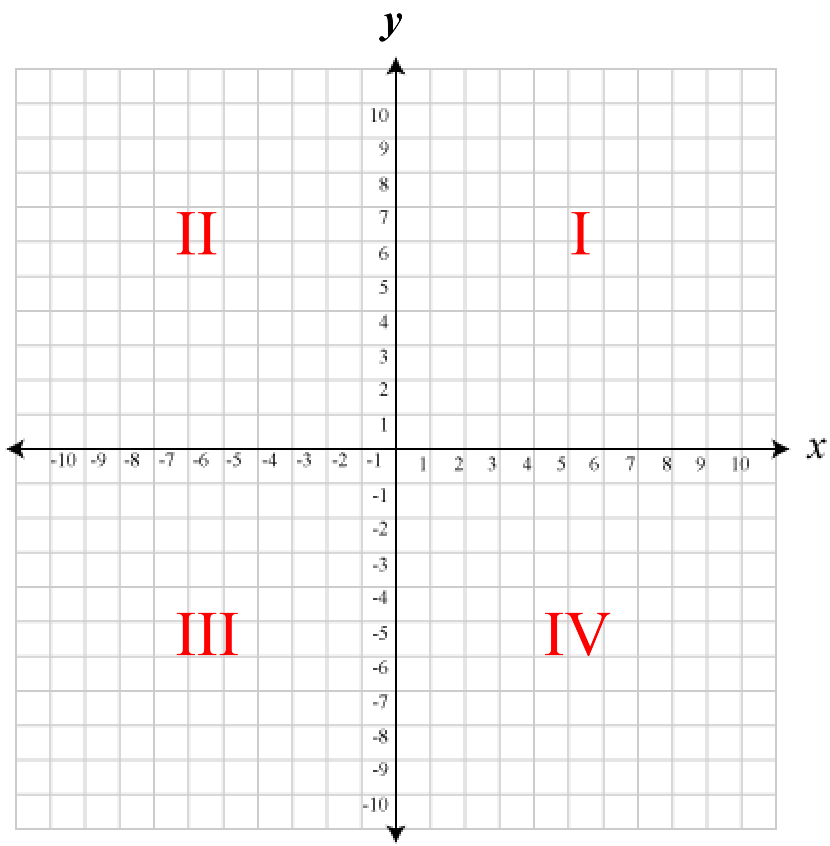
7.1 The Rectangular Coordinate System and Graphing Points

Graphing equations can help you to see mathematical relationships clearly and visually explain many trends found in fields such as Physics, Chemistry, Biology, and more. Graphs help to describe and communicate mathematical relationships.

In this chapter you will learn how linear relationships can be shown (or plotted) on a graph and how graphing can be used to find the solution to a linear equation or find the solution to a system of linear equations.

The Rectangular Coordinate System

If two lines are perpendicular to each other and positioned horizontally and vertically, then you have a **rectangular (or Cartesian) coordinate system**. In the following figure the horizontal line with arrows is called the abscissa- or ***x*-axis** while the vertical line with arrows is called the ordinate- or ***y*-axis**. The point which the axes intersect is called the **origin**. Please note that the horizontal line is labeled as the "*x*" axis on the right. Likewise, near the top of the vertical line is the label for the "*y*" axis".



An infinite amount of values and points can be placed on both axes. The origin (where the lines forming the *x*- and *y*-axis intersect) divides each axis into positive and negative sections. Notice that the right side of the origin on the *x*-axis (horizontal) and the values above the origin on *y*-axis (vertical) are positive, while the values on the left side of the origin on the *x*-axis and the values below the origin on the *y*-axis are negative. Another relevant feature you can notice in the rectangular coordinate system is that the *x* and *y* axes divide the plane into four regions that are called **quadrants**. Quadrant I consists of coordinates where both the *x* and *y* values are positive. Quadrant II consists of coordinates where the *x*-values are negative and the *y*-values are positive. Similarly, Quadrant III consists of coordinates where both the *x* and *y* values are negative. Finally, quadrant IV consists of coordinates where the *x*-values are positive and the *y*-values are negative.

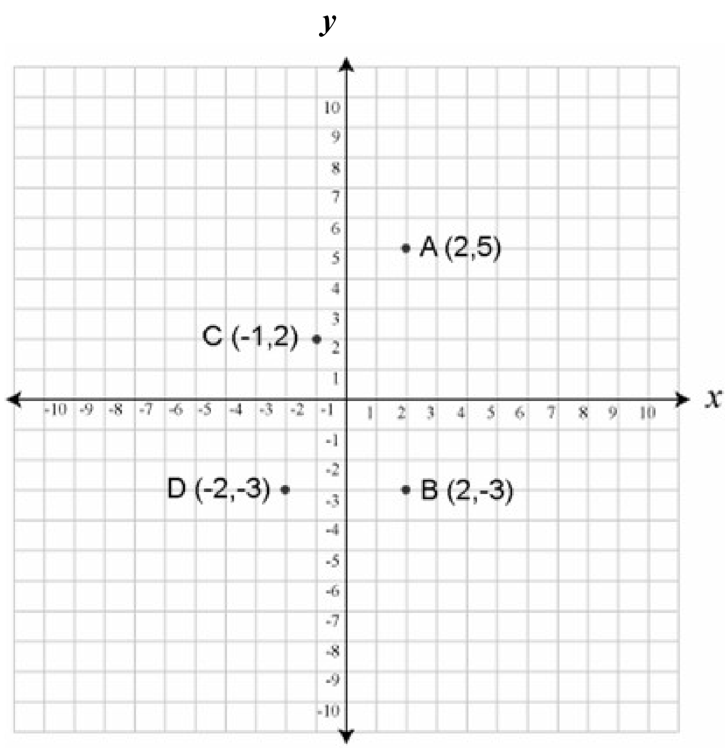
Ordered Pairs

A rectangular coordinate system can be used to show the location of one or more points. A point is represented on a graph as a solid dot (or very small black circle). The location of the point on the graph depends on its assigned *x* and *y* values that are often expressed inside parentheses like this: (*x*,*y*). A point that is expressed in this manner is called an **ordered pair** (the order is always the *x*-value first, then a comma, followed by the *y*-value next). The *x* and *y* values are called the coordinates of the point. The *x*-value tells us where to go along the *x*-axis and the *y*-value tells us where to go along the *y*-axis. Where the *x* and *y* values intersect (or meet) is the location of the point that is being specified. We will demonstrate this clearly with several examples in the following section.

Graphing Ordered Pairs

To identify or plot (or graph) points on the Cartesian plane (or coordinate system) you need two numbers that comprise a **coordinate pair** (an *x* and *y* value) that corresponds to each point. By inspecting the signs of the coordinate pair or by the location of the point on the graph, it is easy to identify the quadrant that any given point lies in. Let´s plot the following four points on the graph below:

Name of point	Coordinates	Quadrant
A	(2, 5)	I
B	(2, -3)	IV
C	(-1, 2)	II
D	(-2, -3)	III

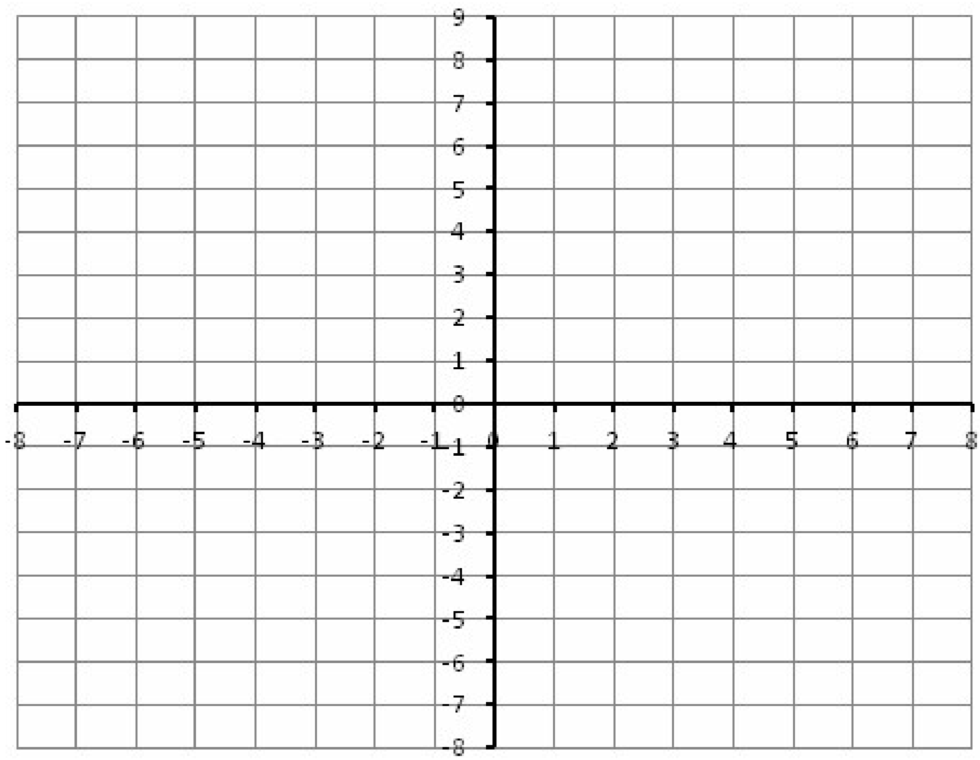


Notice, that a point (shown as a small black dot) has been placed for point A, given by the coordinates (2, 5) at the intersection of the vertical light-colored line going through 2 on the *x*-axis and hori-zontal line going through 5 on the *y*-axis. The same is true for the positioning of the other three points labeled B, C, and D.

Exercise 7.1

Show the following ordered pairs on the graph.

1. (2, 8)
2. (4, 4)
3. (-2, 6)
4. (5, -7)
5. (-6, -2)
6. (-2, 3)
7. (6, -4)
8. (7, 3)
9. (3, -4)
10. (1, 3)
11. (4, -4)
12. (-3, 5)
13. (6, 8)
14. (-1, 7)
15. (-7, -5)



7.2 Linear Equations

Up until now, you have mostly seen expressions and equations with only one variable, such as:

$$x^2 + 17x - 5 \quad \text{or} \quad \frac{3}{4}x \quad \text{or} \quad 19t + 5 = 12$$

However, now we are going to consider equations that have two different variables, such as:

$$y = 3x + 1 \quad \text{or} \quad y = -6x - 5$$

Notice in the examples above that for the equation $y = 3x + 1$ that if we set x equal to 1 (or $x = 1$), we can solve the equation for y by substituting 1 for x. Doing this substitution, we have

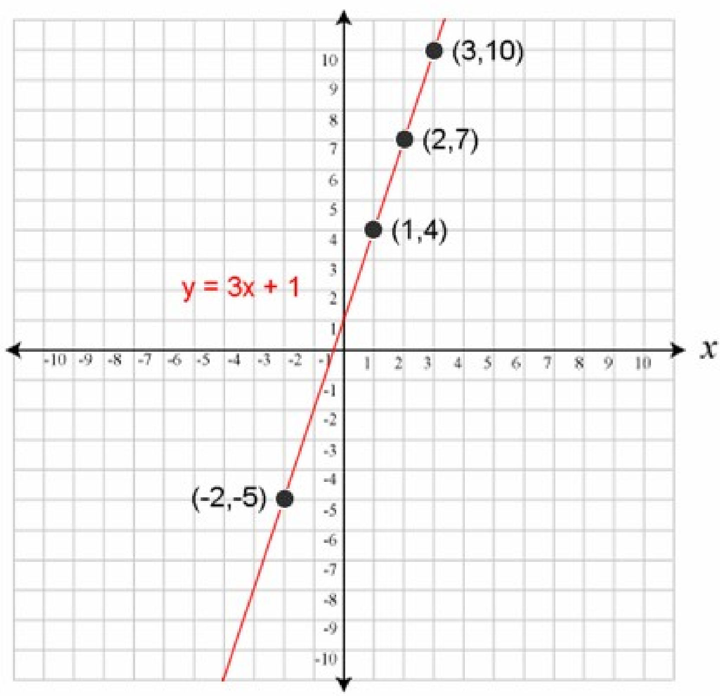
$$y = 3(1) + 1$$

or $y = 4$. When we have a pair of x and y values such as $x = 1$ and $y = 4$ that satisfy the equation (or make it true), this is called a solution that can be written as an ordered pair, (1, 4). Remember that the order of the x and y values in an ordered pair is always the x-value listed first, followed by a comma, and then the y-value next, or (x-value, y-value). Sometimes an ordered pair is also referred to as a **coordinate point** or simply as a **coordinate** or as a **point**.

Actually, there are an infinite number of solutions to the equation $y = 3x + 1$. As an example of another solution, let us set x equal to 2 (or $x = 2$). Substituting the x with 2, we can solve for y,

$$y = 3(2) + 1$$

or $y = 7$. So that the ordered pair (2, 7) is another solution. Still other solutions include (3, 10), or even (-2, -5), since when $x = -2$, we obtain $y = 3(-2) + 1 = -6 + 1 = -5$.



These points four points (1, 4), (2, 7), (3, 10), (-2,- 5) are shown on the graph. Notice that each of these coordinate pairs lie directly on the line since they are solutions to $y = 3x + 1$.

What we have just graphed are points that are solutions to a **linear equation**. Linear equations are equations where if you were to graph all the ordered pairs that are solutions to the equation, they would lie along a straight line, and the line would extend into infinity because there are an infinite number of solutions as x increases to infinity or as x decreases to minus infinity. A linear equation can always be represented in the general form y = $mx + b$; where m is called the "slope" or the coefficient (that is, the number that is multiplied by the variable x) and b is a constant (the y-coordinate value where the line intersects the y-axis). Notice that the variable x in a linear equation is raised to the power of 1 (in other words x as shown in the general form is the same as x^1). We learned earlier that this can be referred to as a first degree polynomial. An equation that contains x raised to the 2nd power, or x^2, is not linear (or is non-linear) and, in fact, is referred to as a quadratic equation (or 2nd degree polynomial) which we will address at a later time.

Finding Solutions to Linear Equations and Graphing Linear Equations

It is straightforward to find solutions and graph linear equations. We simply substitute at least two different values for x into the equation and solve for the two corresponding values of y. Then, we plot the two points and draw a line through them.

Let's demonstrate the procedure with the linear equation,

$$y = -2x - 1$$

Usually $x = 0$ is an easy substitution to use, so that we have

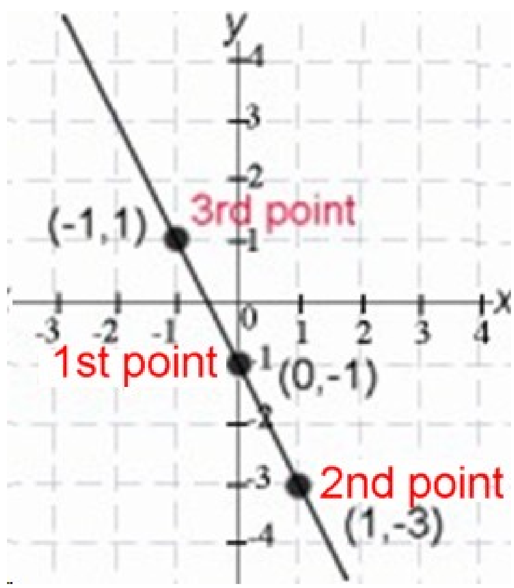
$$y = -2(0) - 1 = 0 - 1 = -1$$

So, the first coordinate point we have on the line is (0, -1). Now, let's try substituting $x = 1$ into our equation, so that we have

$$y = -2(1) - 1 = -2 - 1 = -3$$

So, our second coordinate point is (1, -3). Even though we just need two points to define a line, let's go ahead and determine a third coordinate point using $x = -1$. Then we have

$$y = -2(-1) - 1 = 2 - 1 = 1$$



So, our third coordinate point is (-1, 1). Wasn't that easy! Now we can simply plot our points,

(0, -1), (1, -3), and optionally (-1, 1)

and draw a line through them to graphically represent the linear equation $y = -2x - 1$.

Note: The expression on the right side of the equation $y = -2x - 1$ contains the variable x, so we can say that y is a **function** of x (or y is dependent on the value that is assigned to x), where the function f(x) is equal to $-2x - 1$. This equivalent notation is sometimes used: f(x) = $-2x - 1$, where f(x) is the same a y and indicates that the expression is a function of (or depends on) x. When $x = 4$, we can compute that $y = -9$. Equivalently, f(4) can be computed by making the assignment $x = 4$ and evaluating the expression on the right side—so that f(4) = -2•4 – 1, or f(4)= -9. Both of these conventions are commonly used so it is important to be familiar with this alternate notation.

Exercise 7.2

List three coordinate points that are solutions to each linear equation by substituting $x = -1$, $x = 0$, and $x = 1$ into each equation.

1. $y = 2x - 9$
2. $y = -4x + 3$
3. $y = 3x - 4$
4. $y = -x + 10$
5. $y = 12x - 20$
6. $y = -5x - 7$
7. $y = -3x + 9$
8. $y = 4x - 11$
9. $y = -2x - 13$
10. $y = 5x - 3$
11. $y = 7x + 4$
12. $y = 2x + 27$
13. $y = -3x + 15$
14. $y = 4x - 7$
15. $y = -3x + 19$

Now, graph each linear equation given in problems #1 through #15.

7.3 Graphing a System of Linear Equations

In the last lesson you learned how to find solutions to a single linear equation as well as graph both points and lines.

Systems of Linear Equations

When we are interested in two (or more) linear equations taken together at the same time, this is called a **system of linear equations**. Since each linear equation represents a line on the graph, two linear equations will usually intersect at some point that is common to both lines. This point of intersection is called the solution of the system of linear equations and it can be specified as a coordinate in the form (x,y). Let's look at an example.

As the temperature increases in Chicago, ice cream sales increase, but hot dog sales decrease. This intuitively makes sense since something cold, such as ice cream, is a nice treat in the summer heat; yet, a hot snack, such as a hot dog, is not very popular in such heat.

After some research, it was discovered that the estimated number of ice cream sales per day, y, is given by the linear equation:

$$y = 3x - 25$$

where x is the temperature in degrees centigrade (^{o}C). Similarly, the estimated number of hot dog sales is given by the linear equation:

$$y = -4x + 115$$

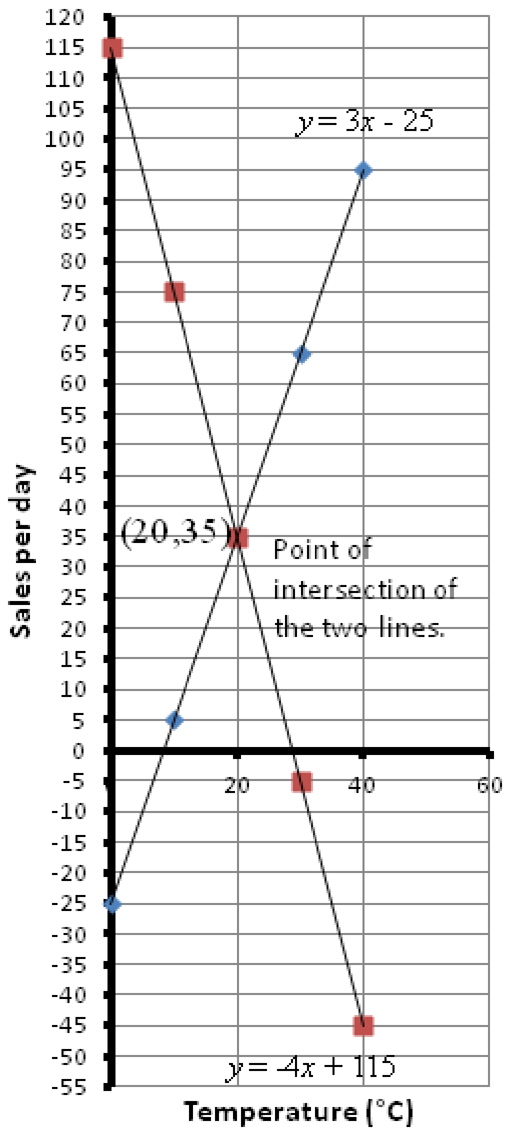
The charts and graph below show how ice cream sales and hot dog sales vary with temperature.

Ice cream sales $y = 3x - 25$	
Temperature (^{o}C) x-value	**Sales y-value**
0	-25
10	5
20	35
30	65
40	95

Hot dog sales $y = -4x + 115$	
Temperature (^{o}C) x-value	**Sales y-value**
0	115
10	75
20	35
30	-5
40	-45

We have set the x-value to 0, then 10, then 20, 30, and 40 in both equations that describe ice cream and hot dog sales and show the corresponding y-values that we obtained using the equations. Notice that at 0^{o}C (and lower negative temperatures), ice cream sales is -25—and such a negative number of sales is not practical; also, at 30^{o}C (and higher), a negative number of hot dog sales is not practical. Both of the lines that represent sales as a **function of temperature** (that is, sales is **dependent on temperature**) are plotted below. Notice the point of intersection of the two lines below.

Graph showing ice cream sales and hot dog sales as a function of temperature (°C).



At what temperature are ice cream sales and hot dog sales exactly the same? The point of intersection occurs at a temperature of 20°C. Thus, if we substitute $x = 20$, into either the ice cream or hot dog sales equation we obtain the same number of sales in each case,

$$y = 3(\mathbf{20}) - 25 = 60 - 25 = 35$$
$$\text{or} \quad y = -4(\mathbf{20}) + 115 = -80 + 115 = 35$$

The solution can be obtained by inspecting the graph above of the two equations. The two lines intersect at (20, 35), which is called the solution of the system of equations. Please note carefully that while the point (10, 5) is located directly on the line representing ice cream sales, this point is not a solution to the system of equations—since the solution to the system of equations must consist of a point that lies on BOTH lines. In other words, the solution to the system of equations is usually only one point located at the intersection of both lines.

Now suppose we had chosen to substitute $x = -1$, $x = 0$, and $x = 1$, which represent temperatures of

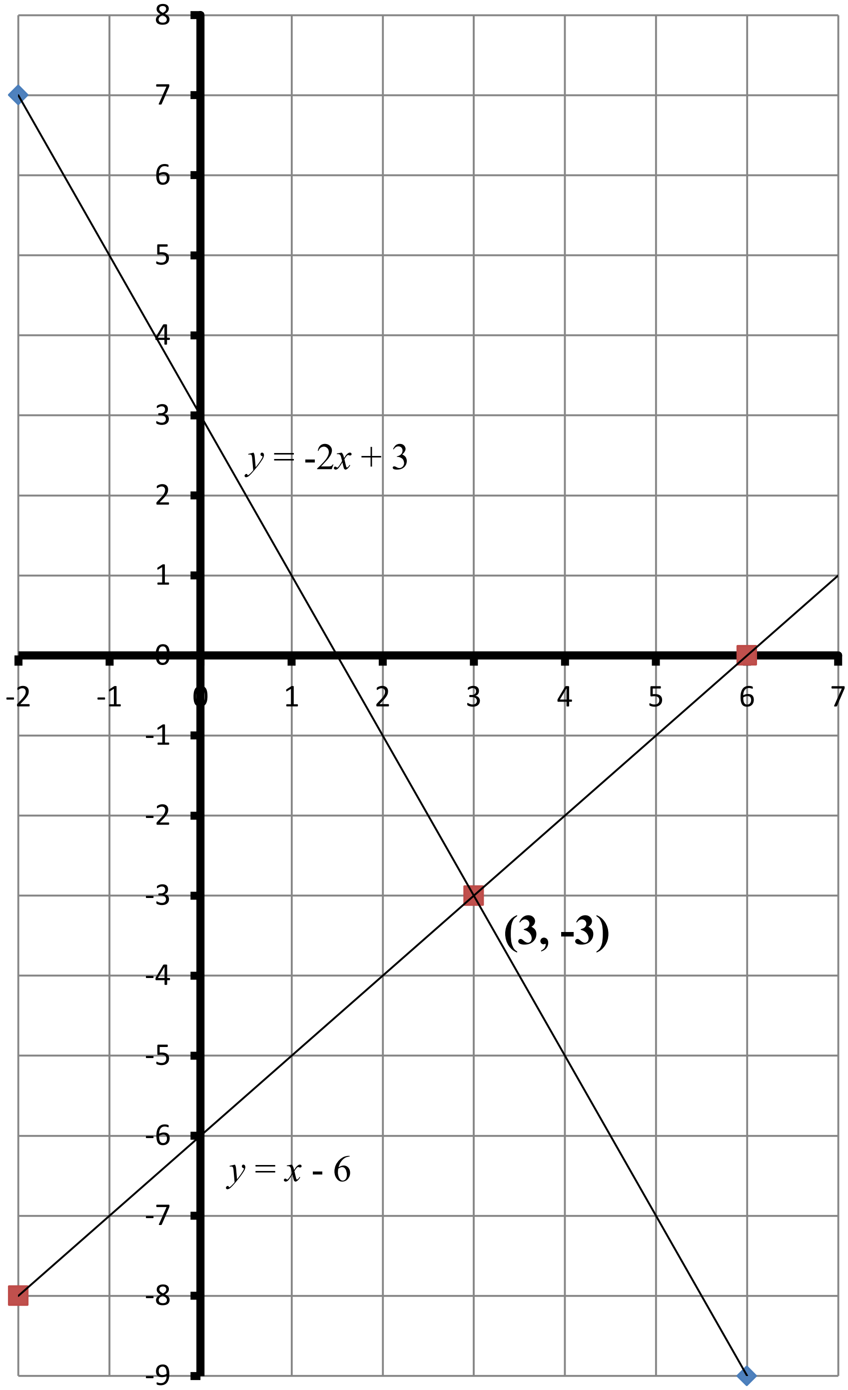
-1°C, 0°C, and 1°C,

into the two linear equations, as we have done in earlier problems. Then for ice cream sales, using $y = 3x - 25$ we would have the points (-1, -28), (0, -25), and (1, -22). For hot dog sales, using y = -4x + 115, we would have the points (-1, 119), (0, 115), and (1, 111). If we plotted and extended the lines we obtained from these points, this would yield the exact same solution (point of intersection) that we obtained, (20, 35). Because of the large *y*-coordinates, however, our graphing of these lines would take a rather large sheet of paper, and it would take extra care to draw a rather lengthy *y*-axis and then extend the lines accurately. Notice that by labeling the axes 0, 5, 10, 15, 20, etc.—using 5 unit increments—we were able to plot the lines and find the intersection in a practical size on paper. Some problems may call for you to determine the scale to use on the *x*- and/or *y*-axes; it may not be practical to always increment the axes values by 1 unit when graphing a system of linear equations and attempting to find the point of intersection of the lines—especially if either the *x*- or *y*-coordinates of that point exceed perhaps 20.

Let's work one more example and find the solution to the following system of linear equations:

$$y = -2x + 3$$
$$y = x - 6$$

We plot (or graph) the two linear equations using the methods we have previously described in Section 7.2 to obtain the following graph that shows the solution, (3,-3), at the point of intersection.



Exercise 7.3

Solve the following systems of linear equations by graphing. Write the solution as an ordered pair (x,y).

1. $y = 7x - 5$

   $y = -3x + 5$

2. $y = -3x - 11$

   $y = x + 5$

3. $y = x + 7$

   $y = 2x + 9$

4. $y = 4x + 5$

   $y = \frac{3}{2}x$

5. The perimeter of a rectangular field is 260 meters, and the length of the field is 50 meters longer than the width. What are the dimensions of the rectangular field?

Hint: Let's let the unknown width be x and the unknown length be y.

Since the perimeter is equal to the sum of all the sides, we have $x + x + y + y = 260$, or

$2x + 2y = 260$. We can put this equation into our linear form by first subtracting $2x$ from each side of the equation, thus we obtain

$$2y = -2x + 260$$

Finally, we can divide both sides by 2, to obtain our linear equation in slope and y-intercept format:

$$y = -x + 130$$

We can obtain our second equation using the fact that the "longest side of the field is 50 meters longer than the shorter side." In algebra terms, we could say that the length is equal to the width plus 50, or

$$y = x + 50$$

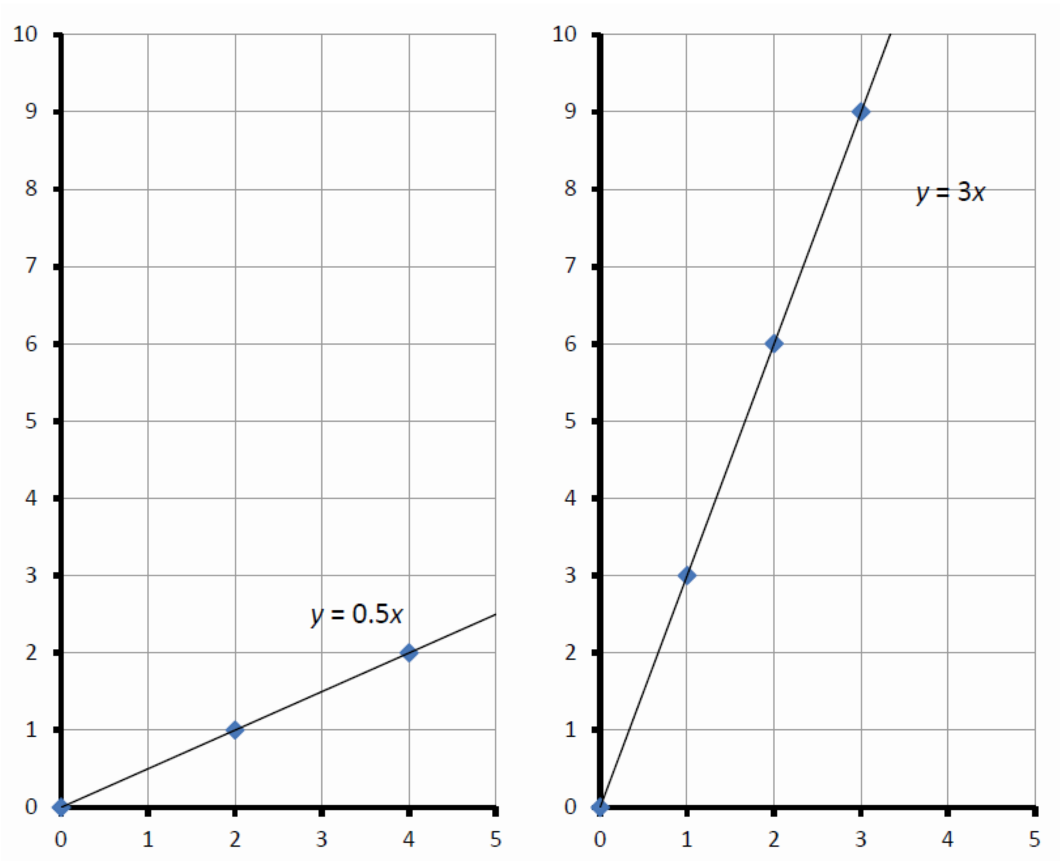
Thus, the system of equations we need to solve is given as follows:

$$y = -x + 130$$
$$y = x + 50$$

The x- and y-coordinates where these two lines intersect will define the width and length dimension of the rectangular field in units of meters.

7.4 Slope and *y*-intercept

At this point in your introduction to graphing, you may have noticed that some linear equations increase more rapidly than others as the *x*-value increases. The following linear equations are plotted on graphs with the same scale on the *x*-axes and same scale on the *y*-axes so that the two slopes or rates of increase may be compared.



The difference in the angles of the two lines is due to what is called **slope**. A horizontal line has a slope of 0; however, the slope increases as the line gets steeper and approaches vertical (the slope of a vertical line is infinity). The slope of a line is defined as the ratio of the change in *y* to the change in *x* between any two points on the line. If we have any two points on a given line, such as (x_1,y_1) and (x_2,y_2), then the formula for the slope is given by

slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{change\ in\ y}{change\ in\ x} = \dfrac{\Delta y}{\Delta x}$, where Δ is the Greek letter "delta" meaning "change" (or difference). Let's select two points on the line $y = 0.5x$, (2, 1) and (4, 2), notice that we compute the slope using the above formula we obtain

$$\text{slope} = \frac{2-1}{4-2} = \frac{1}{2} \text{ or } 0.5$$

Please notice that the slope is the same as the coefficient of *x* in our equation of the line $y = 0.5x$.

Also, if you happen to use the coordinates in the reverse order, the slope is unchanged:

$$\text{slope} = \frac{1-2}{2-4} = \frac{-1}{-2} = \frac{1}{2} \text{ or } 0.5$$

Now, let's select two points on the line *y* = 3x, (0, 0) and (1, 3). Applying the formula for slope we have

$$\text{slope} = \frac{3-0}{1-0} = \frac{3}{1} = 3 \quad \text{or using the points in the reverse order slope} = \frac{0-3}{0-1} = \frac{-3}{-1} = 3$$

Again, please note that the slope is the same as the coefficient of *x* in the linear equation *y* = **3***x*.

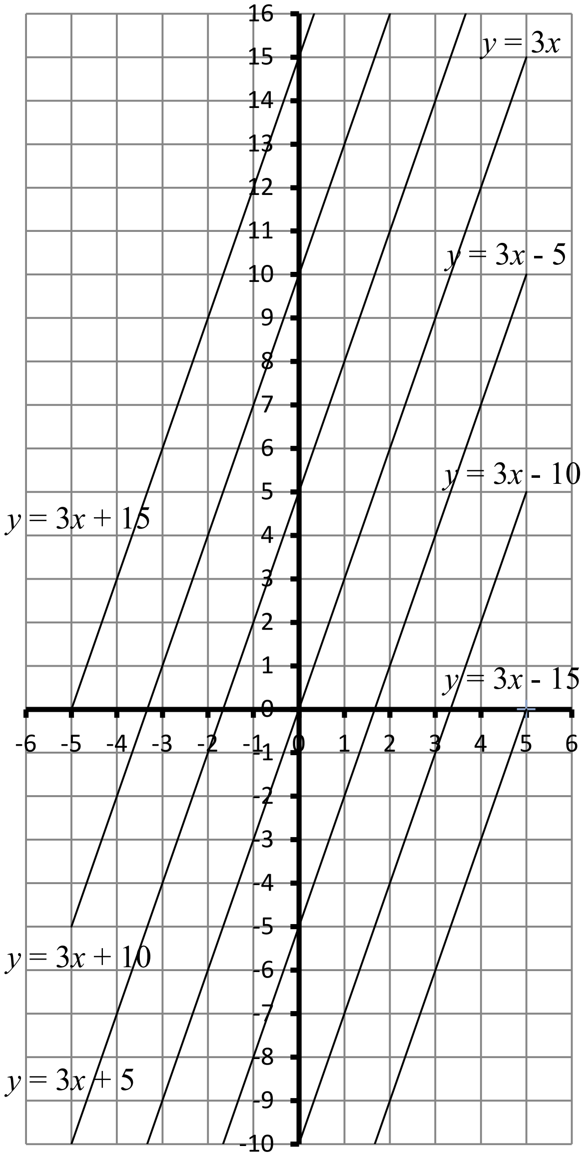
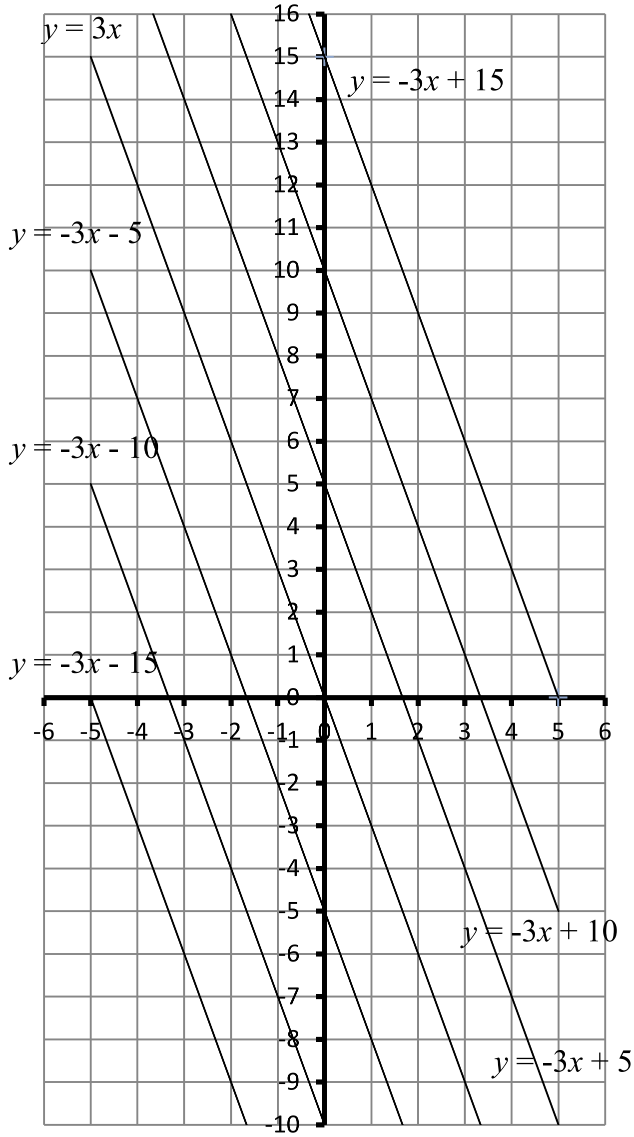
In summary, we conclude that the general equation of a line can be described by

$$y = mx + b$$

where *m* will always be the slope of the line. Interestingly, the reason why the letter "*m*" is chosen to represent the slope—and not the letter "s"—is not known; but *m* is the standard that has been chosen in the mathematics field. We have not yet discussed *b*; however *b* represents the y-intercept which is defined to be the value where the line intersects the *y*-axis. Notice that the line described by this equation: *y* = 0.5*x* and the line described by this equation: *y* = 3*x* both intersect the *y*-axis at 0 [actually at the origin of the graph or (0,0)]. In other words, the line intersects the *y*-axis at the value of 0, therefore the *y*-intercepts for both lines are zero, or *b* = 0. Instead of writing the equations as *y* = 0.5*x* + 0 or *y* = 3*x* + 0, it is unnecessary to show the + 0. Any linear equation with the *b* value missing (or zero) represents a line that will go through the origin (0,0) of the graph—intersecting the *y*-axis at zero.

Notice in the graph below that the linear equation *y* = 3*x* passes through 0 on the *y*-axis as expected. Additional lines are plotted that have the same slope (3), but different *y*-intercepts. If the *y*-intercept is given as +10, the line intersects the *y*-axis at 10; if the *y*-intercept is given as -5, then the line intersects the *y*-axis at -5. When *m* (or the slope) is positive, then the *y*-values of the line increase as the *x*-values increase. This is known in statistics as a positive correlation between the *x* and *y* values.

Various parallel lines.

Lines all having the same positive slope.	Lines all having the same negative slope.
	

Notice that when the slope of the line is negative, the *y*-values decrease as the *x*-values increase. This is known in statistics as a negative correlation between *x* and *y* values.

Lines that have the same slope, regardless of the value of the *y*-intercept are parallel lines as demonstrated in the above graphs. Since parallel lines with different y-intercepts never intersect each other, a system of equations comprised of lines under these conditions has no solution. There is the special case where two equations in a system are identical (with the same slope and y-intercept). In this unique case the lines are both parallel to each other and lie on top of each other forming what appears to be a single line with every point in common between the lines. In this instance, there are infinite solutions to this unique system of equations; that is any point that lies on the line or lines is a valid solution.

Let's say we did not know the equation of a line. If the line was shown on a graph, we could find the slope using any two points on the line to compute the slope (m) and note where the line intersected the y-axis (b). Then, we could write the equation of the specific line by supplying the values for m and b in the general equation for any line:

$$y = mx + b$$

Let's try to figure out the equation of the line shown in the graph on the next page. We note that the y-intercept is -6, so b = -6. For the slope, we need to locate any two points that lie directly on the line. Let's use the points (0, -6) and (-1, -4). Starting at the lower point (0, -6) we note that we travel 2 units up—for a positive change in the y direction, and 1 unit to the left—for a negative change in the x direction. Thus, by simply inspecting the vertical and horizontal distances between the two selected points we observe that the slope is

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{2}{-1} = -2$$

Alternatively, if we start at the higher point (-1, -4), we travel 2 units down—for a negative change in the y direction, and 1 unit to the right—for a positive change in the x direction. Thus, again, by simply inspecting the vertical and horizontal distances between the two selected points we observe the slope is

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{-2}{1} = -2$$

Alternatively, we could have calculated the slope using our selected points (0, -6) and (-1, -4) as follows:

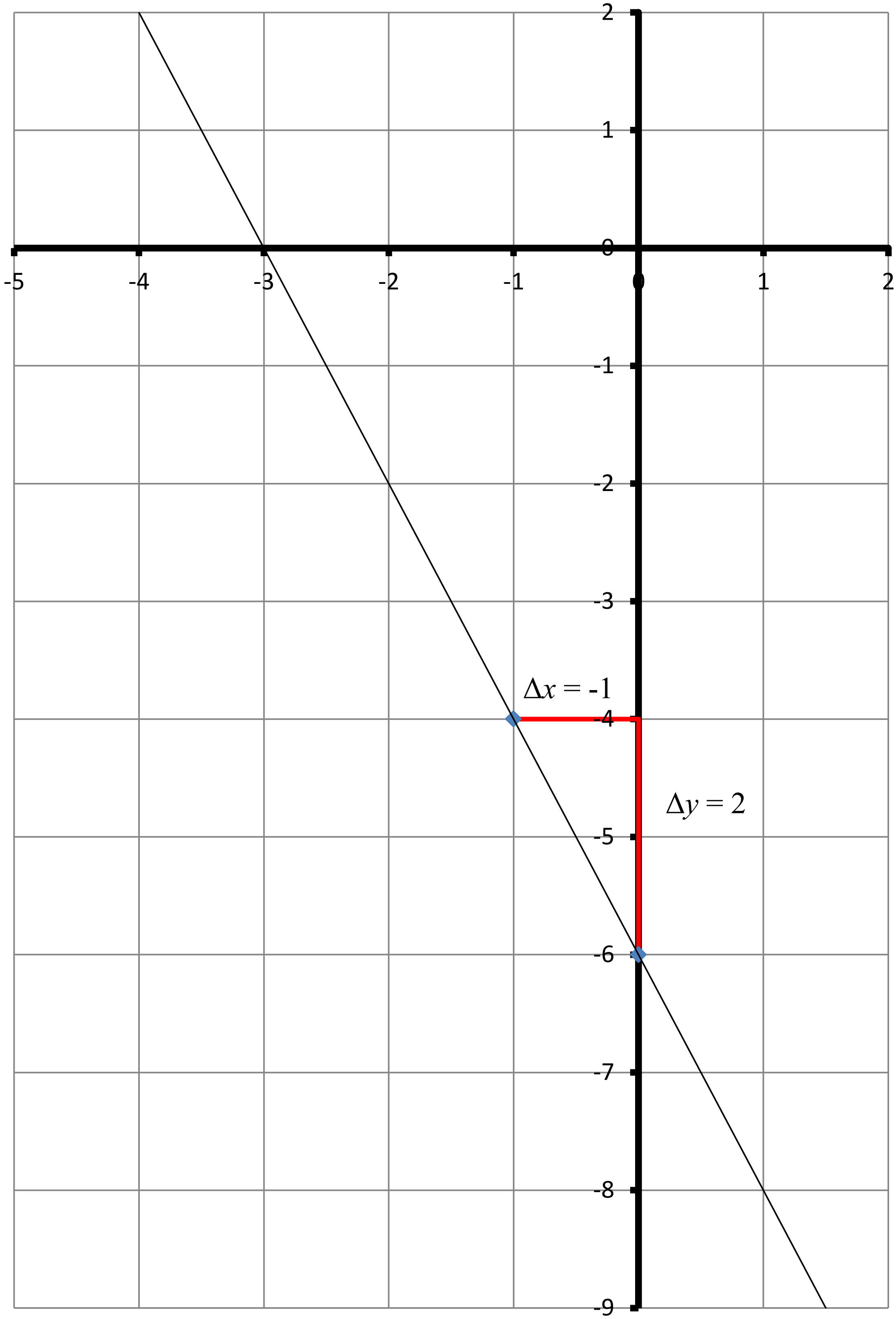
$$m = \frac{-4-(-6)}{-1-0} = \frac{-4+6}{-1} = \frac{2}{-1} = -2$$

Now the y-intercept of the line shown just happens to lie directly on a whole number -6. Let's say this was not the case and we could not precisely determine from looking at the graph if the y-intercept was for example -1.95 or some other decimal or fractional number. Then we can always compute the y-intercept using the slope we computed above along with any one of the points that we found on the line—let's use (-1, -4). The formula for the y-intercept is

$$b = y - m\text{x}$$

Substituting our known point coordinate values, x = -1 and y = -4, and using our slope m = -2, we have

$$b = -4 - (-2)(-1) = -4 - 2 = -6$$

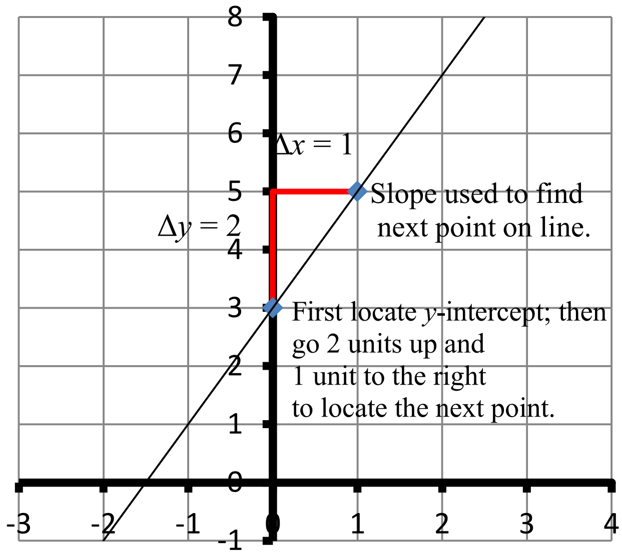


We have now determined that the equation of the line shown above is $y = -2x - 6$.

Description	Definition	Comments
Point or coordinate	(x,y)	Also called an ordered pair.
Equation of a line in slope and y-intercept format	$y = mx + b$	Where m is the slope and b is the y-intercept
Slope	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x}$	Slope is undefined for a vertical line and 0 for a horizontal line.
y-intercept	$b = y - mx$	y-value where the line intersects the y-axis (corresponding to x=0)
Origin	(0, 0)	Point at which the x and y axes intersect

So, at this point we have learned so much about lines, that we should no longer have to calculate various y-values that satisfy the equation of the line using different x-values. Instead, by inspecting the slope and the y-intercept, given a linear equation written in slope and y-intercept format, we instantly know the slope and y-intercept. Given, for example the linear equation $y = 2x + 3$, we instantly know that the slope or $m = 2$ and the y-intercept or $b = 3$. Since the slope is defined to be the change in the y-values divided by the change in the x-values, we can consider a slope of 2 the same as $\frac{2}{1}$ (since we can divide any number by 1 and not change its original value). This means that once we have located one point on the line, we simply go up 2 units (since the 2 is positive) and then go over to the right 1 unit (since 1 is positive). Let's demonstrate this quick process. First let's put a point on the y-axis at the y-intercept of 3, then using the slope of 2, we will go up 2 units and over to the right 1 unit and mark our next point.

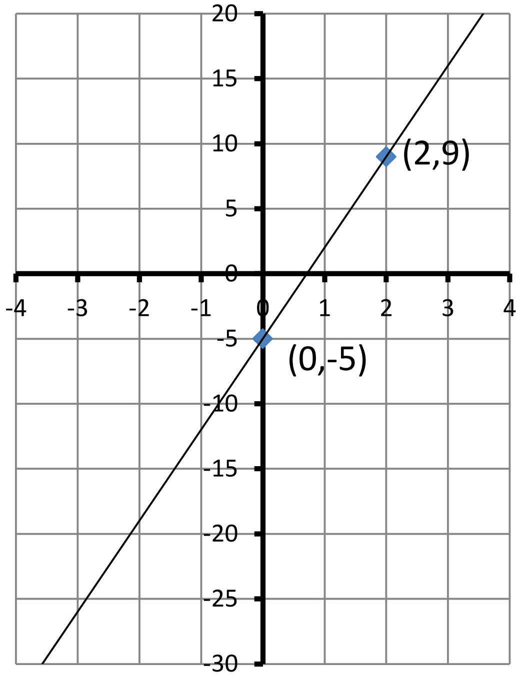
Graphing a line given the y-intercept (b = 3) and slope (m = 2).



Exercise 7.4

Determine the equations of the lines shown with two points on the given lines.

1.



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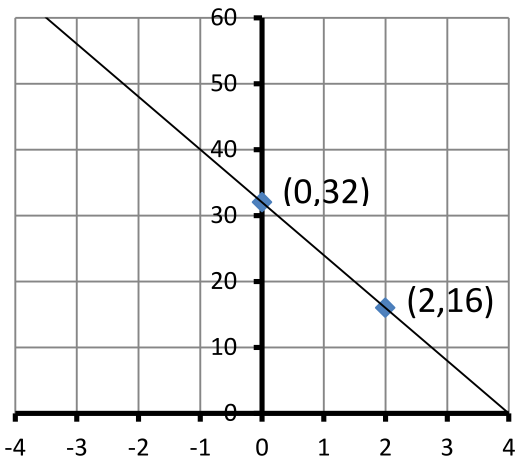
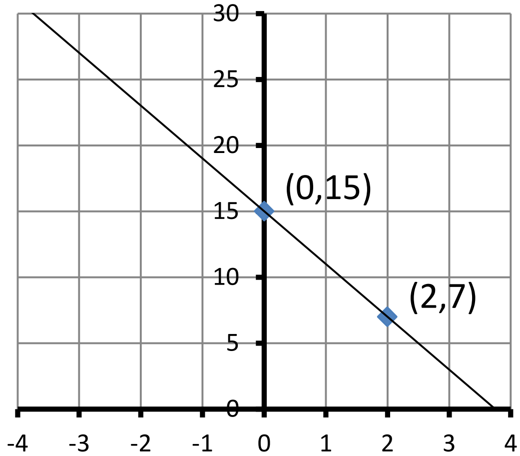


3.



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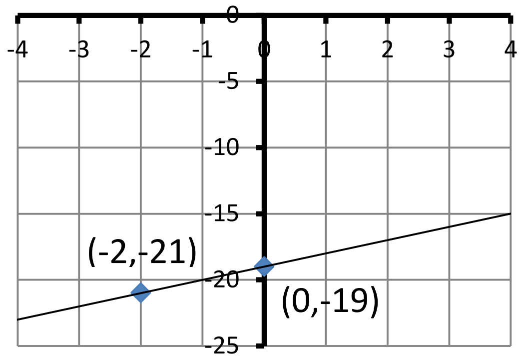

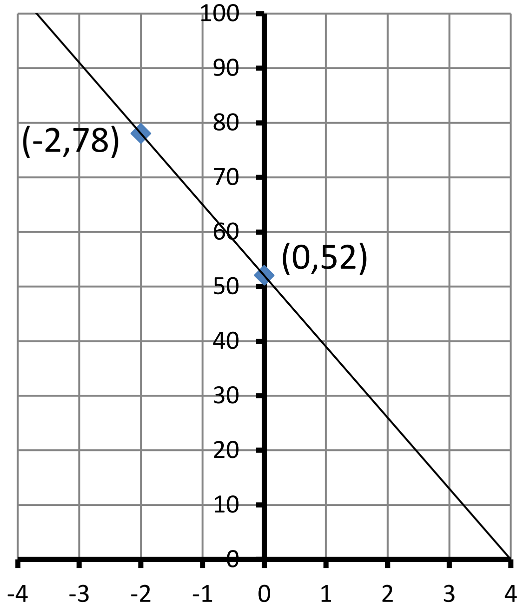

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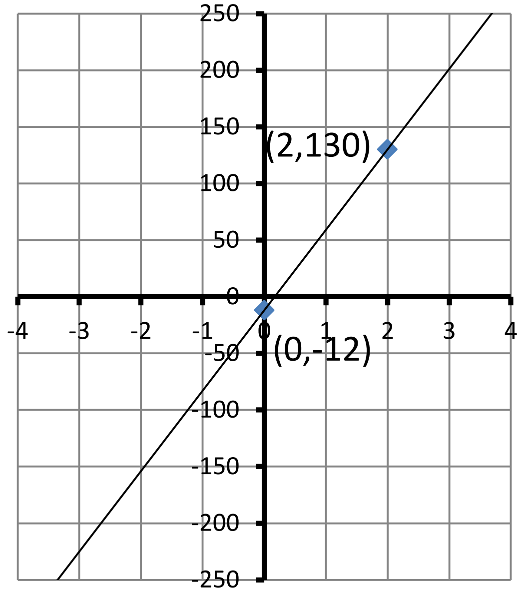

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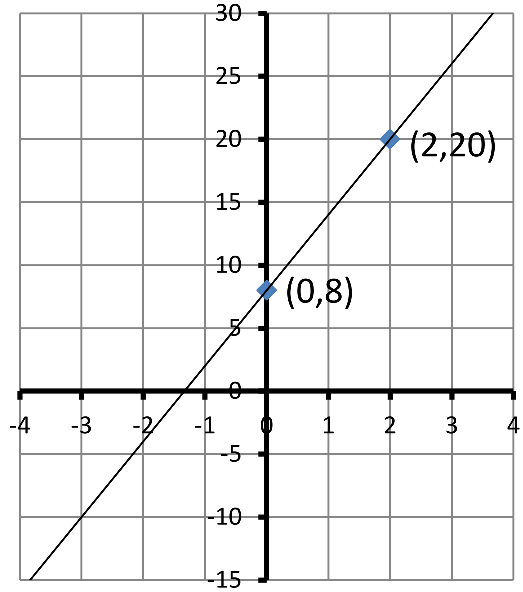

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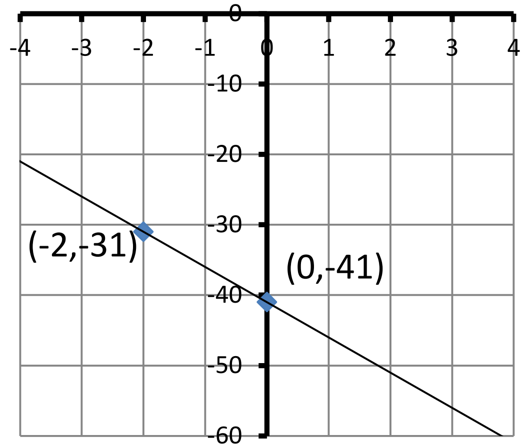

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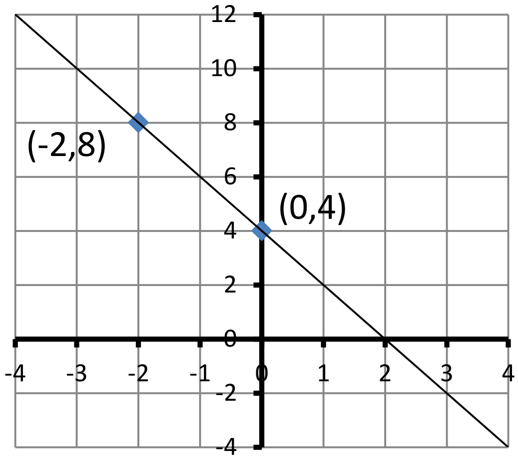
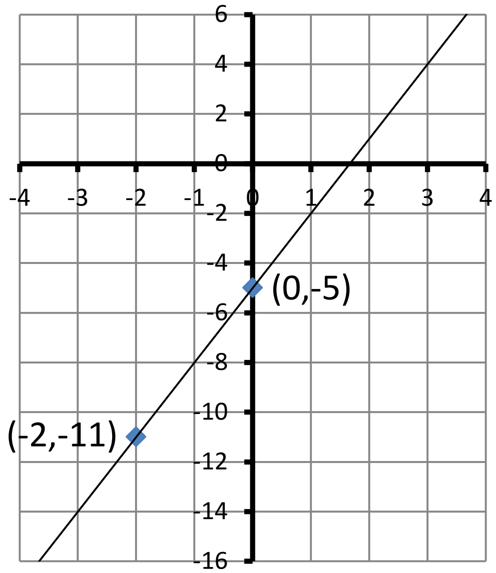
9. 

10. 

Find the equation of the line, by finding the slope (m) and y-intercept, given these two points on the line.

11. (-11, 5), (8, 57)
12. (10, 4), (-6, 41)
13. (-1, 3), (4, 21)
14. (6, 2), (-8, 13)
15. (-70, -5), (9, 78)
16. (-6, 8), (8, 92)
17. (-21, -15), (-4, 39)
18. (-8, 14), (-24, 9)
19. (-7, 11), (7, 9)
20. (52, 61), (-8, -41)
21. (-8, 5), (39, -70)
22. (31, 7), (8, 13)
23. (-1, 6), (-4, 3)
24. (13, 61), (9, -41)
25. (17, -8), (14, 16)
26. (6, 4), (-4, 8)
27. (-33, 15), (6, -18)
28. (7, -2), (9, 3)
29. (-1, 17), (-2, 15)
30. (81, 5), (90, 95)

7.5 Solving a System of Linear Equations by Elimination

In the last section you learned how to solve a system of linear equations by graphing. In this section, however, we are going to see how to add (or subtract) the linear equations so as to solve the system of equations algebraically—without a graph. The method of elimination that we are going to consider is so named because we add the linear equations in such a way so as to eliminate one of the two variables. Let's demonstrate this new method by solving this example system of linear equations:

$$y = 8x - 15$$

$$7x - \frac{1}{5}y = 30$$

Our first step is to re-write the equations in what is called standard form: ax + by = c (or ax – by = c). Basically, the standard form consists of listing the x-term first, then the y-term, and then the equal sign and the constant. The second equation is already in the proper standard form; however, the first equation can be modified to that form by subtracting $8x$ from both sides of the equation. Thus, the two equations in standard form are:

$$-8x + y = -15$$

$$7x - \frac{1}{5}y = 30$$

Now, our goal is to get one of the variables to cancel when we add the two equations together. Notice that if we multiply the first equation by $\frac{1}{5}$, we will have a term $\frac{1}{5}y$ in the first equation that will cancel the term $-\frac{1}{5}y$ in the second equation. So let's go ahead and multiply the first equation (each and every term) by $\frac{1}{5}$ and we now have this equivalent system of equations:

$$-\frac{8}{5}x + \frac{1}{5}y = -3$$

$$7x - \frac{1}{5}y = 30$$

Now, let's add the two equations together. Let's add the x-terms:

$$-\frac{8}{5}x + 7x = -\frac{8}{5}x + \frac{35}{5}x = \frac{-8+35}{5}x = \frac{27}{5}x$$

The sum of the y-terms is of course 0 (since $\frac{1}{5}y - \frac{1}{5}y = 0$). And finally we add the constants: -3 + 30 = 27. We now have one equation in one unknown, x, since we have eliminated the y terms:

$$\frac{27}{5}x = 27$$

Of course, to solve for x we now multiply by the reciprocal coefficient, $\frac{5}{27}$, to obtain $x = 5$.
As our final step, we find the corresponding y-value by substituting $x = 5$ into any of the initial equations and solve for y. Performing this substitution using the first equation, $y = 8x - 15$, we obtain,

$$y = 8(5) - 15 = 40 - 15 = 25$$

So, our solution to the system of linear equations is (5, 25).

Now, let's solve the same system of equations, but this time let's eliminate the x terms:

$$-8x + y = -15$$

$$7x - \frac{1}{5}y = 30$$

Notice that if we multiply the $7x$ term in the second equation by $\frac{8}{7}$, we will obtain $8x$ which will cancel with the $-8x$ that appears in the first equation. So, let's multiply each term of the 2nd equation by $\frac{8}{7}$, to obtain

$$-8x + y = -15$$

$$8x - \frac{8}{35}y = \frac{240}{7}$$

Now we add the like terms in both equations. Of course the $-8x + 8x$ is zero. Next we have the y-terms, $y - \frac{8}{35}y = \frac{35}{35}y - \frac{8}{35}y = \frac{35-8}{35}y = \frac{27}{35}y$ and then add the constants:

$-15 + \frac{240}{7} = \frac{-105}{7} + \frac{240}{7} = \frac{135}{7}$. So that now we have the one equation in one unknown:

$$\frac{27}{35}y = \frac{135}{7}$$

To solve for y, we simply multiply both sides of the equation by the reciprocal coefficient $\frac{35}{27}$, thus our solution is

$$y = \frac{135}{7} \bullet \frac{35}{27} = 25.$$

Next, we substitute our value for y into one of the original equations to solve for the value of x. Again, using the first equation, $-8x + y = -15$, and substituting $y = 25$, we have $-8x + 25 = -15$.

To isolate the variable x, we subtract 25 from each side of the equation, $-8x = -40$. Now we divide both side of the equation by -8 to obtain $x = 5$. Thus, our solution is the same as we determined previously, (5, 25).

As another example of an alternative strategy to eliminate the x terms, we could have multiplied the first equation by 7 and the second equation by 8, transforming the system as follows:

$$-8x + y = -15 \quad \text{multiply by 7 to obtain:} \quad -56x + 7y = -105$$

$$7x - \frac{1}{5}y = 30 \quad \text{multiply by 8 to obtain:} \quad 56x - \frac{8}{5}y = 240$$

Let's practice this elimination method again, but this time using a word problem.

A farmer bought 3 cows and 6 horses at \$8,700. Later, he bought 6 cows and 2 horses at \$7,900. If the price of the horses and cows did not change for these purchases, what did the farmer pay for each horse and cow?

Since the price paid for each cow and each horse is unknown, let's assign these unknown prices the following variables:

$$x = \text{price for a cow}$$
$$y = \text{price for a horse}$$

Please note that to solve for two unknowns requires having two equations.

Then we can write two equations in two unknowns based on the information given in the word problem:

$$3x + 6y = 8700$$
$$6x + 2y = 7900$$

As we inspect this system of equations, we notice that they are already in standard form. So we can either multiply the first equation by -2 to cancel the x terms or we can multiply the second equation by -3 to cancel the y terms. Either approach will lead the same final solution. Let's go ahead and multiply all terms in the second equation by -3. Then we have the equivalent system of equations:

$$3x + 6y = 8700$$
$$-18x - 6y = -23700$$

Now we add the like terms to obtain one equation in one unknown:

$$-15x = -15000$$

To solve for x, we divide both sides of the equation by -15 to obtain,

$$x = 1{,}000$$

So we have determined that each cow is priced at \$1,000. Next, we substitute this value of x into any of the initial equations and solve for y. Using the first equation, $3x + 6y = 8700$, we have

$$3(1000) + 6y = 8700$$

To isolate the y, we first subtract 3000 from each side of the equation to obtain,

$$6y = 5700$$

Finally, we divide both sides of the equation by 6 to obtain

$$y = 950$$

We have now determined that each horse is priced at \$950.

Exercise 7.5

Solve the system of linear equations and express the answer as an ordered pair (x,y).

1. $3x + 4y = 165$
   $2x - 6y = -150$

2. $-4x + 5y = -175$
   $x - y = 47$

3. $-7x + 8y = 322$
   $21x + 5y = -154$

4. $-9x + 9y = -198$
   $-5x - 7y = 82$

5. $3x - 6y = 75$
   $4x - 4y = 64$

6. $-6x + 8y = -136$
   $3x - 7y = 47$

7. $7x + 2y = 134$
   $4x - 7y = -13$

Solve each word problem for the requested unknowns.

8. Karla spent \$625 on 6 pair of pants and 5 sweaters, while her friend Martha spent \$530 on 8 pairs of pants and 2 sweaters. What is the price of one pair of pants and a sweater?

9. Three times the age of Al plus the age of Ed is 70 years, but if you multiply 7 times Al´s age and add nine times the age of Ed the sum is 330. What are the ages of Al and Ed?

10. Last week Laurie and 7 friends went to the movie theater and spent \$42, which included the purchase of 2 bags of popcorn. This week Laurie went with 6 friends and spent \$48, which included the purchase of 4 bags of popcorn. What is the price of one movie ticket and one bag of popcorn?

11. The difference between two numbers is -3, while their sum is 1. Find the numbers.

12. The sum of two numbers is 76, and one sixth of their difference is -7. Find the numbers.

13. The sum of two numbers is 2090, while one tenth of their difference is 59. Find the numbers.

14. One-fifth of the sum of two numbers is 11, while one-third of their difference of the same number is -5. Find the numbers.

15. Two-thirds of the sum of two numbers is 6, while four-fifths of the difference of the same numbers is 44/5. Find the numbers.

16. The sum of two numbers is 75, while double of the first number minus the second is 21. Find the numbers.

17. The difference between the first number and three times the second number is 61, while four times the first number plus twice the second is 440. Find the numbers.

18. The sum of two numbers is 28. Also, 3/4 of the first number is 9/16 of the second one. Find the numbers.

19. A hundred marbles are divided into two bags. Twice the amount in the first bag is equal to half the amount of marbles in the other bag. How many marbles are in each bag?

20. The sum of two numbers is 30, while eight times the difference of the same numbers is -32. Find the numbers.

7.6 Solving a System of Linear Equations by Substitution

You have already learned to solve a system of linear equations by graphing and more recently, algebraically using the method of elimination by adding (or subtracting) the equations to obtain one equation in one unknown. The method of substitution is yet another approach to solving a system of linear equations. Consider these two equations in two unknowns:

$$3.5x - y = -310$$
$$13x - y = -250$$

Using the method of substitution, we solve the first equation for *y* and then substitute the expression we obtain for every occurrence of the variable *y* into the second equation. (Note: nothing of any significance is accomplished by solving the first equation for *y* and then trying to substitute the expression back into the ***first*** equation—**you must utilize the second equation**.) So, solving the first equation for *y*, we obtain $-y = -301 - 3.5x$ and divide by -1 to obtain,

$$y = \mathbf{3.5x + 310}$$

Next, every place that *y* occurs in the second equation, we substitute $3.5x + 310$, thus,

$$13x - \mathbf{y} = -250 \text{ becomes } 13x - \mathbf{(3.5x + 310)} = -250$$

So now we have just one equation with one unknown—and can solve for *x*. Next we use the distributive property to obtain $13x - 3.5x - 310 = -250$ and combine like terms

$$9.5x - 310 = -250$$

Then isolate the variable by adding 310 to each side of the equation to obtain,

$$9.5x = 60$$

Finally, we divide both sides of the equation by 9.5 to obtain

$$x = 6\frac{6}{19}$$

Now we substitute this value for x into any of the equations and solve for y. if we use the first equation we have these steps:

$3.5x - y = -310$; $3.5(6\frac{6}{19}) - y = -310$; $3.5\,(\frac{120}{19}) - y = -310$; $\frac{420}{19} - y = -310$ and finally,

$$y = 310 + \frac{420}{19} \text{ or } y = 310 + 22\frac{2}{19}$$

Simplifying, we have $y = 332\frac{2}{19}$. Thus, the solution expressed as an ordered pair is $(6\frac{6}{19}, 332\frac{2}{19})$.

Alternatively, we could have solved the first equation ($3.5x - y = -310$) for *x* to obtain $x = \frac{y - 310}{3.5}$ and then substituted this into the second equation, $13x - y = -250$, and solved for the value of *y*.

Exercises 7.6

Solve the following linear equations by substitution and express the solution as an ordered pair (x,y).

1. $4x - 4y = -4$
   $7x - y = 35$

2. $-4x + 5y = 25$
   $7x + 3y = 62$

3. $9x - 3y = -15$
   $-4x - 9y = -76$

4. $-3x - 5y = -24$
   $-8x + 6y = -6$

5. $4x - 8y = -6$
   $-2x + 6y = 20$

6. $-8x + 2y = 6$
   $x + y = 8$

7. $8x - 5y = 36$
   $-7x + 2y = -41$

8. Mark has \$255 in 37 bills of \$5 and \$10. How many bills of \$5 and \$10 does Mark have?

9. If Mary gives Charles \$7, both individuals will have the same amount of money. Mary and Charles have a combined sum of \$46. How much money does each of them have?

10. Seven years ago Alice´s grandma was 4 times older than Alice. The sum of the current age of Alice and her grandmother is 109. How old are both right now?

7.7 Solving and Graphing Inequalities

So far we have dealt with equations where the terms or expression are equal to other terms or expressions. However, sometimes based on the information available, all we know is that some terms or expressions are different—perhaps either greater or less than other terms or expressions. Mathematical statements can be strict inequalities such as a is greater than b (written as: $a > b$) or a is less than b (written as: $a < b$), or not strict inequalities, such as a is greater than or equal to b (written as: $a \geq b$) or a is less than or equal to b (written as: $a \leq b$). All such relationships are known as inequalities and we will consider this topic for the remainder of this chapter.

Solving Inequalities with One Variable

The process of solving an inequality is almost exactly the same as solving an equation. The only difference being that when you multiply or divide both sides of an inequality by a negative number, **you must reverse the inequality sign.**

Consider solving this strict inequality:

$$x + 5 < -7$$

Just pretend that the "less than" sign is an equal sign. What first step would you do? Of course, you would isolate the variable, x, by subtracting 5 from each side of the equation, thus we have

$$x + 5 - 5 < -7 - 5$$

Please note that we do not change the sign when subtracting a number from each side of the equation—only when multiplying and dividing both sides of the equation by a negative number.

This inequality simplifies to

$$x < -12$$

So, that is our solution. Wasn't that rather easy? Let's try another example,

$$\frac{x}{6} \geq 14$$

Again, pretend that the "greater than or equal to" sign is an equal sign. What first step would you do? Of course, since x is divided by 6, you would do the inverse and multiply by 6, thus

$$x \geq 14 \bullet 6$$

Simplifying the right side of the equation we have the solution,

$$x \geq 84$$

Let's do one more example,

$$-4y \leq 15$$

To solve for y, we must divide both sides of the equation by -4 to obtain,

$$y \leq -\frac{15}{4}$$

However, please note the above inequality is **INCORRECT**—in other words—**WRONG**! **Anytime that you divide or multiply both sides of an inequality by a negative number, you MUST remember to change the direction of the inequality, so that the correct inequality now is shown with a "greater than or equal to" sign:**

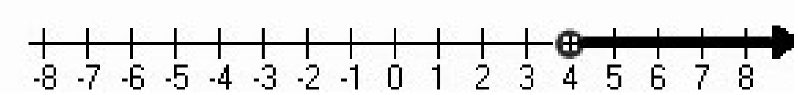
$$y \geq -\frac{15}{4}$$

Graphing an Inequality with One Variable

Imagine that we need more than $4, or the temperature must be kept at -5°C or cooler. If we let x represent the amount of money and y represent the temperature, then this information can be easily described by the following inequalities.

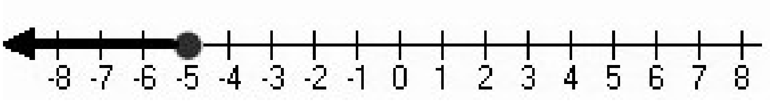
$$x > 4 \quad \text{and} \quad y \leq -5$$

An inequality with a single variable can be shown on a number line (or graph with just one axis). The open circle at 4 indicates the strict inequality: x is "greater than" 4, or $x > 4$.



Money ($)

If we were to place a dot at 4 (instead of a circle), we would have the inequality x is "greater than or equal to" 4, or $x \geq 4$ instead of $x > 4$. In the case of a temperature at -5° C or cooler, since -5 is included, we will use a solid dot at -5 on the number line and then draw a line pointing to the left—representing all temperatures cooler than -5°C.



Temperature (°C)

Solving Linear Inequalities Algebraically and Graphing the Solution Region

Now let's consider a linear equality that had two variables, x and y, such as

$$3x - 4y < 4$$

For the moment, let's transform this inequality into a slope and y-intercept form (remember this is $y = mx + b$) by solving for y. To isolate the y term, we must first subtract $3x$ from each side of the inequality. This gives us

$$-4y < 4 - 3x$$

Notice how we have preserved the "less than" (<) sign at this step. Next, we divide both sides of the equation by -4 and we now have the linear equation in slope and y-intercept format (after rearranging the terms on the right side of the equation):

$$y < \frac{3}{4}x - 1$$

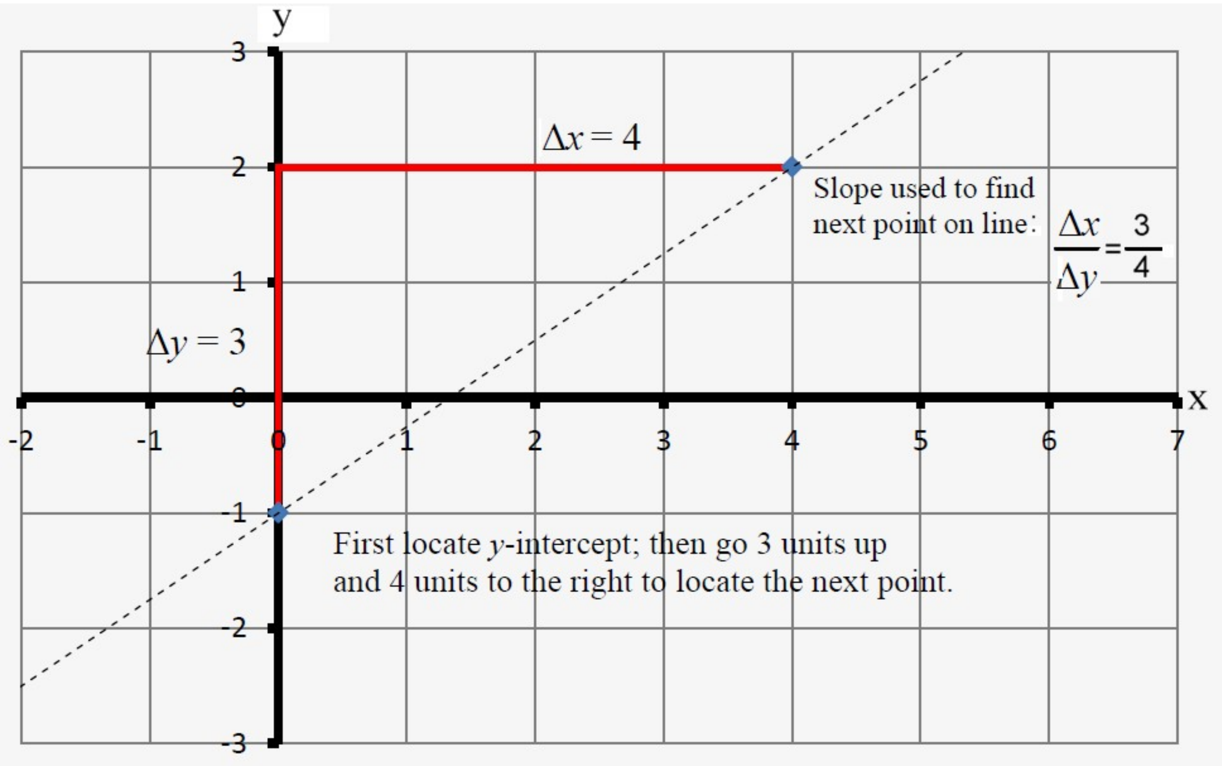
However, please note this inequality is **INCORRECT**—in other words—**WRONG**! **Anytime that you divide or multiply both sides of an inequality by a negative number, you MUST remember to change the direction of the inequality, so that the correct inequality now is shown with a "greater than" sign:**

$$y > \frac{3}{4}x - 1$$

Thus, so far, we have determined a boundary line with

slope of $\frac{3}{4}$ ($m = \frac{3}{4}$) and a y-intercept of -1 (b = -1)

as shown in the graph below:



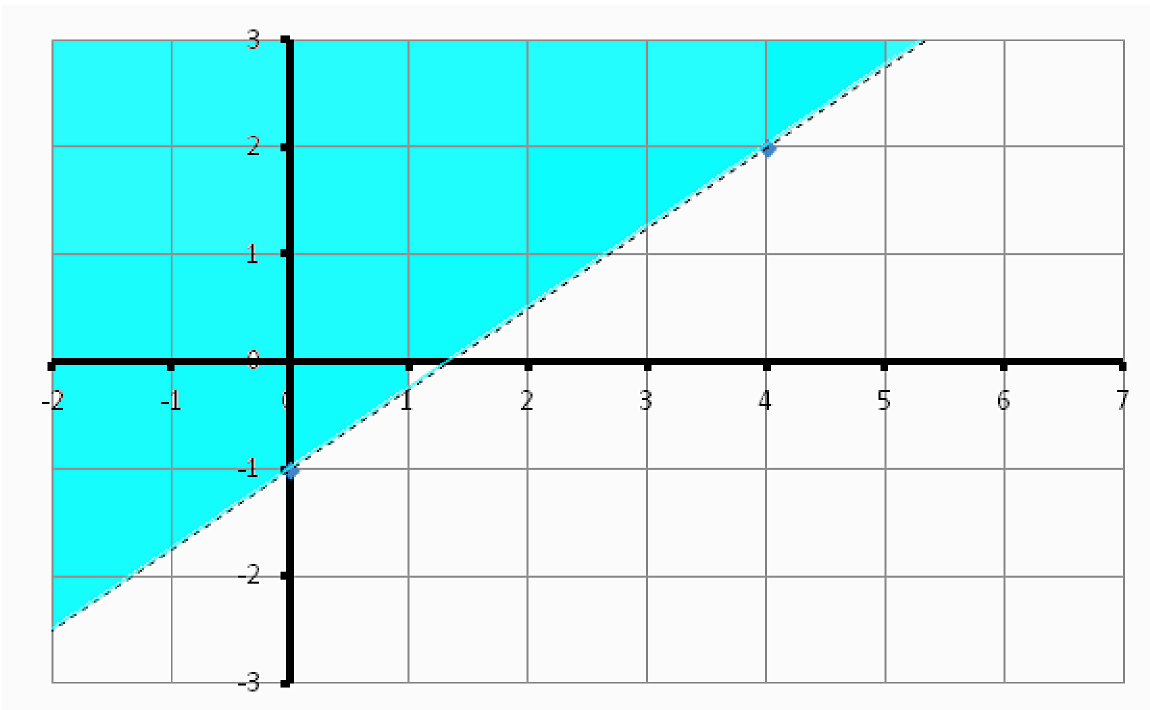
The line is shown as dashed since the linear inequality is strictly "greater than", therefore any points that lie directly on the line are not included in the solution set (or solution region). Now we are ready to find the region containing all the points that satisfy the inequality. There are several ways this can be done. Since the inequality is "greater than", it is easiest to note that the solution includes all points above the dashed line. (If the inequality was "less than", than the solution would include all points below the dashed line.) To double-check our solution, let's choose any point in the region above the dashed line, say (1, 2) which is clearly above the dashed line. [Note: we could have selected (1,0) which is very close, but still above the line or (-1,-1), or (4,3), or even (0,0), etc.] Let's substitute x = 1 and y = 2 into our inequality and see if the statement is true:

$$y > \frac{3}{4}x - 1$$

$$2 > \frac{3}{4}(1) - 1$$

$$2 > -\frac{1}{4}$$

Since this is a true statement, we have verified that the region we have selected to shade is correct. All points in the shaded region shown below satisfy the original inequality $3x - 4y < 4$ which is equivalent to our slope and y-intercept form: $y > \frac{3}{4}x - 1$:



Anytime that we desire to check to see if a point is a valid solution to an inequality, we simply substitute the x and y-values of the point into the inequality and see if the resulting inequality is true.

For example to check whether (5, 3) is a solution of $3x - 4y < 4$, simply substitute into the inequality $x = 5$ and $y = 3$, thus we have

$$3(5) - 4(3) < 4 \text{ or } 15 - 12 < 4, \text{ which simplifies to } 3 < 4$$

The ordered pair (5, 3) satisfies the inequality.

Let's do one more example of graphing the solution of a linear inequality. Suppose we have the linear inequality,

$$2y - 8x < 10$$

As the first step, let's change this problem to be in slope and y-intercept form to make the line easier to graph. To isolate the y-term, we must first add $8x$ to each side of the equation. This gives,

$$2y < 8x + 10$$

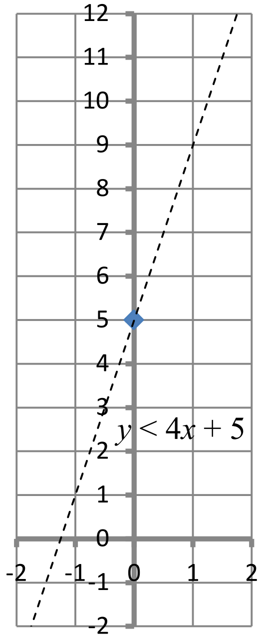
Next, we divide both sides of the inequality by 2 to obtain

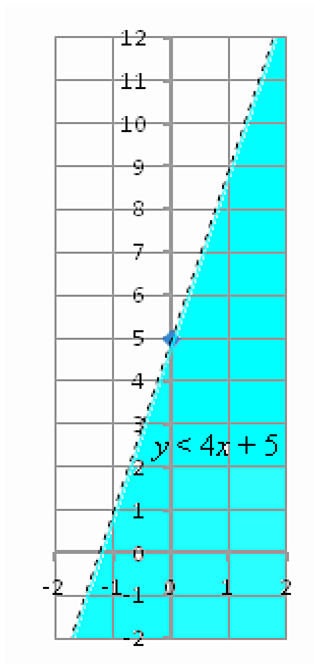
$$y < 4x + 5.$$

Notice that since we did not multiply the inequality by a negative number, the sign remains unchanged. We are now ready to graph this line since we observe the slope is 4 and the y-intercept is 5. Remember that the slope of 4 is the same as $\frac{4}{1}$ when graphing the line.

First Step—graph the line by drawing a dashed line for a strict inequality (> or <); solid line for not a strict inequality (such as ≥ or ≤).

Second Step—Shade the proper region (everything below the line) to reflect the solutions to the inequality.





Next, we must shade (or color) the region (or area) that is below the line to show that the solution of the inequality includes that entire region: To check to see if we have shaded the proper region, we can try substituting any point within the shaded region into the inequality and see if the result is true. Using the point (0, 0), let's substitute $x = 0$ and $y = 0$ into our inequality $y < 4x + 5$, and we obtain

$$0 < 4(0) + 5$$

Or 0 < 5 (0 is less than 5) which is a true statement. Therefore, we have shaded the proper region. Please notice that we did not draw a solid line since those points that lie directly on the line are not included in a strict inequality where the operator is a greater than (>) or less than (<) sign.

Exercise 7.7

Solve the following inequalities:

1. $4x + 3 < 2$
2. $5y - 12 > 7$
3. $-x/4 + 5 \geq 13$
4. $-4y - 6 \leq 15$
5. $x - 9 \geq 6$
6. $-8y - 7 > 11$
7. $3x + 1 \leq -15$
8. $2y - 3 > 6$
9. $-9x - 4 \geq 17$
10. $5y + 4 > -6$
11. $8x - 9 \geq 7$
12. $-9y + 2 < 9$
13. $-5x - 6 > 3$
14. $-5y + 3 \geq 4$
15. $9x - 9 \leq 7$

Identify the following points as
A. a solution of the inequality
B. not a solution of the inequality.

16. (5,7) $6x + 7y > 8$
17. (9,8) $-8x + 2y < 5$
18. (5,9) $5x - 6y < 9$
19. (8,5) $9x + 6y > 7$
20. (-8,2) $2x - y > 6$
21. (7, -9) $-x + 9y < 7$
22. (5, 5) $x - y > 9$
23. (6, -8) $8x + 7y < 8$
24. (7, -4) $4x - 3y < 11$
25. (-1, 0) $7x - 3y > -10$

Graph the following inequalities (one variable).

26. $x > 4$
27. $x \leq -7$
28. $x < -7$
29. $x > -3$
30. $x \geq -8$

Graph the following inequalities (two variables).

31. $y > 8x + 10$
32. $y \geq 4x - 1$
33. $y \leq -6x + 4$
34. $y < -3x - 7$
35. $y \leq 5x + 2$

7.8 Cramer's Rule (Optional)

As we have seen in this chapter, a system of linear equations in two unknowns can be solved by various methods that include (1) graphing, (2) elimination, and (3) substitution. You might ask me (as the author of this text), "Gary, which method do you prefer to use?" My preference is to use none of the methods we have discussed—in favor of a very easy approach that is presented in higher math courses. This method is called Cramer's Rule and it involves the use of matrices (plural form of the word "matrix"). I will share with you my "secret" method, but you must promise not to utilize this method on any of your exercises or tests—since you are not suppose to know or understand this particular approach until after you have taken an upper-level math course.

Let's consider the system of linear equations that we solved earlier when we were graphing the solutions,

$$y = -2x + 3$$
$$y = x - 6$$

We are going to rewrite these equations in standard form, rather than work with the equations in slope and y-intercept format. The equivalent system of equations in standard form is

$$2x + y = 3$$
$$-x + y = -6$$

Rewriting these equations showing the implied coefficients, we have

$$\mathbf{2}x + \mathbf{1}y = 3$$
$$\mathbf{-1}x + \mathbf{1}y = -6$$

Next, we can form a matrix of the coefficients of x and y as shown,

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

Next, we compute what is called the determinant of the coefficient matrix by multiplying the number in the top left of the matrix times the number in the bottom right and then subtracting the product of the number in the bottom left of the matrix and the number in the upper right. Thus, we have the determinant computed as follows:

$$D = \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = (2)(1) - (-1)(1) = 2 + 1 = 3$$

We will also need to compute D_x and D_y. So based on the general system of equations:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

The determinants are computed as follows:

$$\mathrm{D} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1 \quad D_\mathrm{x} = \begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix} = c_1b_2 - c_2b \quad D_\mathrm{y} = \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = a_1c_2 - a_2\, c_1$$

Then, the values of x and y that satisfy the equations are simply given by

$$x = D_\mathrm{x}/\mathrm{D}$$
$$y = D_\mathrm{y}/\mathrm{D}$$

In summary, the x and y solutions can be computed for any system of two equations in two unknowns using these equivalent formulas (derived using Cramer's Rule):

$$x = (c_1b_2 - c_2b_1)/(a_1b_2 - a_2b_1) \quad \text{or} \quad x = (c_1b_2 - c_2b_1)/\mathrm{D}$$
$$y = (c_2a_1 - c_1a_2)/(a_1b_2 - a_2b_1) \quad \text{or} \quad y = (c_2a_1 - c_1a_2)/\mathrm{D}$$

The solution to our system of linear equations is

$$x = [(3)(1) - (-6)(1)]/3 = (3 + 6)/3 = 9/3 = 3$$
$$y = [(-6)(2) - (3)(-1)]/3 = (-12 + 3)/3 = -9/3 = -3$$

Thus, the solution is (3,-3).

It should be noted that if the determinant of the coefficient matrix is 0, then either the lines are parallel or they are identical. What is interesting about Cramer's rule is that it can be used for systems of equations involving more than two unknowns.

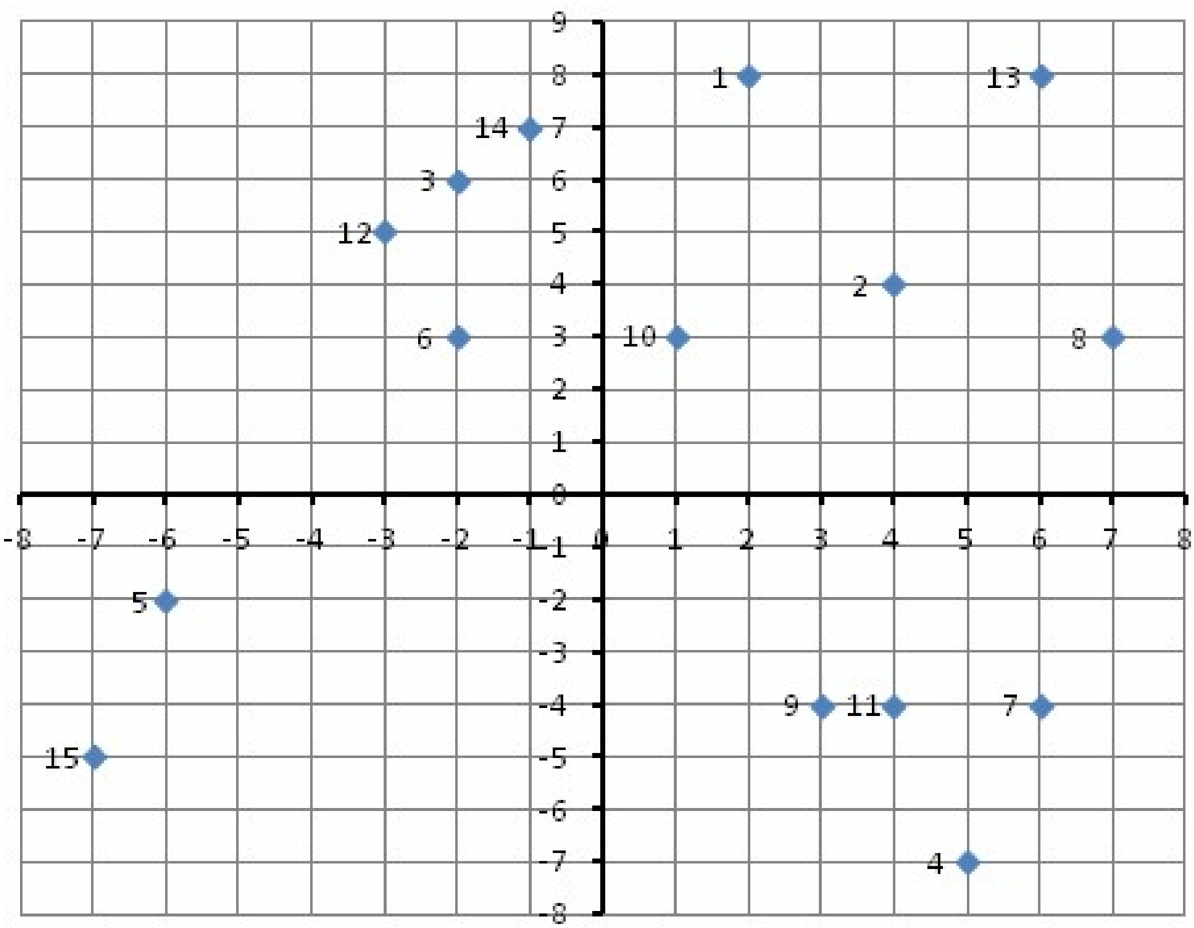
Cramer's Rule Summary

Description	**Formula**
Given the system of equations in two unknowns $a_1x + b_1y = c_1$ $a_2x + b_2y = c_2$	
The coefficient matrix determinant is	$\mathrm{D} = a_1b_2 - a_2b_1$
The solution for x is	$x = (c_1b_2 - c_2b_1)/\mathrm{D}$
The solution for y is	$y = (c_2a_1 - c_1a_2)/\mathrm{D}$

Answers to Chapter 7 Exercises

Exercise 7.1

Problems 1 through 15 shown on graph:

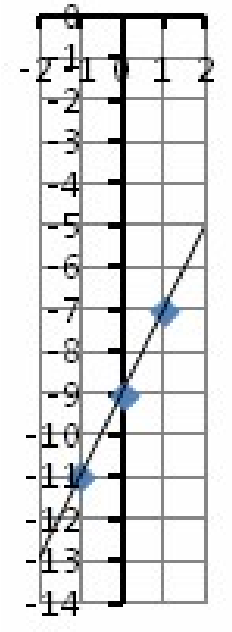


Exercise 7.2

1. (-1, -11), (0, -9), (1, -7); **2.** (-1, 7), (0, 3), (1, -1); **3.** (-1, -7), (0, -4), (1, -1);
4. (-1, 11), (0, 10), (1, 9); **5.** (-1, -32), (0, -20), (1, -8); **6.** (-1, -2), (0, -7), (1, -12);
7. (-1, 12), (0, 9), (1, 6); **8.** (-1, -15), (0, -11), (1, -7); **9.** (-1, -11), (0, -13), (1, -15);
10. (-1, -8), (0, -3), (1, 2); **11.** (-1, -3), (0, 4), (1, 11); **12.** (-1, 25), (0, 27), (1, 29);
13. (-1, 18), (0, 15), (1, 12); **14.** (-1, -11), (0, -7), (1, -3); **15.** (-1, 22), (0, 19), (1, 16)

Linear graphs for problems #1 through #15 are shown below.

1.



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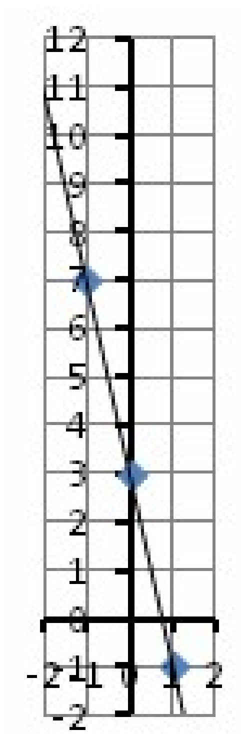

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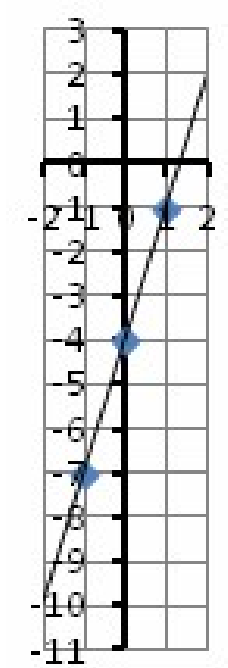

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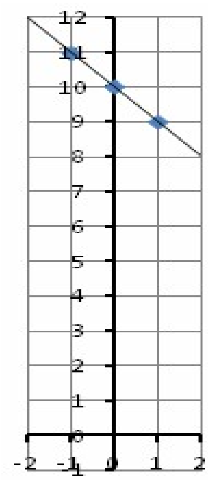

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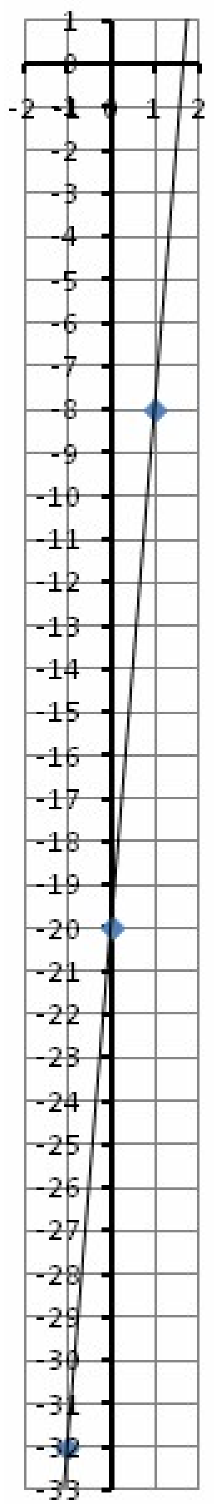

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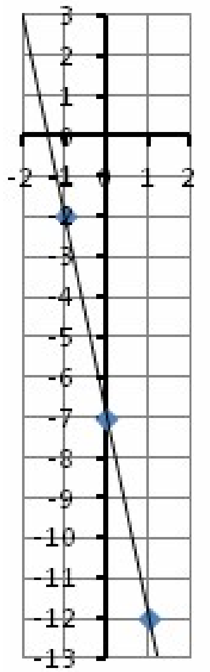

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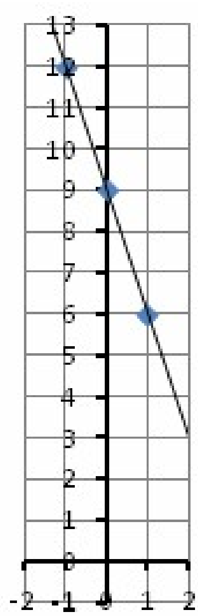

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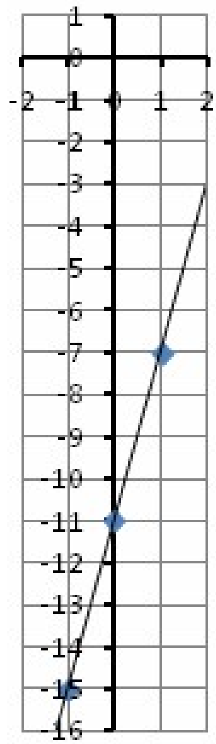

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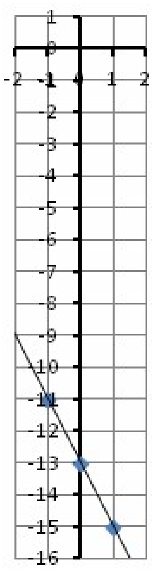

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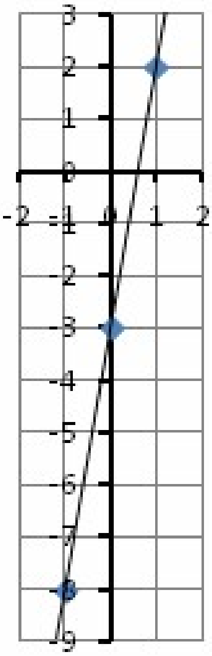

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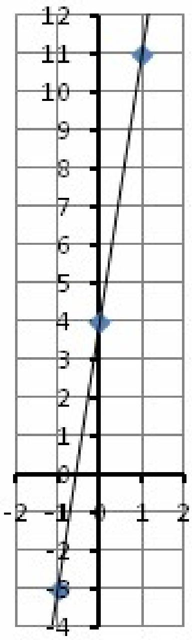

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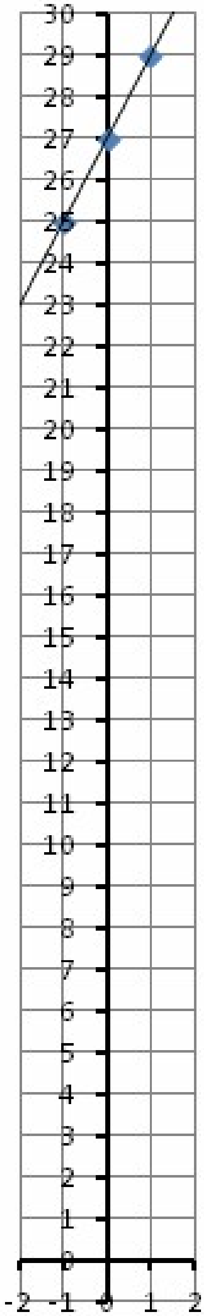
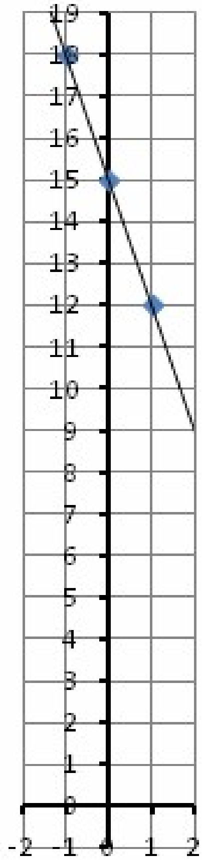


13.



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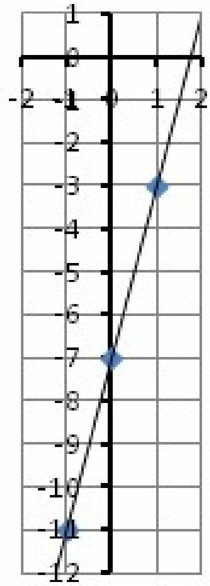

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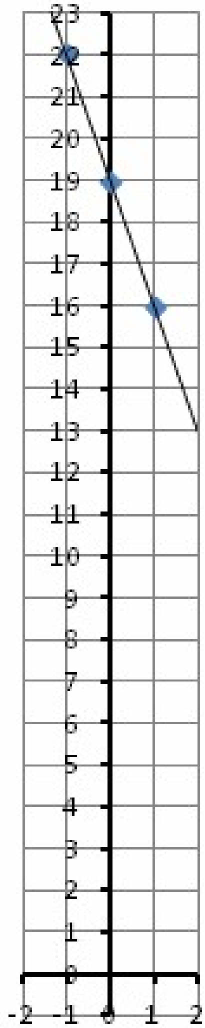
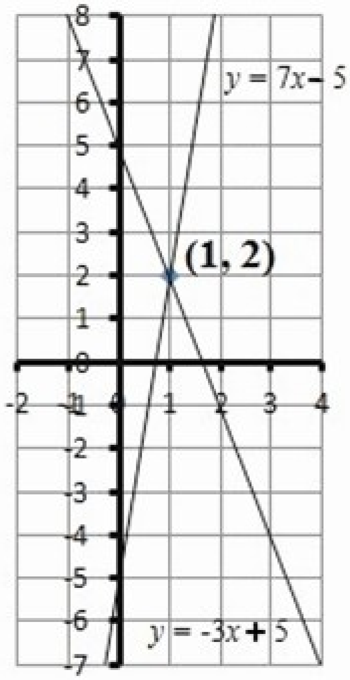
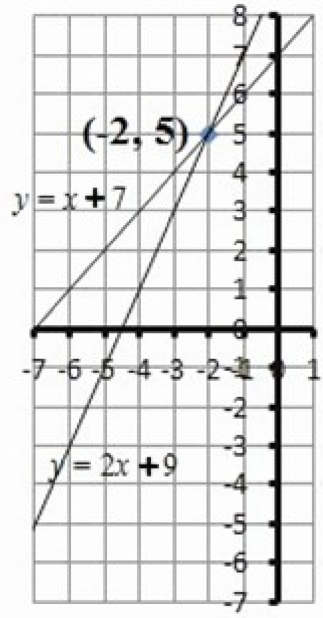


Exercise 7.3

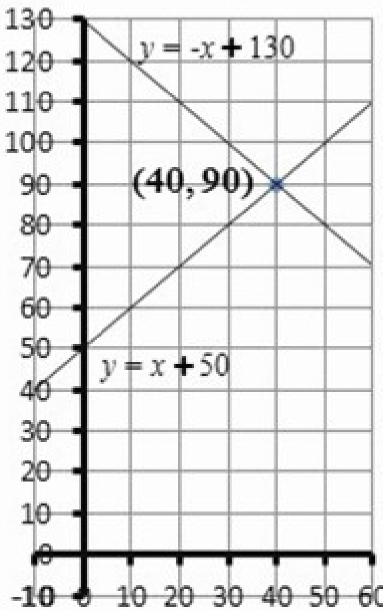
1. (1,2)



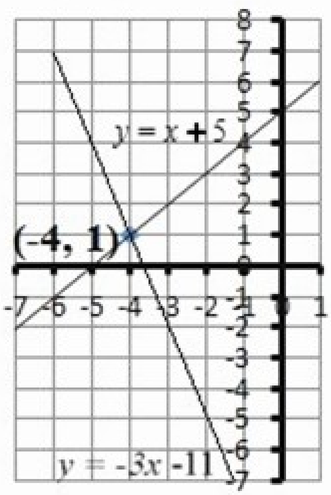
3. (-2,5)



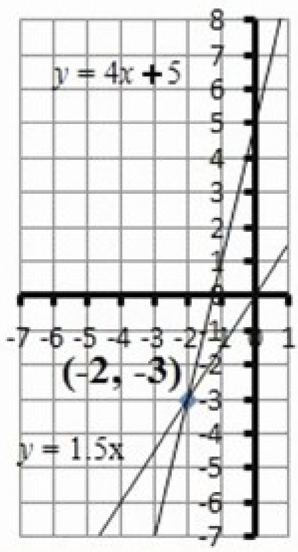
5. 40 m x 90 m



2. (-4,1)



4. (-2,-3)



Exercise 7.4

1. $y = 7x - 5$; **2.** $y = -8x + 32$; **3.** $y = -4x + 15$; **4.** $y = x - 19$; **5.** $y = -13x + 52$; **6.** $y = 71x - 12$;

7. $y = 6x + 8$; **8.** $y = -5x - 41$; **9.** $y = -2x + 4$; **10**. $y = 3x - 5$; **11.** $y = \frac{52}{19}x + 35\frac{2}{19}$;

12. $y = -\frac{37}{16}x + 27\frac{1}{8}$; **13.** $y = \frac{18}{5}x + 6\frac{3}{5}$; **14.** $y = -\frac{11}{14}x + 6\frac{5}{7}$; **15.** $y = \frac{83}{79}x + 68\frac{43}{79}$; **16.** $y = 6x + 44$;

17. $y = \frac{54}{17}x + 51\,\frac{12}{17}$; **18.** $y = \frac{5}{16}x + 16\frac{1}{2}$; **19.** $y = -\frac{1}{7}x + 10$; **20.** $y = \frac{17}{10}x - 27\frac{2}{5}$; **21.** $y = -\frac{75}{47}x - 7\frac{36}{47}$

22. $y = -\frac{6}{23}x + 15\frac{2}{23}$; **23.** $y = x + 7$; **24.** $y = \frac{51}{2}x - 270\frac{1}{2}$; **25.** $y = -8x + 128$; **26.** $y = -\frac{2}{5}x + 6\frac{2}{5}$;

27. $y = -\frac{11}{13}x - 12\frac{12}{13}$; **28.** $y = \frac{5}{2}x - 19\frac{1}{2}$; **29.** $y = 2x + 19$; **30.** $y = 10x - 805$;

Exercise 7.5

1. (15,30); **2.** (60,13); **3.** (-14, 28); **4.** (6, -16); **5.** (7, -9); **6.** (32, 7); **7.** (16, 11);
8. pair of pants—\$50 ea., sweater—\$65 ea.; 9. Al—15 years old, Ed—25 years old;
10. movie ticket—\$4 ea., bag of popcorn—\$5 ea.; **11.** -1, 2; **12.** the two equations are:
$x + y = 76$ and $\frac{1}{6}(x - y) = -7$; then we multiply each side of the 2nd equation by 6 to obtain
6$\cdot\left[\frac{1}{6}(x - y)\right]$ = **6**$\cdot$(-7), which simplifies to $x - y = -42$, to obtain $x = 17$, $y = 59$; **13.** 1340, 750;
14. 20, 35; **15.** 10, -1; **16.** 32, 43; **17.** 103, 14; **18.** 12, 16; **19.** 20, 80; **20.** 13, 17.

Exercise 7.6

1. (6, 7); **2.** (5, 9); **3.** (1, 8); **4.** (3, 3); **5.** (15½, 8½); **6.** (1, 7); **7.** (7, 4);
8. Let x = no. of \$5 bills, y = no. of \$10 bills, then $x + y = 37$ and $5x + 10y = 255$, \$5 bills—23, \$10 bills—14; **9.** Let x = amount Mary has, y = amount Charles has, then $x - 7 = y + 7$, $x + y = 46$, Mary—\$30, Charles—\$16; **10.** Let x = Alice's age now, y = grandma's age now, then $y - 7 = 4(x - 7)$ and $x + y = 109$, Alice—26 years old, Grandma—83 years old.

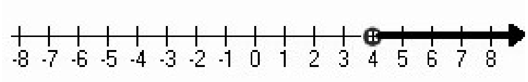
Exercise 7.7

1. $x < -1/4$; **2.** $y > 19/5$ or $3\frac{4}{5}$; **3.** $x \leq -32$; **4.** $y \geq -21/4$ or $-5\frac{1}{4}$; **5.** $y \geq 15$; **6.** $y < -9/4$ or $-2\frac{1}{4}$;

7. $x \leq -16/3$ or $-5\frac{1}{3}$; **8.** $y > 9/2$ or $\frac{1}{2}$; **9.** $x \leq -7/3$ or $-2\frac{1}{3}$; **10.** $y > -2$; **11.** $x \geq 2$; **12.** $y > -7/9$;

13. $x < -9/5$ or $-1\frac{4}{5}$; **14.** $y \leq -1/5$; **15.** $x \leq 16/9$ or $1\frac{7}{9}$; **16.** A; **17.** A; **18. A; 19.** A; **20.** B; **21.** A;

22. B; **23.** B; **24.** B; **25.** A;

26.


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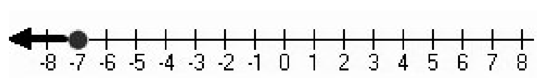

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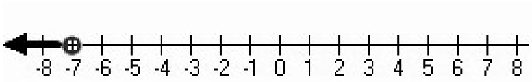

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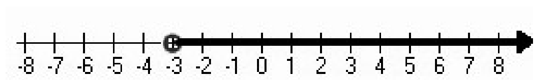

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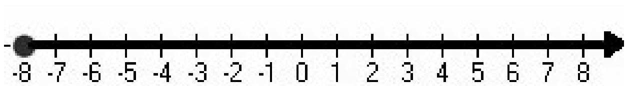
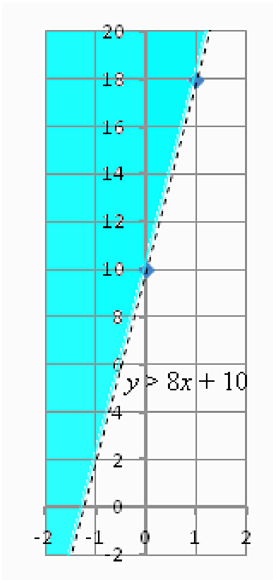
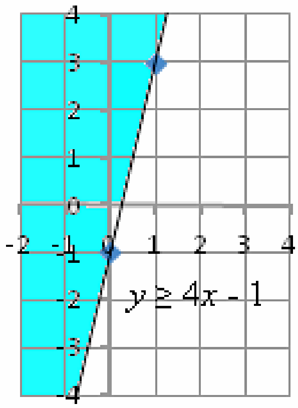
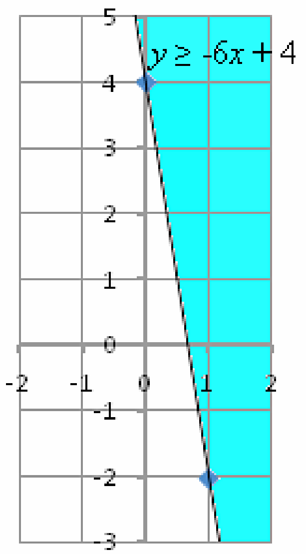
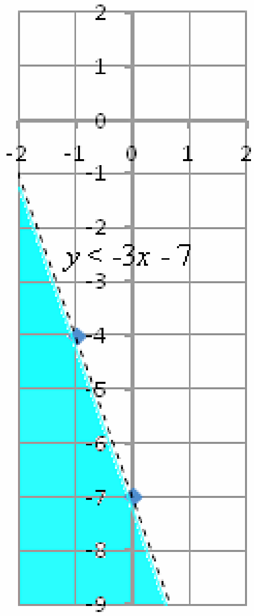
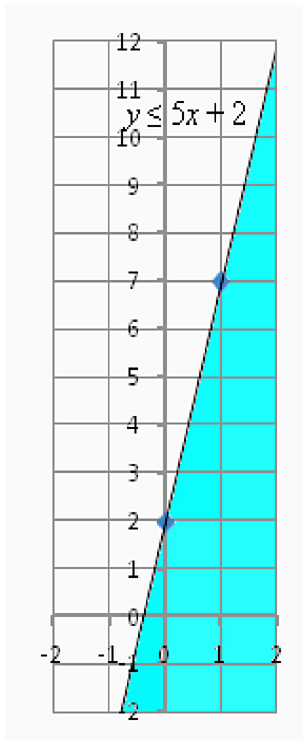


31. 

32. 

33. 

34. 

35. 

Chapter 8: Geometry

The word “geometry" comes from the Greek *geos* which means “earth” and *metron* which means “measurement”. In this chapter we will learn the differences between polygons and polyhedrons and how to calculate various lengths (perimeters), areas, and volumes (in the case of the polyhedrons) of various geometric figures.

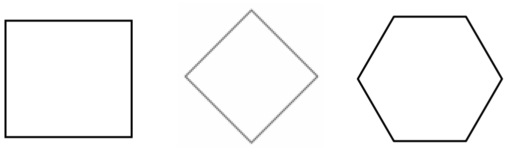
8.1 Polygons and Polyhedrons

Polygons: An Introduction

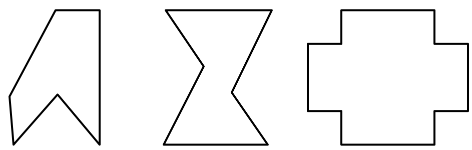
A polygon is a closed plane figure made up of segments (also called sides). Polygons are named according to the number of segments from which they are comprised. The term *n*-gon is used to denote a polygon with *n* sides. The *n* is substituted, according to the number of sides that comprise the figure, by the Greek cardinal prefixes: **penta** (or 5), **hexa** (or 6), **hepta** (or 7), etc. The only two exceptions are the three- and four-sided polygons which are the triangles (never “trigons”) and quadrilaterals (never “tetragons”). The number of sides is equal to the number of angles in all polygons.

Polygons can be classified in two different ways: internal angles and lengths of sides. Depending on the measure of the internal angles, polygons can be **convex,** with all internal angles less than 180° (see page 211), or **concave** (with at least one or more internal angles greater than 180°). You can distinguish a convex polygon from a concave polygon by noting that convex polygons have all their angles pointing out, while concave polygons have at least one angle that points inward. It is impossible to construct a 3-sided (triangular) concave polygon.

Examples of Convex Polygons

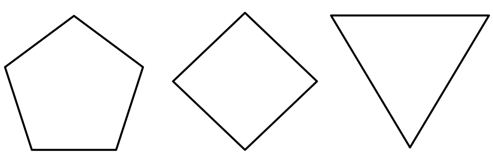


Examples of Concave Polygons

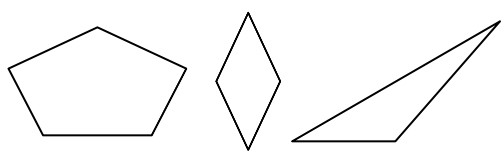


Convex polygons can also be classified as **regular** (when they are equilateral**—**that is, all sides have the same length; and equiangular**—**that is, all angles have the same measure) and **irregular** (when at least one angle or side differs from the rest).

Examples of Regular Polygons



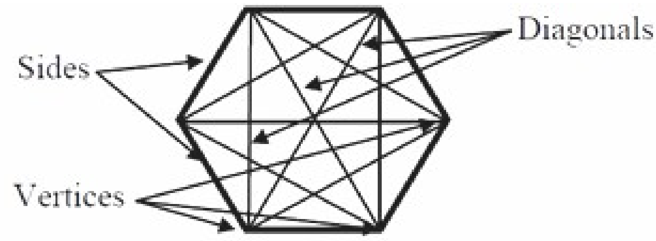
Examples of Irregular Polygons



The parts that make up a polygon are the following:

- **Sides**: the segments or straight lines that form the polygon.
- **Vertices**: the point where the sides meet. The sides always form an angle at each vertex.
- **Diagonals**: are the lines that join two non-consecutive angles.

Consider the following convex hexagon:



A "hexagon" is a 6-sided polygon having 6 vertices and 9 different diagonals. In general, the number of diagonals can be computed given the number of vertices, n, using this formula:

$$\text{no. of diagonals} = n(n-3)/2$$

Diagonals that connect the vertices of a concave polygon, will sometimes lie partially or completely outside the boundaries of the object.

The polygons are named according to the number of sides they contain as given in the table below. In actual practice, instead of, for example, identifying a 40-sided polygon as a tetracontagon, most mathematicians will simply call it a 40-gon.

Name of Polygon	**Number of sides**	**Name of Polygon**	**Number of sides**
triangle	3	enneadecagon	19
quadrilateral	4	icosagon	20
pentagon	5	triacontagon	30
hexagon	6	tetracontagon	40
heptagon	7	pentacontagon	50
octagon	8	hexacontagon	60
nonagon	9	heptacontagon	70
decagon	10	octacontagon	80
hendecagon	11	enneacontagon	90
dodecagon	12	hectagon	100
triskaidecagon	13	chiliagon	1,000
tetrakaidecagon	14	myriagon	10,000
pentadecagon	15	Megagon	1,000,000
hexakaidecagon	16	googolgon	10^{100}
heptadecagon	17		
octakaidecagon	18	n-gon	n

Optional Formulas: The **sum of the interior angles** of a general n-sided polygon is given by the following:

$$\textit{Sum of interior angles} = (n-2)180°$$

Given an **interior angle**, a, the **exterior angle** is 180 - a and the **number of sides** of a regular n-gon having that interior angle is given by $n = \frac{360}{180-a}$. Knowing n, we can find $a = 180 - \frac{360}{n}$. The sum of the exterior angles of a polygon is always 360°.

Polyhedrons: An Introduction

All polygons by definition are two-dimensional (i.e., flat) objects. Let´s consider now three-dimensional figures (i.e., solid figures that have height, width, and depth).

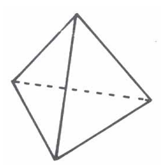
A **polyhedron** is a geometric solid in three dimensions with flat faces and straight edges. A polyhedron is a three-dimensional figure assembled with polygonal-shaped surfaces. In a manner similar to that of polygons. Polyhedrons are comprised of

- **Faces**: Plane polygon-shaped surfaces.
- **Edges**: Intersections between faces.
- **Vertices**: The vertices of the faces are the vertices of the polyhedron also. Three or more faces can meet at a vertex.

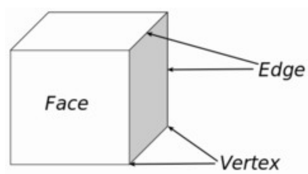
Interestingly, the number of vertices (V), edges (E), and faces (F) of a polyhedron are related according to this formula:

$$V - E + F = 2$$

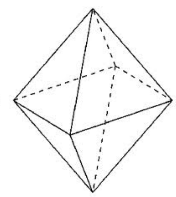
Here are some examples of polyhedrons (or polyhedra):



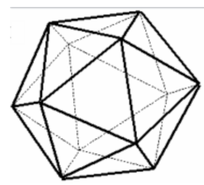
Tetrahedron



Cube
(or hexahedron)



Octahedron



Icosahedron

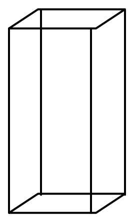
Just as polygons are named based on the number of sides that comprise them, polyhedrons are named after the number of faces, that comprise them. The *n*-hedron names are generated by substituting the *n* with the Greek cardinal prefixes (tri—or 3, tetra—or 4, octa—or 8, dodeca—or 12, icosa—or 20) that match the number of faces that comprise the figure.

Name of polyhedron	Number of faces	Name of polyhedron	Number of faces
trihedron	3	dodecahedron	12
tetrahedron	4	tetradecahedron	14
pentahedron	5	icosahedron	20
hexahedron	6	icositetrahedron	24
heptahedron	7	triacontrahedron	30
octahedron	8	icosidodecahedron	32
nonahedron	9	hexecontahedron	60
decahedron	10	enneacontahedron	90
undecahedron	11	*n*-hedron	*n*

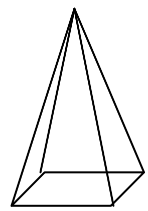
Prisms and pyramids

There are other types of non-polyhedral solids that we are going to consider in this chapter, known as prisms and pyramids. The main difference between these and polyhedrons consists in the different shaped-faces that constitute them.

Prisms are three-dimensional figures with two similar (or congruent—same shape and size, but in different positions) polygon bases that rest in parallel planes. **Pyramids**, on the other hand, have only one polygon base in a plane with the remainder of the faces connected to a point outside this plane.



Regular square prism

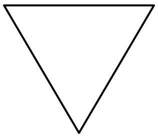


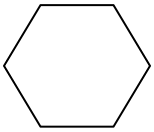
Regular square pyramid
(named as such because base is a square)

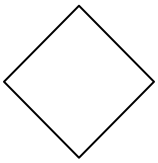
Regular prisms and pyramids are named after the shape of the base they have. In the figures above, for example, since the base is a square, a regular square prism and a regular square pyramid are shown.

Exercise 8.1

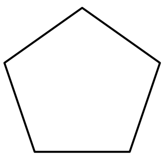
Name the following polygons, and tell whether they are
A. convex or
B. concave
Also give the number of diagonals for each figure.

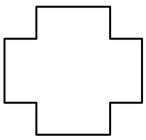
1. 

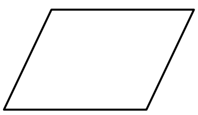
2. 

3. 

4. 

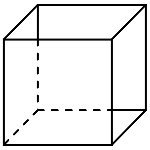
5. 

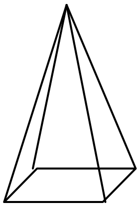
6. 

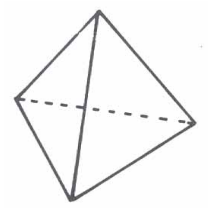
7. 

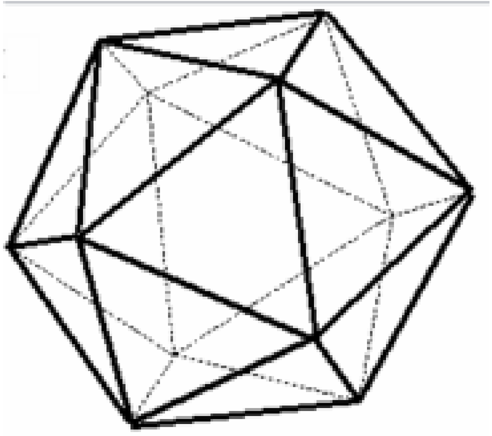
8. 

Name the following 3-dimensional figures. If the figure is a polyhedron, also provide the name of the face polygon and give the number of faces, edges, vertices, and number of faces that meet at each vertex.

9. 

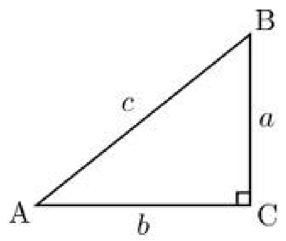
10. 

11. 

12. 

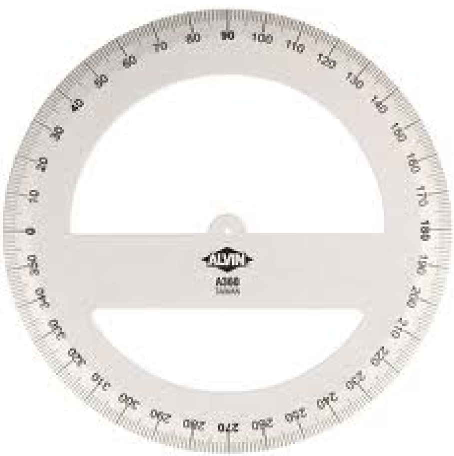
8.2 Triangles and Quadrilaterals

Triangles: An Introduction



Triangles are plane (i.e., flat, two-dimensional) figures with three straight sides (labeled a, b, and c in the figure shown on the left) and three angles. Triangles are formed by three straight lines that join three vertices (labeled A, B, and C). The symbol for a triangle is Δ. So the triangle shown is referred to as ΔABC. The small square at vertex C is used to denote an angle of 90^o, or a right angle. Triangles have many applications in physics, architecture, and many other disciplines. They are so important within geometry, that they have a specialized field of study termed "trigonometry", which will be considered in Chapter 9. The longest side, *c*, of a right triangle is called the hypotenuse.

Classification of a Triangle: By Angle

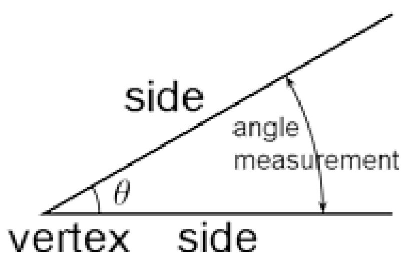


An angle (shown below) is a figure formed by non-parallel segments that intersect each other at a given point. The word angle comes from the Latin *angulus* which means corner. As you may already know, the word triangle angle describes a 3-sided polygon comprised of 3 angles. An angle is the arc formed by two straight segments of a line, called the sides of the angle. The sides of the angle meet at the vertex or intersection point of both lines.

A protractor (shown on the right) is a tool or instrument that is used to measure an angle in units of degrees, minutes, and seconds. Here is one such protractor that can measure angles from 0 to 360 degrees (usually written at 360°) or a full circle.

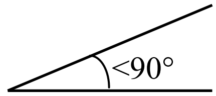
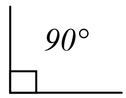
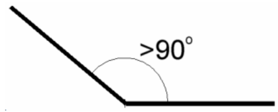
Degrees can be further divided into units of minutes of arc and seconds of arc. There are 60 *minutes of arc* in 1 degree, and 60 *seconds of arc* in one minute. Thus, 5-degrees 30-minutes 18 seconds (usually written as 5° 30' 18") is equivalent to 5.505°—since 18"=18/60=0.3' and 0.3'=0.3/60=0.005°, and 30'=30/60=0.5°). The Greek letter theta (Θ) is commonly used to denote the measure of an angle, such as

$$\theta = 30°$$

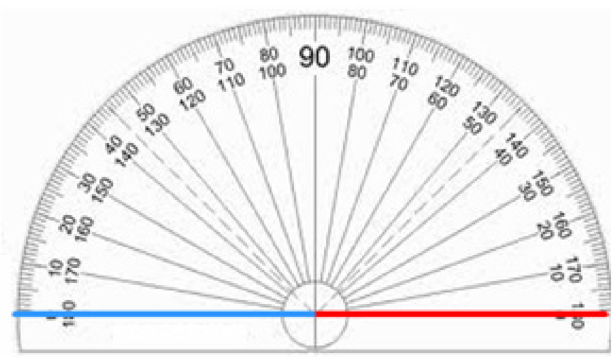


The Component Parts of an Angle

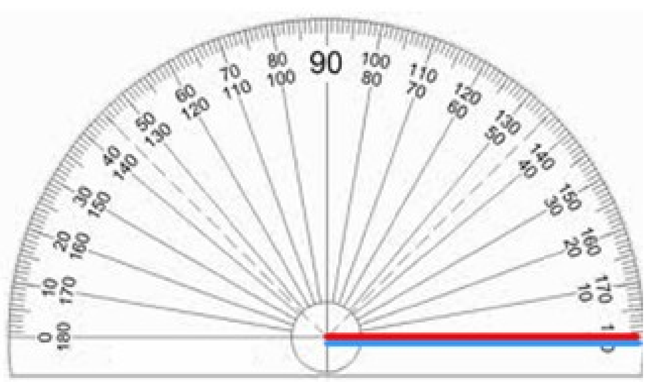
Angles can be classified according to the measure of their angle. There are three specific types of angles: acute, right, and obtuse as shown in the table below.

Type of Angle	Description	Example
acute	Angle of less than 90°	
right	90° angle is formed when two lines are perpendicular to each other.	
obtuse	Angle of greater than 90°	

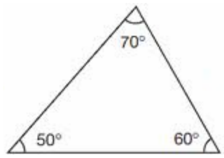
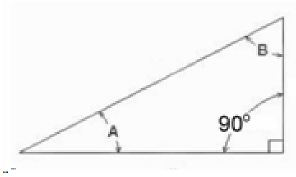
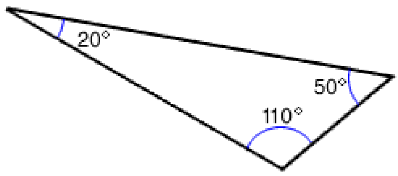
There are two special cases of angles shown below over which a protractor has been placed. Notice in this 1st case that the red line, forming one side of the angle, is parallel (or coincident lines—on top of one another) to the blue line, forming the other side of the angle. These sides form an obtuse angle of 180° on the protractor.



In the 2nd case, notice that the red line, forming on side of the angle, is again parallel and directly on the blue line, forming the other side of the angle. The angle formed here on the protractor is an acute angle of 0°. Alternatively, the outside angle (measured from the red line, counter-clockwise around to the blue line) could be considered an obtuse angle of 360°.

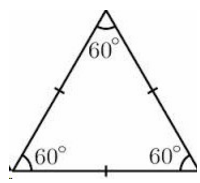
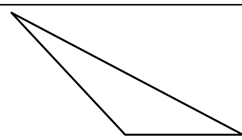


Triangles can be classified by their interior angles as right (with at least one 90° angle) or oblique (without any 90° angle).

Triangle	Description	Example
acute (oblique)	Contains interior angles that are all less than 90°.	
right	Contains a 90° angle (or right angle).	
obtuse (oblique)	Contains one obtuse angle, i.e. one angle greater than 90°.	

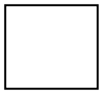
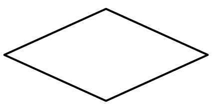
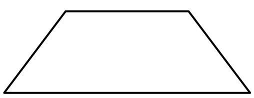
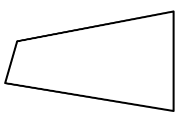
Classification of a Triangle: By Congruent Sides

Congruent sides are simply sides that have the same length or measure.
In addition to classifying triangles like we did previously by considering their angles, we can also classify triangles based on the number of congruent sides of which they are comprised. The table below shows three different classifications of a triangle based on the congruency (comparison of the lengths) of its sides.

Triangle	Description	Example
equilateral	All three sides have equal length and the three interior angles are acute and 60°. An equilateral triangle has three congruent sides. Notice the “tick” marks placed on each side. These denote that the all the sides have the same length.	
isosceles	Two of the three sides of the triangle have equal length and the angles facing those sides are equal. An isosceles triangle has two congruent sides. Notice only two of the sides have tick marks.	
scalene	The three sides have different lengths and the angles have different measurements. A scalene triangle has no congruent sides.	

Quadrilaterals: An Introduction

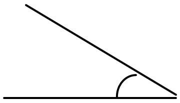
As we saw previously, quadrilaterals are polygons with four sides (or edges) and four vertices (or corners) that can have different shapes. The interior angles of a quadrilateral add up to 360-degrees. In fact, the word "quadrilateral" comes from two Latin words—*quadri* meaning four and *latus* meaning sides. Let's identify six different types of quadrilaterals.

Quadrilateral	Shape	Description
square (regular quadrilateral)		A regular quadrilateral having all four sides of equal length (equilateral), and all four angles are right angles (90°)-- opposite sides are parallel (a square is a parallelogram), the diagonals perpendicularly bisect each other (a square is a rhombus), and are of equal length (a square is a rectangle).
rectangle (irregular quadrilateral)		Opposite sides are parallel and of equal length. Also, all four angles are right angles. An equivalent condition is that the diagonals bisect each other and are equal in length. When sides are of equal length, this special condition is called a square.
parallelogram (irregular quadrilateral)		A quadrilateral with two pairs of parallel sides—opposite sides are of equal length; that opposite angles are equal; or that the diagonals bisect each other. Parallelograms also include the square, rectangle, rhombus and rhomboid (similar to rhombus, but with different length sides).
rhombus (irregular quadrilateral)		A quadrilateral with four sides of equal length—opposite sides are parallel and opposite angles are equal, or the diagonals bisect each other at right angles (i.e., the diagonals are perpendicular). Every rhombus is a parallelogram, but not every parallelogram is a rhombus.
trapezoid (irregular quadrilateral)		A trapezoid has a pair of opposite sides that are parallel. If the sides that are not parallel are of equal length and both angles from a parallel side are equal, this is a special case called an isosoles trapezoid.
irregular quadrilateral		Only the square is a regular quadrilateral, so all other quadrilaterals are irregular.

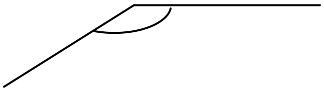
Exercise 8.2

Identify the following angles as to whether they are
A. acute B. right C. obtuse

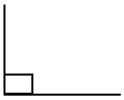
1.



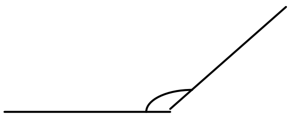
2.



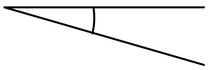
3.



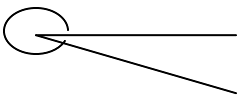
4.



5.

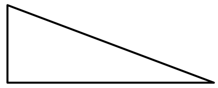


6.

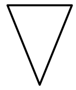


Identify the following triangles as
A. acute B. right C. obtuse
Also identify the following triangles as
D. equilateral E. isosceles F. scalene

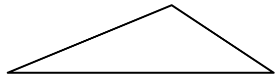
7.



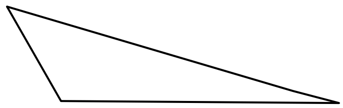
8.



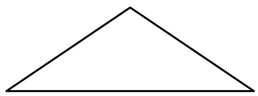
9.



10.



11.



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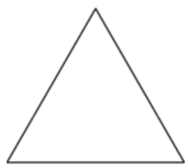


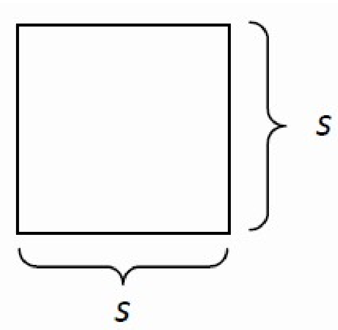
Identify the following quadrilaterals by the description.
13. It has four different length sides.
14. It has two sets of parallel sides and opposite angles with equal measurements.
15. It has four same-length sides and four right angles.
16. It has two sets of parallel sides and four right angles.
17. It has four same-length sides and opposite angles with equal measurements.
18. It has one set of parallel sides or bases.

8.3 Areas of Parallelograms and Trapezoids

Knowing how to calculate areas is important and has many useful applications, including estimating the amount of paint required for a room, the amount of tile needed in a kitchen, the amount of carpet needed for a house, etc.

Perimeter and Areas of a Parallelograms

A parallelogram is a quadrilateral with two sets of parallel sides and opposite angles equal.
A square is a special type of parallelogram having four sides (s) of equal length and right angles.



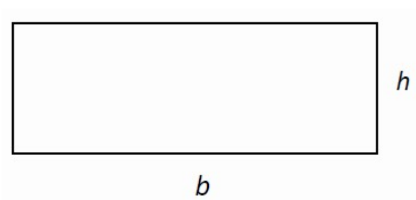
The perimeter, P, of a square is the sum of the lengths of each side (s), and since all four sides are of equal length then,

$$P = s + s + s + s \text{ or } P = 4s$$

The area (A) follows the regular formula for parallelograms, which is base (b) times height (h). In the case of a square the base and the height are of equal lengths, therefore:

$$A = bh = ss \text{ or } A = s^2$$

A rectangle is a parallelogram with two sets of parallel sides of differing lengths and four right angles.



The perimeter (P) of a rectangle is computed in a manner similar to that of the square—the sum of the lengths of its sides, and since it has two sets of equal length sides then:

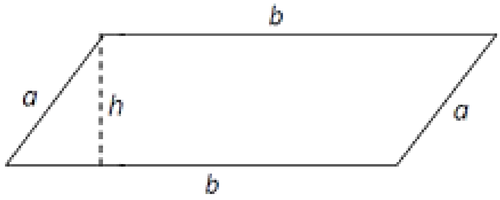
$$P = b + h + b + h \text{ or } P = 2b + 2h$$

The area (A) follows the regular formula of parallelograms, which is base (b) times height (h):

$$A = bh$$

Instead of using base and height as the two side measurements of a rectangle, it is also common practice to refer to the sides as length (l) and width (w)—with the usual convention that the length is the longer side. Another variation is to always call the horizontal dimension the length and the vertical dimension the width—regardless of the relative sizes. This practice is used when specifying the dimensions of windows to a building manufacturer or contractor. In this book, it is sufficient to define the rectangle as being comprised to two different side measurements—whether we call them base and height or length and width, or reverse the order, does, not matter.

Other kinds of parallelograms have angles other than 90°. As can be seen in the diagram below, the base (b) of a parallelogram is the length of any one of its sides, while the height (h) of a parallelogram is the perpendicular distance between the side whose length is the base and the opposite side.



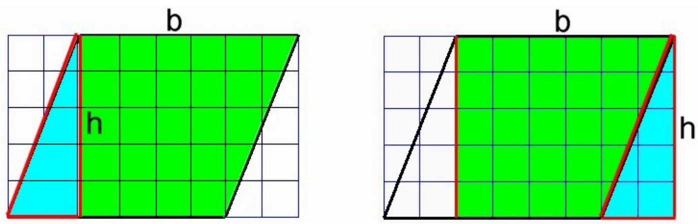
The perimeter (P) of a parallelogram is the sum of all its sides, and since it has two sets of equal length sides (a and b) then:

$$P = a + b + a + b \quad or \quad P = 2a + 2b$$

The area (A) follows the regular formula of parallelograms, which is base (b) times height (h), or

$$A = bh$$

This can be easily proven if we divide the parallelogram through the perpendicular line of the height and reposition the blue-colored triangle on the other side (as shown below) to form a rectangle with the same base (b) and height (h).



Now, let´s consider the area of quadrilaterals with just one set of parallel sides.

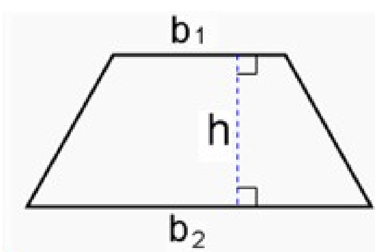
<u>Perimeter and Areas of a Trapezoids</u>

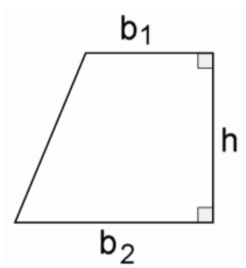
A trapezoid has just one pair of parallel bases (b_1 and b_2) or sides. The height (h) of a trapezoid is the perpendicular distance between the sides whose lengths are the bases (b_1 and b_2).

The perimeter of a trapezoid is the sum of its sides. The area (A) of a trapezoid is the average length of the bases times the height, h. Therefore,

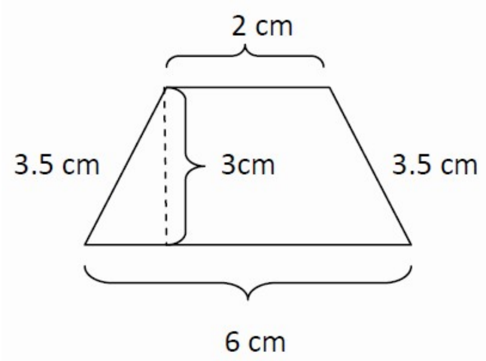
$$A = \frac{1}{2}(b_1 + b_2)h.$$

where b_1, b_2, and h are shown in the generalized trapezoid figures:





Now let's consider this example trapezoid:



The perimeter is given by

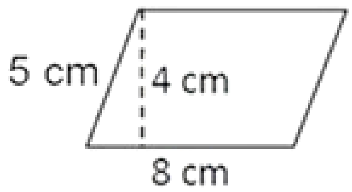
$$P = 2 + 3.5 + 3.5 + 6 = 15 \text{ cm}$$

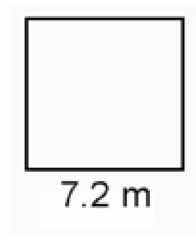
The area is given by

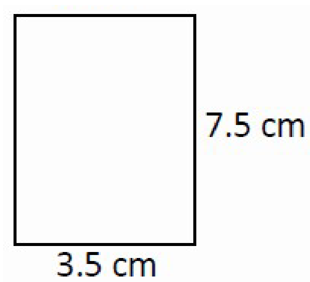
$$\text{A} = \frac{1}{2}(6 + 2)3 = 12 \text{ cm}^2$$

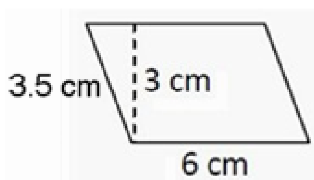
Exercise 8.3

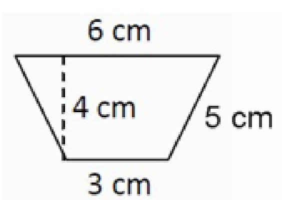
Find the perimeter and area of the following parallelograms and trapezoids.

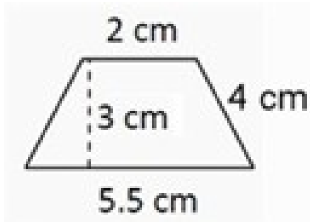
1. 

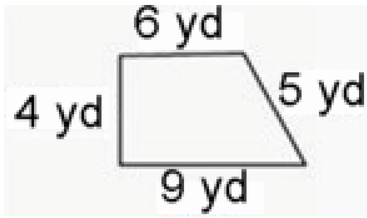
2. 

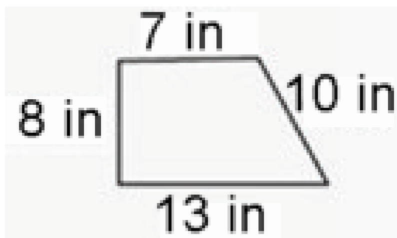
3. 

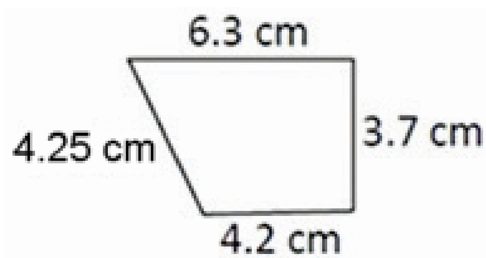
4. 

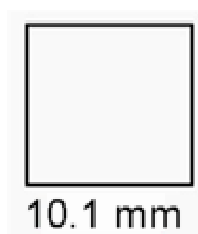
5. 

6. 

7. 

8. 

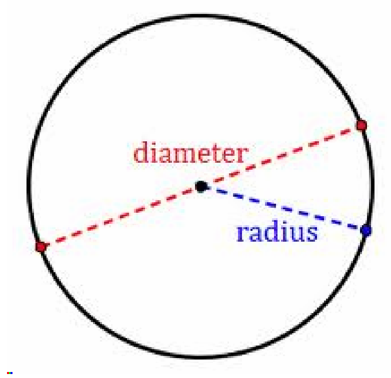
9. 

10. 

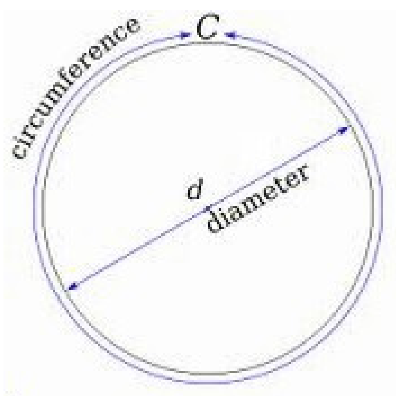
8.4 Circumference and Area of a Circle

The studies of the properties of circles have numerous applications in various fields. It was important in the initial study of geography when the earth's circumference was first estimated by Eratosthenes in the third century B.C. Also the knowledge of the properties of circles are invaluable in physics and mechanics disciplines. Let´s first review some definitions of terms that are frequently associated with circles.

A **circle** is a planar shape (that is, it lies in a single plane) where all points in the plane are the same distance from a fixed point known as the center.



The **radius (*r*)** of a circle is the distance between the center and any point on the circle. Radius can refer to the segment itself or to the length of the segment.
The **diameter (*d*)** of a circle is the distance across the circle through the center. The length of the diameter is twice the length of the radius.
The **circumference** of a circle is the distance around the circle. It is similar to the **perimeter** which is used to describe the distance associated with objects comprised of boundary made up of line segments. The ratio of a given circle's circumference to its diameter is a constant known as pi, usually represented by the Greek letter $\boldsymbol{\pi}$.



Pi is an irrational number that is shown here to an accuracy of 100 decimal digits:

3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679 ...

Often times, pi is simply approximated as the decimal 3.14 or the fraction $\frac{22}{7}$.

Finding the Circumference of a Circle

The circumference, C, (or perimeter) of a circle is the product of π and the diameter d, or twice the product of π and the radius r (the radius being exactly half the value of d, the diameter). This can be written as:

$$C = \pi d \text{ or } C = 2\pi r$$

It is easy to prove that with algebra, if the diameter is the length of two radii (plural of radius):

$$d = 2r$$

then substituting for d, the circumference is:

$$C = \pi d = \pi(2r) = 2\pi r$$

Finding the Area of a Circle

The area A of a circle is the product of π and the square of the radius r. This can be written as

$$A_{\text{circle}} = \pi r^2$$

That means that if you have a circle with a 5 cm radius, the area would be

$$A = \pi(5)^2 = 25\pi \textit{ or } \text{approximately } 78.5\ cm^2 \text{ (rounded to the nearest tenth)}$$

There is a caution that should be noted at this time concerning the evaluation of the area of a circle. Consider a circle with a radius of 500 km. Find the area to the nearest hundredth. Often a student will perform the calculation ***INCORRECTLY*** as follows:

$$\text{A} = \pi(500)^2 = 3.14(500 \text{ km})^2 = 3.14(250000 \text{ km}^2) = 785000 \text{ km}^2$$

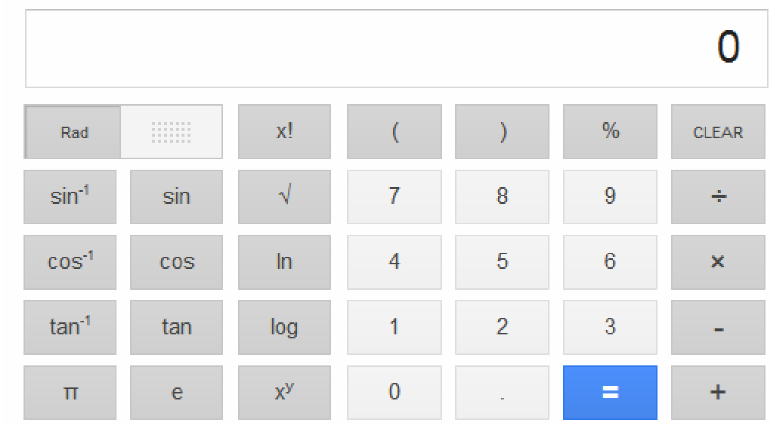
Do you see the problem here? Notice that the calculation was performed assuming $\pi = 3.14$ which is already an approximation (or rounded value of pi) to begin with. Let's redo the calculation using a more accurate value for pi, $\pi = 3.1415926535$, then we have,

$$\text{A} = \pi(500)^2 = 3.1415926535(500 \text{ km})^2 = 3.1415926535(250000 \text{ km}^2) = 785398.163 \text{ km}^2$$

Now, when we round to the nearest hundredth, we accurately have the area A = 785398.16 km^2

It was essential that we used a fairly accurate approximation of pi (to 10 significant digits: 3.1415926535), especially when were concerned with computing the area or the circumference of a circle having a large radius. If we had chosen to approximate pi, using $\pi = 3.1416$, the area that we would have computed, A = 785400 km^2, would not have been accurate even to the nearest whole number (785398). Therefore, it is a preferred practice to use an accurate value of pi (to 10 or more decimal digits) and then after performing all calculations, round your answer. Please remember that area is always expressed in square units, such as km^2, m^2, in^2, ft^2, etc.—not km, m, in, or ft which are units of length (not units of area).

Some scientific calculators have the constant pi readily available on the key pad (see the π-key in the lower left corner of the scientific calculator below). You can find the calculator shown below online by doing a Google search for "Online Calculator" or go to http://web2.0calc.com/ for a slightly different scientific calculator. To find the square root of 25, first strike the $\sqrt{\ }$ key, then key in "25" and press the equal sign (=).



In the PHP programming language, the variable M_PI is predefined with the value of π given to a precision of twenty digits after the decimal point, or M_PI= 3.14159265358979323846

<u>Finding the Radius of a Circle</u>

Since the formula to compute the area of a circle is a function of—or depends upon—the specified radius, we can use that formula to determine the radius of a circle given its area. Given $A = \pi r^2$, we can solve for the radius, r, by first dividing both sides of the equation by π

$$\frac{A}{\pi} = r^2$$

Next, we take the square root of both sides to solve for the radius, r, to obtain, $r = \sqrt{\frac{A}{\pi}}$

Now, given the area of a circle is 64 mm^2. What is the radius of that circle rounded to the nearest tenth of a mm? We substitute A = 64, and solve for r:

$$r = \sqrt{\frac{64}{\pi}} = \sqrt{20.37} = 4.51$$

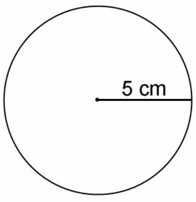
or $r = 4.5$ mm (rounded to the nearest tenth).

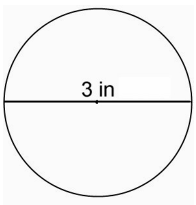
Summary of Circle Formulas

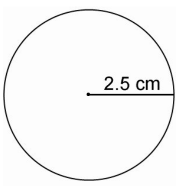
Circle measurements	**Formula**	**Formula to find radius, *r***
Circle Circumference, C	$C = \pi d$ or $C = 2\pi r$	$r = \frac{C}{2\pi}$
Circle Area, A_{circle}	$A_{\text{circle}} = \pi r^2$	$r = \sqrt{\frac{A}{\pi}}$

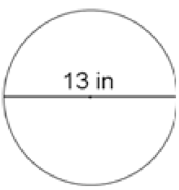
Exercise 8.4

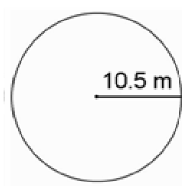
Find the circumference and area of the given circles to the nearest hundredth.

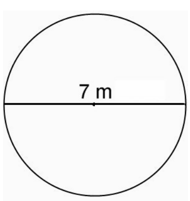
1. 

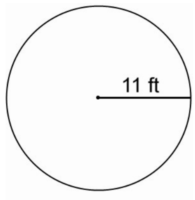
2. 

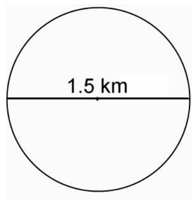
3. 

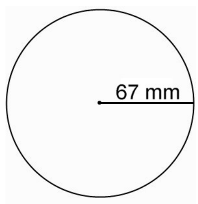
4. 

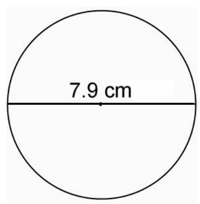
5. 

6. 

7. 

8. 

9. 

10. 

Find the radius of the circles with the following areas. Round to the nearest tenth.

11. 537 cm^2
12. 357 in^2
13. 219 m^2
14. 55 km^2
15. 273 ft^2
16. 49π yd^2
17. 96 in^2
18. 48 cm^2
19. 100π m^2
20. 89 mm^2

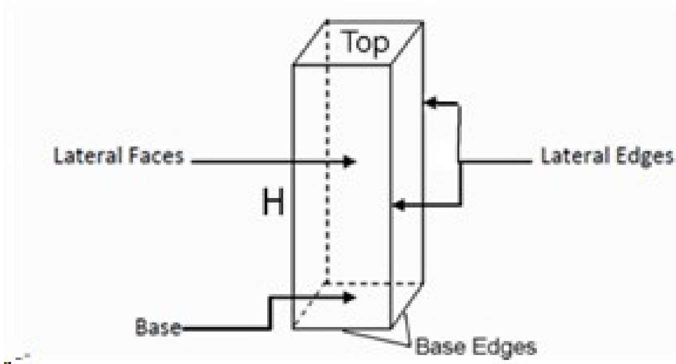
8.5 Volumes of Prisms and Cylinders

The computation of perimeter and areas of polygons and circles (also known as plane figures) has many practical applications. Once we compute the area of a polygon or circle, this area calculation can assist us in calculating the volume of a three-dimensional object, such as a prism or cylinder. In this section we are going to consider the regular solids: Prisms and Cylinders.

Volume of a Prism

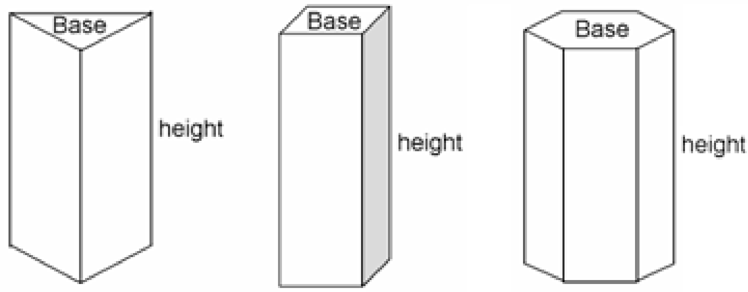
A **prism** is a solid object with two congruent polygonal bases lying in parallel planes. Prisms are comprised of the following:

Bases: Sides of the prisms which are always parallel to each other.
Base Edges: The edge formed where the base meets the lateral face.
Lateral Edges: Parallel segments that connect the vertices of the polygon base.
Lateral faces: Are faces that connect the bases and are rectangles or parallelograms.
The altitude or **height** (H) is the perpendicular distance between the two bases, and it is important for calculating the volume. In the square prism below, there are 4 lateral edges and two sets of 4 base edges, for a total of 12 edges. There are also a total of 6 faces—the 4 lateral faces and the top and bottom base.



The volume (V) of a prism is the product of the base area, B, and the height, H, or $V = BH$. This formula is applicable for computing volumes of three-dimensional objects having a circular (e.g., forming the base of a cone or cylinder) or trapezoidal (e.g., trapezoidal prism) base.

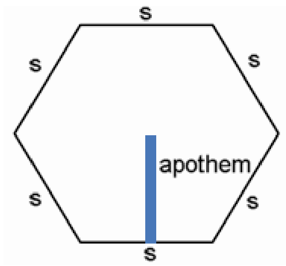
Each of the following prisms has a different polygon as a base:



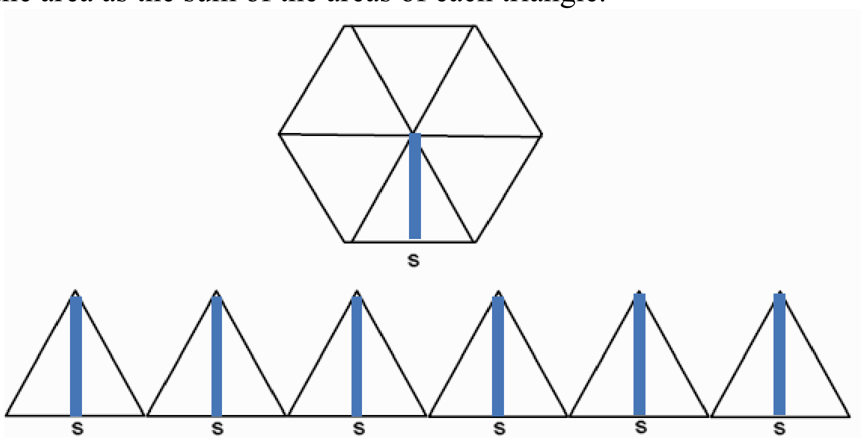
Triangular Prism Rectangular Prism Hexagonal Prism

Calculating the Base Area

The volume of each of the prisms shown at the bottom of the previous page is obtained by using the formula for the area of the base (or area of the base polygon), B, times the height (H) of the prism. Prisms with a triangular base (having 3 sides) or rectangular or square base (having 4 sides) are relatively easy to compute. So, let's consider how we can determine the volume of prisms having bases comprised of 5 or more sides. To find the volume of such prisms, we need to first evaluate the area of these polygons. In order to calculate the area of a polygon with 5 or more sides, we have to know what the **apothem** is. The apothem is a segment that connects the center of regular polygon with a side at a 90° angle. In the following hexagon, the apothem is the thick line segment shown:



To compute the area of this six-sided polygon (or hexagon) we can divide it into 6 triangles and then calculate the area as the sum of the areas of each triangle.



Notice that the apothem is actually the height of the triangles whose base length is equal to the side length (s) of the polygon's side. Using this information, the base area (B) of the hexagon is given by

$$B = s\bullet\text{apothem}/2 + s\bullet\text{apothem}/2 + s\bullet\text{apothem}/2 + s\bullet\text{apothem}/2 + s\bullet\text{apothem}/2 + s\bullet\text{apothem}/2$$

$$\text{or } B = 6s\bullet\frac{apothem}{2}$$

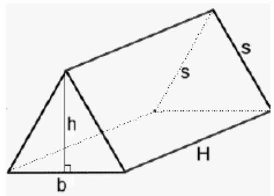
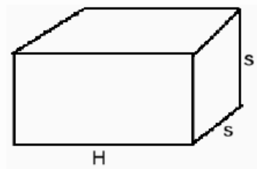
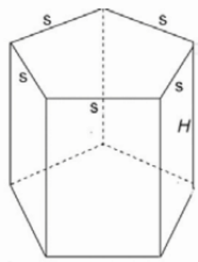
We can generalize this procedure to all polygons. Furthermore, note that the perimeter (P) of the polygon is the number of sides (n) times the length of each side (s), or

$$P = ns$$

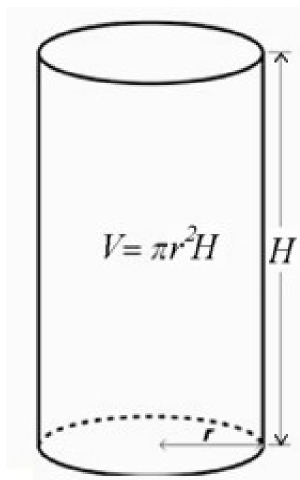
Thus, we can write the general formula for the base area (B) of a polygon with n sides, where n is 5 or higher, as

$$B = \frac{P \bullet apothem}{2}$$

Table 8.5 Formulas for Base Area and Volume of Regular Prisms

Prism	**Shape of Base**	**Base Area, B**	**Volume** (**B•H where H is height of prism**)
Triangular Prism	Triangular	$\frac{bh}{2}$	$\frac{bh}{2} \bullet H$ 
Square Prism	Square	s^2	$s^2 \bullet H$ 
Pentagonal Prism	Pentagon	$\frac{P \bullet apothem}{2} = \frac{5 \bullet s \bullet apthothem}{2}$	$\frac{5 \bullet s \bullet apothem}{2} \bullet H$ 
General n-sided regular Prism	n-sided	$\frac{P \bullet apothem}{2} = \frac{n \bullet s \bullet apothem}{2}$	$\frac{n \bullet s \bullet apothem}{2} \bullet H$

<u>Volume of a Cylinder</u>



The same principle applies to the cylinder. The volume (V_{cylinder}) of a cylinder is the product of the base area B and the height H. The area of the circular base is given by

$$B_{\text{circle}} = \pi r^2$$

*So, s*ubstituting the area of the circle into the formula for volume, we have the volume of the cylinder given by

$$V_{\text{cylinder}} = B_{\text{circle}} H = \pi r^2 H$$

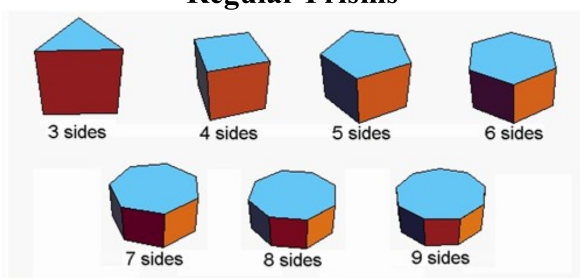
Exercise 8.5

Find the area of the base (to the nearest hundredth of a square centimeter) and volume (to the nearest tenth of a cubic centimeter) of each regular prism having the following measures (see figures below). Please use a calculator.

Exercise	no. of sides	length of sides (cm)	apothem[a] (cm)	height of prism (cm)	B, base area (cm^2)	V, volume (cm^3)
1.	5	5	3.440955	9	______	______
2.	3	2	0.577350	6	______	______
3.	5	1	0.688191	6	______	______
4.	8	9	10.863961	2	______	______
5.	3	9	2.59808	3	______	______
6.	4	5	2.5	2	______	______
7.	9	8	10.989910	3	______	______
8.	6	9	7.794229	5	______	______
9.	5	7	4.817337	8	______	______
10.	7	6	6.229564	4	______	______

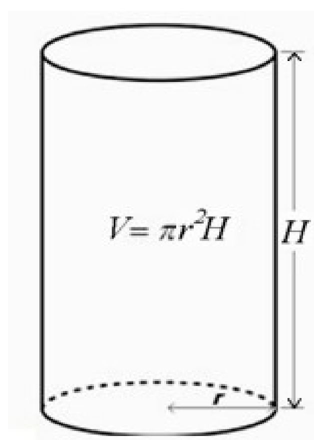
[a]The apothem (shown to six significant digits) can be calculated based on the side length (s) and number of sides (N) using this trigonometric formula: apothem = $\frac{s}{2\bullet\tan(\pi/N)}$ where π/N represents an angle expressed in radians. Trigonometric functions, including sine, cosine, and tangent, will be considered later in another chapter so there is no need to be concerned with this calculation at this time.

Regular Prisms



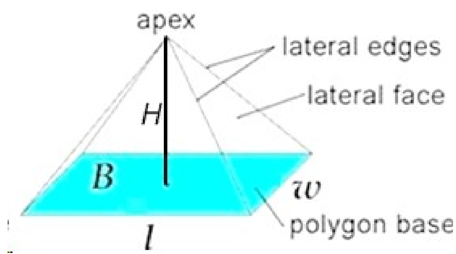
Find the volume of the following cylinders with the following measures. All measurements are in centimeters (cm) and round your answer to the nearest hundredth of a cubic centimeter.

Exercise	radius (cm)	height (cm)	V, volume (cm3)
11.	2	6	______
12.	3	9	______
13.	6	8	______
14.	4	5	______
15.	5	1	______
16.	4	4	______
17.	2	5	______
18.	6	7	______
19.	7	2	______
20.	8	7	______



8.6 Volumes of Pyramids and Cones

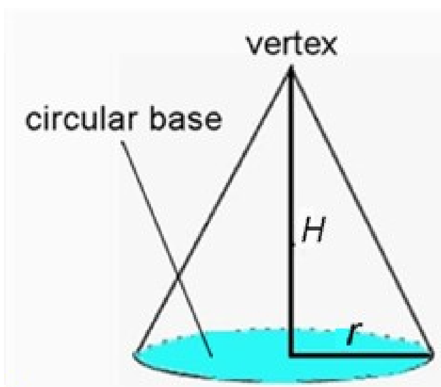
A **pyramid** by definition is a three-dimensional object with a polygon as a base, connected to a point outside the plane where the base lies. All pyramids have a polygon base (shown shaded) and an apex, or vertex point, outside the base plane. The lateral faces that connect the base with the apex are usually triangles which intersect at the lateral edges. Extremely important for calculating the volume is the height, *H*, (or altitude) of the pyramid, which is the perpendicular segment between the base and the vertex. The various parts of the pyramid are labeled below.



The general formula for the value of a pyramid is

$V = \frac{1}{3}(BH)$ where *B* is the area of the base which for a rectangular base is equal to the length times the width ($B = lw$) and *H* is the height of the pyramid. If the base is some other polygon figure, simply find the area as discussed previously in section **8.5** (subheading **Calculating the Base Area**)

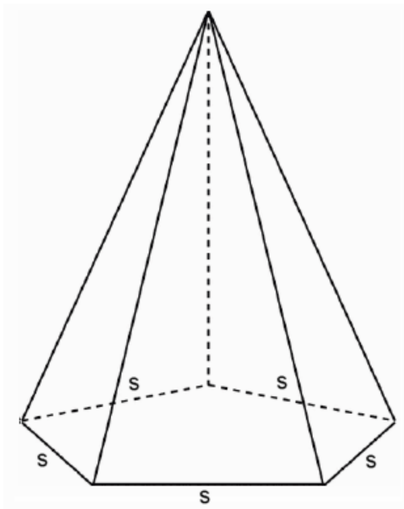
A **cone** is a three-dimensional object with a circular base and one vertex point outside the base plane shown below, where again, the volume is equal to one-third the area of the base ($B = \pi r^2$) times the height, *H*.



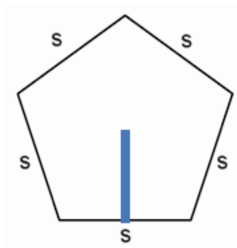
Since a cone has a circle as the base, the volume of a cone is given by

$$V_{\text{cone}} = \frac{1}{3}BH = \frac{1}{3}\pi r^2 H$$

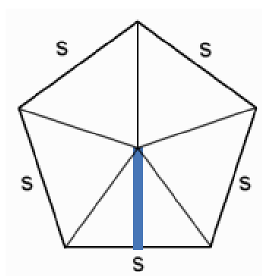
Let´s estimate the volume of the following pentagonal pyramid:

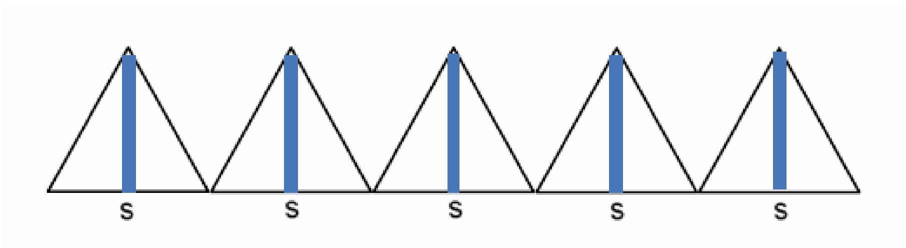


First we need to calculate the base area which is represented by the following pentagon; the apothem is represented by the thick line segment:



To estimate the area of this polygon we can divide it in 5 triangles and then calculate the area as the sum of the areas of each triangle, or equivalently, the area of one triangle times the number of triangles (5).





As you can see now the apothem represents the height of the triangles with the length of the polygon´s side as the base of the triangle. With this data you can calculate the area of the pentagon:

$$Area = \text{(number of sides)}\bullet\text{(area of triangle)} = n \bullet \frac{1}{2}\text{(base length)(triangle height)} =$$

$$5\bullet \frac{\text{(polygon side length)}*\text{(apothem)}}{2}$$

Since the same principle applies to all polygons and the perimeter (P) of the polygon is the number of sides (n) times the length of each side (s), we can rewrite the formula for *Area* as:

$$Area = \frac{n\bullet s\bullet apothem}{2} = \frac{perimeter\bullet apothem}{2} = \frac{P\bullet apothem}{2}$$

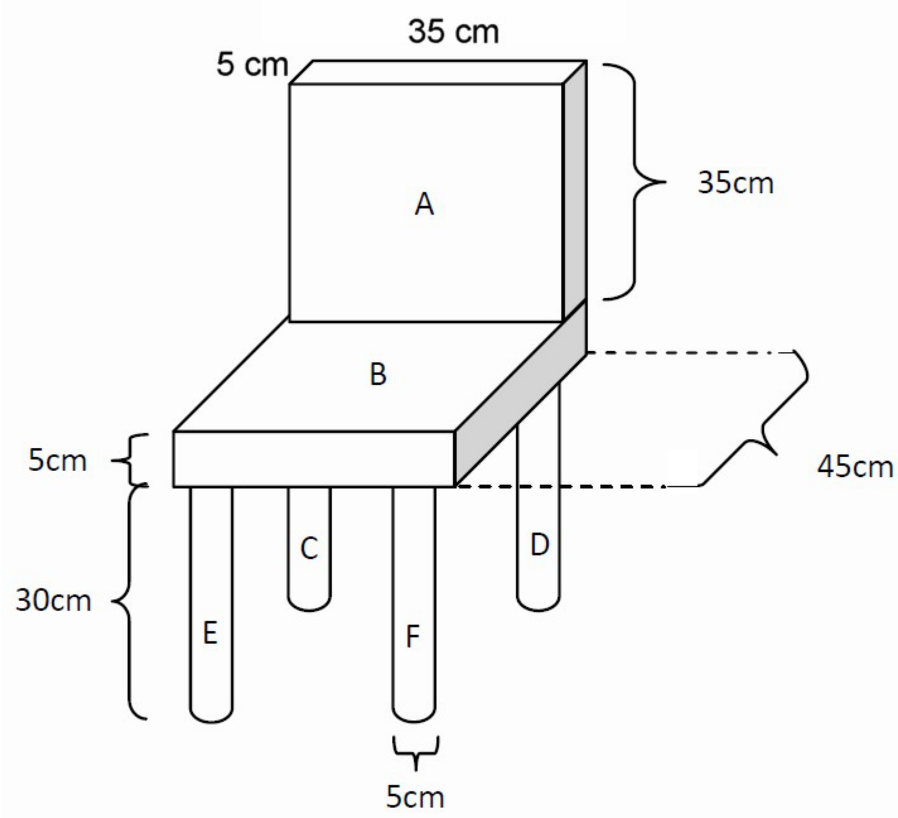
Table 8.6 Formulas for the Base Area and Volume of Cones and Regular Pyramids

3-dimensional Shape	**Shape of Base**	**Base area, B**	**Volume = $\frac{1}{3}BH$** (where H is the pyramid height)
Cone	circle	πr^2 (where r is the radius of the circle)	$\frac{1}{3}\pi r^2 H$
Triangular pyramid	triangular	$\frac{bh}{2}$ (where h is height of the base triangle, or use the General n-sided regular pyramid formula below with $n = 3$)	$\frac{1}{3}\left(\frac{bh}{2}\right)\bullet H = \left(\frac{bh}{6}\right)\bullet H$
Square pyramid	square	s^2 (where s is the side length)	$\frac{s^2\bullet H}{3}$ (where s is the side length)
General n-sided regular pyramid	hexagon or n-gon for $n \geq 3$	$\frac{P\bullet apothem}{2}$ (where P is the perimeter, or $P = n\bullet s$; n is the number of sides, s is the side length)	$\frac{1}{3}\left(\frac{n\bullet s\bullet apothem}{2}\right)\bullet H = \left(\frac{n\bullet s\bullet apothem}{6}\right)\bullet H$ (where $n\bullet s$ is the Perimeter P; n is the number of sides, s is the side length)

Volumes of Solids

Sometimes, you'll find a solid shape that is made up of various other shapes of which you know how to find the volume for. For example, a chair could be made up by a combination of cylinders and prisms. The best strategy to find the volume of wood needed is to split a complex shape into simple ones, then calculate the volume separately and add them together at the end.

Consider again the example of the chair shown below.



In order to calculate the overall volume of wood to the nearest whole cubic centimeter (cm^3), we can break the shape into six separate pieces consisting of the volume of two prisms (A and B) and four same volume cylinders (C, D, E, and F).

Prism A: The dimensions are 35cm x 35cm x 5cm = $6125cm^3$

Prism B: The dimensions are 35cm x 45cm x 5cm = $7875cm^3$

Cylinder C: $V_{\text{cyl}} = \pi r^2 H$, so $\pi\left(\frac{5\ cm}{2}\right)^2 (30\ \text{cm}) = 589\ \text{cm}^3$ (rounded to the nearest cm^3)

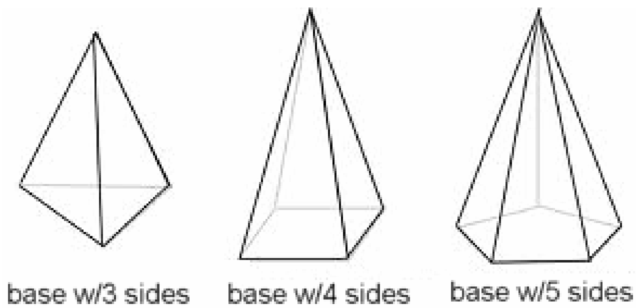
The overall volume of a chair is thus the sum of all six separate volumes:

$$6125\ \text{cm}^3 + 7875\ \text{cm}^3 + 4*589\ \text{cm}^3 = 16356\ \text{cm}^3$$

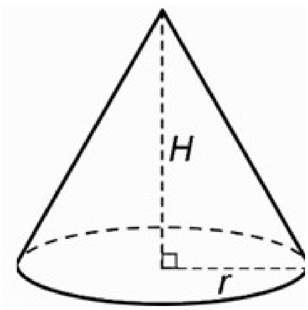
Exercise 8.6

Find the area of the base (to the nearest hundredth of a square centimeter) and the volume (to the nearest tenth of a cubic centimeter) of each pyramid with the following measurements. Please use a calculator.

Exercise	no. of sides	length of sides (cm)	apothem (cm)	height of prism (cm)	B, base area (cm^2)	V, volume (cm^3)
1.	4	9	4.5	1	_____	_____
2.	5	2	1.3764	4	_____	_____
3.	3	8	2.3094	9	_____	_____
4.	7	6	6.2296	4	_____	_____
5.	7	5	5.1913	3	_____	_____
6.	5	8	5.5055	8	_____	_____
7.	7	21	21.8035	3	_____	_____
8.	8	5	6.0355	7	_____	_____
9.	9	5	6.8687	8	_____	_____
10.	6	7	6.0622	7	_____	_____



Find the volume of each cone to the nearest hundredth of a cubic centimeter with the given radius and height. Please use a calculator.

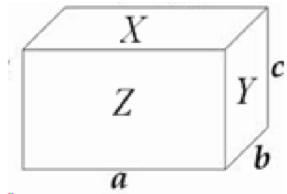


Exercise	radius (cm)	height (cm)	V, volume (cm^3)
11.	7	4.5	_____
12.	7	1.9	_____
13.	3	2	_____
14.	8	8	_____
15.	2	1	_____

8.7 Surface Areas of Prisms and Cylinders

We have previously learned how to calculate the volume of many three-dimensional objects. It is also important to know how to calculate the **surface area** (SA). The surface area of a prism is the sum of the areas of its faces.

Surface Area of a Rectangular Prism



For example, if you have a rectangular prism, the surface area is sum of the two bases (top and bottom rectangles—labeled as X), the two sides (right and left rectangles labeled as Y), and the other two sides (front and back rectangles—labeled as Z). Thus the combined areas of these six surfaces yields the surface area of the rectangular prism given by

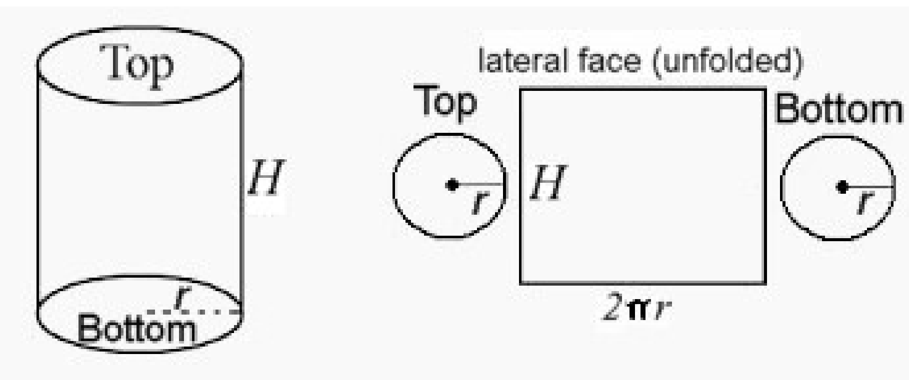
$$SA_{\text{rect}} = X + X + Y + Y + Z + Z \quad \text{or} \quad SA_{\text{rect}} = 2X + 2Y + 2Z = 2(X + Y + Z)$$

Having the length, width, and depth dimensions of the box edges (a x b x c), we can add up the area of each of the six faces. The top and bottom are the same ($X=ab$), so we have $2ab$ as the area of those 2 faces. The right and left sides are the same ($Y=bc$), so we have $2bc$ as the area of those 2 faces. Finally, the front and back faces have the same area ($Z=ac$), so we have $2ac$ as the area of those 2 faces. Therefore, the surface area of the rectangular solid using the dimensions a, b, and c is given by

$$SA_{\text{rect}} = 2ab + 2bc + 2ac = 2\,(ab + bc + ac)$$

Surface Area of a Cylinder (or Cylindrical Prism)

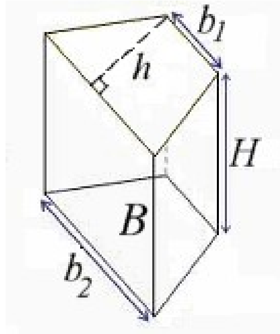
In the case of a cylinder, the surface area is again the sum of the areas of each surface that comprises the cylinder. We have two circular areas that comprise the bases (top and bottom). To that area we must add the curved lateral area that comprises the "side" of the cylinder. Notice that if we unrolled the side of the cylinder, we obtain a rectangular figure whose length is equal to the circumference of the circular base (circumference = $2\pi r$) and whose width is the height of the cylinder, H.



Thus, the surface area of a cylinder is given by

$$SA_{\text{cyl}} = 2\bullet(\text{base area}) + \text{lateral face area} = 2\pi r^2 + 2\pi rH = 2\pi r(r + H)$$

Surface Area of a Prism



By inspecting all sides of the prism shown, the surface area (SA) is seen to be the sum of twice the base area B and the lateral side area. The lateral side area is the sum of the area of each of the four lateral sides, or the product of the base perimeter P and the height H of the prism. Thus, the surface area, SA, is given by $SA = 2B + PH$

The prism shown on the right has a trapezoid-shape base, so to calculate the surface area of this particular prism we have to substitute the trapezoid area for the base area, B, so that $B_{\text{trapezoid}} = \frac{1}{2}(b_1 + b_2)h$. If we add the area of two bases (the top and bottom) to the area of the lateral sides, the surface area becomes $SA_{\text{trapezoidal prism}} = 2[\frac{1}{2}(b_1 + b_2)h] + PH = (b_1 + b_2)h + PH$

Since only the parallel sides of the trapezoid above are labeled (as b_1 and b_2) and there are no dimensions given for the lengths of the angled sides of the trapezoid, it is not possible to determine the perimeter of the trapezoidal base as shown above—which would simply be the sum of all the side lengths. For other regular prisms, the base area B can be computed given the perimeter P and height H as previously shown in Section 8.5 (Table 8.5) and then the surface area becomes $SA_{\text{prism}} = 2B + PH = 2\left(\frac{n \bullet s \bullet apothem}{2}\right) + n \bullet s \bullet H = n \bullet s \bullet (\text{apothem} + H)$

In general, no matter what shape the base of the prism—whether triangular, square, rectangular, trapezoidal, or other—the surface area (SA_{prism}) is always computed by adding up the areas of the 2 bases and areas of each of the sides. Since the base area (B) is needed to compute the volume of a prism and pyramid (see Sections 8.5 and 8.6) as well as to compute the surface area of a prism, the formulas for surface area and volume share some similarity.

Table 8.7 Formulas for the Base Area and Surface Area of Cylindrical and Regular Prisms

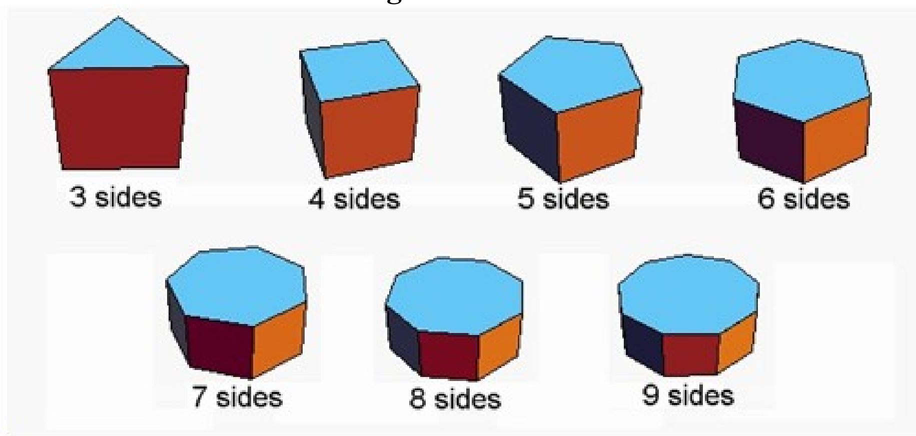
Prism	**Shape of Base**	**Base area, B**	**Surface Area = $2B + PH$** (H is the prism height)
Cylindrical prism	circular	πr^2	$2\pi r^2 + 2\pi rH = 2\pi r(r + H)$
Triangular prism	triangular	$\frac{bh}{2}$ (where h is height of the base triangle; or use the General n-sided regular pyramid formula below with $n = 3$)	$2\left(\frac{bh}{2}\right) + PH = bh + 3sH$ ($3s$ is the Perimeter, P, and s is the length of a base side.)
Square prism	square	s^2 (where s is the base side length.)	$2s^2 + 4sH$ (where $4s$ is the Perimeter, P, s is length of a base side.)
Trapezoidal prism	trapezoid	$\frac{1}{2}(b_1 + b_2)h$ (where b_1 and b_2 are the base lengths; and h is the height)	$(b_1 + b_2)h + PH$ (where P is the perimeter)
General n-sided regular prism	hexagon or n-gon for $n \geq 3$	$\frac{n \bullet s \bullet apothem}{2}$ (where $n \bullet s$ is the perimeter or P; n is number of sides; s is length of a base side.)	$P \bullet (apothem + H)$ (where $n \bullet s$ is the Perimeter, P; n is number of sides; s is length of a base side.)

Exercise 8.7

Find the surface area to the nearest tenth of a square centimeter (cm^2) of each regular prism with the given measures. Please use a calculator.

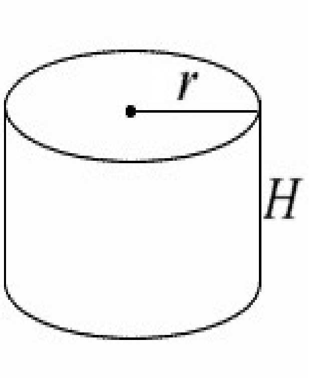
Exercise	no. of sides	length of sides (cm)	apothem (cm)	height of prism (cm)	surface area (cm^2)
1.	5	5	3.44095	9	_____
2.	3	2	0.57735	6	_____
3.	5	1	0.68819	6	_____
4.	8	9	10.86396	2	_____
5.	3	9	2.59808	3	_____
6.	4	5	2.5	2	_____
7.	9	8	10.98991	3	_____
8.	6	9	7.79423	5	_____
9.	5	7	4.81734	8	_____
10.	7	6	6.22956	4	_____

Regular Prisms



Find the surface area to the nearest tenth of a square centimeter (cm^2) of each cylinder having the given radius (r) and height (H). Please use a calculator.

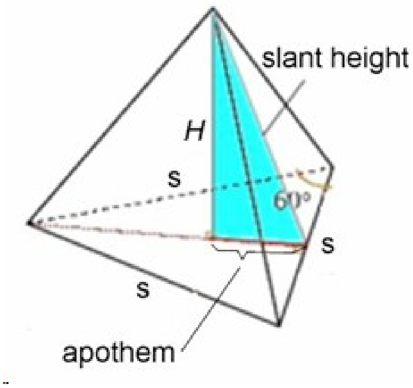
Exercise	radius (cm)	height (cm)	Surface area (cm^2)
11.	2	6	_____
12.	3	9	_____
13.	6	8	_____
14.	4	5	_____
15.	5	1	_____
16.	4	4	_____
17.	2	5	_____
18.	6	7	_____
19.	7	2	_____
20.	8	7	_____



8.8 Surface Areas of Pyramids and Cones

Finding the Slant Height of a Pyramid

Consider the regular triangular pyramid (or tetrahedron—four triangular faces) shown below:



The slant height of this pyramid can be found using the Pythagorean Theorem, where the slant height (or hypotenuse—longest side—of the right triangle shown in blue) squared is equal to the sum of the two sides squared. One side (the bottom) of the right triangle (in blue) is labeled the apothem and the other side (or vertical leg of the blue triangle) is the height of the pyramid or H. So, given $H = 10$ cm and the apothem length is 4 cm, we have

$$(\text{slant height})^2 = H^2 + \text{apothem}^2$$

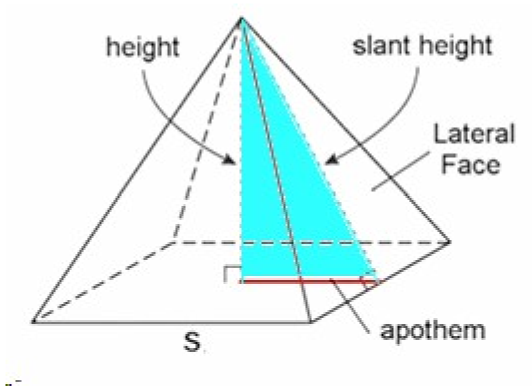
Using the variable l for slant height, and substituting our numerical values for the pyramid height H and the apothem length, according to the Pythagorean Theorem, we have

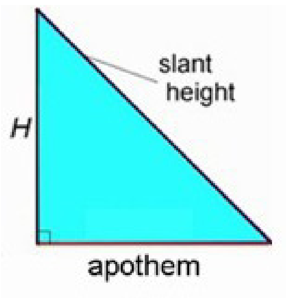
$$l^2 = 10^2 + 4^2 \quad \text{or} \quad l^2 = 116$$

Since the above equation defines the square of the slant height (l^2), we must take the square root of each side, so that we have $l = \sqrt{116} = 10.77$ (to the nearest hundredth). For regular pyramids with 3 or more sides, the slant height is given by this formula:

$$\text{slant height} = \sqrt{H^2 + \text{apothem}^2}$$

The base of a pyramid is a polygon, and the **lateral faces** of the pyramid are triangles with a common vertex. The perpendicular distance between the base and this common vertex is the **height** of a pyramid. For a **regular pyramid**, the base is a regular polygon and all of the lateral faces of the pyramid are congruent isosceles triangles. Recall that an isosceles triangle contains two sides that are of equal length. So the **slant height** of a pyramid is the height of any of these triangular lateral faces.





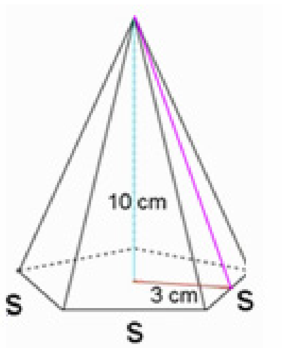
In summary, given the blue-colored right triangle, we find the slant height of a regular pyramid by using the Pythagorean Theorem:

$$(\text{slant height})^2 = H^2 + \text{apothem}^2$$

Next, we take the square root of both sides of the equation to obtain,

$$\text{slant height} = \sqrt{H^2 + \text{apothem}^2}$$

Having found the slant height of a pyramid, in the next section we use that calculation to determine the area. Before continuing, let's consider another example. Let's find the slant height of the pyramid shown (to the nearest hundredth of a centimeter or to the nearest 0.01 cm):



$$\text{slant height} = \sqrt{10^2 + 3^2} = \sqrt{100 + 9} = \sqrt{109} \approx 10.44\ cm$$

Now that we have the slant height, we will proceed to calculate surface area.

Remember that the formula for the slant height of a regular pyramid:

$$\text{slant height} = \sqrt{H^2 + \text{apothem}^2}$$

Surface Area of a Regular Pyramid

The surface area *SA* of a regular pyramid is the sum of the base area *B* and all lateral triangular surfaces. Since the area of each lateral triangle is one-half times the side length (s) times the slant height (l) we have $A_{\text{triangle}} = \frac{1}{2}sl$

The surface area of a regular pyramid is the sum of the areas of the base and all lateral triangles:

$$SA_{regular\ pyramid} = B + \frac{1}{2}Pl$$

where P is the base perimeter (numbers of sides times side length, or ns). Referring to the cone previously shown above, given the side length of the base, s ≈ 4.3593 cm, and there are 5 sides to the base (n=5), then the base area is

$$B = \frac{ns \bullet \text{apothem}}{2} = \frac{5 \bullet 4.3593 \bullet 3}{2} \approx 32.695\ \text{cm}^2$$

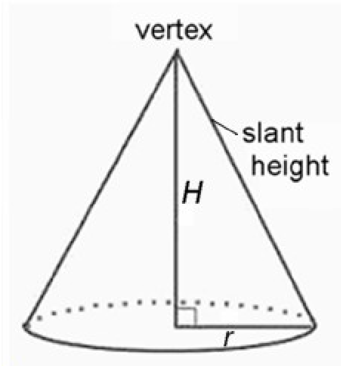
Thus, using the slant height that we previously calculated (10.44 cm), the surface area is

$$\text{SA}_{\text{regular pyramid}} \approx 32.695 + \frac{1}{2}21.7965 \bullet 10.44 \approx 32.695\ \text{cm}^2 + 113.78\ \text{cm}^2 \approx 146.5\ \text{cm}^2 \text{ rounded to the nearest tenth.}$$

The general formula for an n-sided regular pyramid is

$$\text{SA}_{\text{regular pyramid}} = \frac{ns \bullet \text{apothem}}{2} + \frac{ns \bullet (\text{slant height})}{2} = \frac{ns}{2} \bullet (\text{apothem} + \text{slant height})$$

Surface Area of Cones



The **vertex** of a cone is the point directly above the center of its base. The **height** of a cone is the distance between the vertex and the center of the base. The **slant height,** l**,** of the cone is the distance between the vertex and any point on the edge of the base.

To find the surface area of a cone, we need to know the radius r of the circular base and the slant height, l. By definition, the surface area SA_{cone} of a cone is the sum of the base area B and the product of π, the base radius r, and the slant height l. This can be written as:

$$SA_{cone} = B + \pi r(\text{slant height}) = \pi r^2 + \pi rl$$

By factoring out the common terms, we have

$$SA_{cone} = \pi r(r + l)$$

In a manner similar to that of the pyramid, since the slant height, l, of the cone comprises the hypotenuse (longest side) of a right triangle with sides r and H, the square of the slant height can be found using the Pythagorean theorem as follows:

$$l^2 = r^2 + H^2$$

Now, we must take the square root of both sides of the equation to obtain the slant height (l) given the cone radius (r) and height (H),

$$l = \sqrt{r^2 + H^2}$$

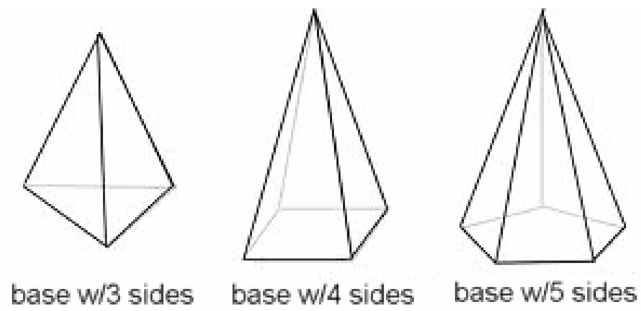
Table 8.8 Surface area of Regular Pyramids and Cones

Pyramid or Cone	Shape of Base	Base area, B	Surface area formula
Cone	circular	πr^2 (where r is the radius of the circle)	$SA_{cone} =$ $\boldsymbol{B} + \pi rl = \pi r(r + l)$ (where l is the slant height)
General n-sided regular pyramid	triangular, square, or n-gon for n ≥ 3	$\frac{n \bullet s \bullet apothem}{2}$ (where $n \bullet s$ is the perimeter or P; n is number of sides; s is length of a base side.)	$SA_{\text{regular pyramid}} = B + \frac{P \bullet l}{2} =$ $\frac{p \bullet \text{apothem}}{2} + \frac{P \bullet l}{2} =$ $\frac{P}{2} \bullet (\text{apothem} + l)$ (where $n \bullet s$ is the Perimeter, P; n is number of sides; s is length of a base side; l is the slant height.)

Exercise 8.8

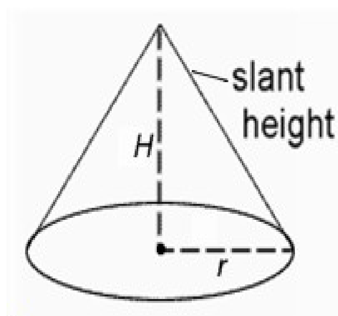
Find the slant height to the nearest hundredth centimeter and surface area to the nearest tenth of a square centimeter (cm^2) of each regular pyramid given the measures shown. Please use a calculator.

Problem	no. of sides	Side length (cm)	apothem (cm)	height of prism (cm)	slant height (cm)	surface area (cm^2)
1.	4	9	4.5	1	_____	_____
2.	5	2	1.37638	4	_____	_____
3.	3	8	2.30940	9	_____	_____
4.	7	6	6.22956	4	_____	_____
5.	7	5	5.19130	3	_____	_____
6.	5	8	5.50553	8	_____	_____
7.	7	21	21.80347	3	_____	_____
8.	8	5	6.03553	7	_____	_____
9.	9	5	6.86869	8	_____	_____
10.	6	7	6.06218	7	_____	_____



Find the slant height to the nearest hundredth centimeter and surface area to the nearest tenth of a square centimeter (cm^2) of each cone with given radius and height. Please use a calculator.

Problem	Radius, r (cm)	Height, H (cm)	slant height (cm)	surface area (cm^2)
11.	7	4.5	_____	_____
12.	7	1.9	_____	_____
13.	3	2	_____	_____
14.	8	8	_____	_____
15.	2	1	_____	_____



8.9 Metric Units of Length, Area, Volume, and Weight

The metric system uses units of meters (m) for length, m^2 for area, liters (L) for liquid volume, and grams for weight. Using the chart of prefixes and corresponding factors below, we see that a millimeter (mm) is 10^{-3} (or 0.001 = 1/1000) of a meter (m). Therefore 1000 mm is equal to 1 meter. Also, since a centimeter (cm) is 10^{-2} (or 0.01 = 1/100) of a meter, 100 centimeters (cm) is equal to 1 meter (m). We also see that 10 mm is equal to 1 cm and 1 kilometer (km) is equal to 1000 m. These same relationships apply to metric volume, such that 1 liter (L) is made up of 1000 milliliters (mL). Also metric weight shares the same relationships, such that 1 gram is equal to 1000 milligrams (mg) and one kilogram (kg) is 1000 grams (g). Presently, computers have disks capable of storing one or more terabytes which is equal to 1000 gigabytes.

Metric prefixes are used to represent powers of 10. The most commonly used prefixes are highlighted in **bold** the table.

Prefix	Symbol	Factor
yotta	Y	10^{24} = 1,000,000,000,000,000,000,000,000
zetta	Z	10^{21} = 1,000,000,000,000,000,000,000
exa	E	10^{18} = 1,000,000,000,000,000,000
peta	P	10^{15} = 1,000,000,000,000,000
tera	**T**	**10^{12} = 1,000,000,000,000**
giga	**G**	10^{9} = 1,000,000,000
mega	**M**	10^{6} = 1,000,000
kilo	**k**	10^{3} = 1,000
hecto	h	10^{2} = 100
deka	da	10^{1} = 10
deci	**d**	10^{-1} = 0.1
centi	**c**	10^{-2} = 0.01
milli	**m**	10^{-3} = 0.001
micro	**μ**	10^{-6} = 0.000,001
nano	n	10^{-9} = 0.000,000,001
pico	p	10^{-12} = 0,000,000,000,001
femto	f	10^{-15} = 0.000,000,000,000,001
atto	a	10^{-18} = 0.000,000,000,000,000,001
zepto	z	10^{-21} = 0.000,000,000,000,000,000,001
yocto	y	10^{-24} = 0.000,000,000,000,000,000,000,001

English and Metric Units and Equivalences

	English units	Metric units	Basic metric unit	Metric and English Equivalences
Length	12 in = 1 ft 3 ft = 1 yd 5280 ft = 1 mi	10 mm = 1 cm 100 cm = 1 m 1000 mm = 1 m 1000 m = 1 km	meter (m)	2.54 cm = 1 in
Area	144 in^2 = 1 ft^2 43,560 ft^2 = 1 acre 640 acres = 1 mi^2	10,000 cm^2 = 1 m^2 10,000 m^2 = 1 hectare 100 hectare = 1 km^2	square meters (m^2)	1 in^2 = 6.4516 cm^2 1 mi^2 = 2.589975 km^2
Volume	1 tablespoon (tbsp) = 3 teaspoons (tsp) 1 cup = 48 tbsp 1 pint = 2 cups 1 qt = 2 pints 1 gal = 4 qt 1728 in^3 = 1 ft^3 231 in^3 = 1 gal (liquid) 4 qt = 1 gal 42 gal = 1 barrel 32 qt = 1 bushel	1 cm^3 = 1 mL 1000 mL= 1 L 1000 L = 1 kL 1 m^3 = 1000 L	liters (L)	16.3870 cm^3 = 1 in^3 3.78541 L = 1 gal
Weight	437.5 grains = 1 oz 16 oz = 1 lb 2000 lb = 1 ton (English)	1000 mg = 1 g 1000 g = 1 kg 1000 kg = 1 metric ton	grams (g}	454 g = 1 lb 2.20462 lb = 1 kg

Using Conversion Factors to Convert Units

We can convert from one type of metric unit to a different metric unit or even English units and vice-versa by multiplying the initial number by the appropriate conversion factor. The conversion factor is always equivalent to multiplying by 1. A conversion factor is a ratio or fraction where the denominator consists of the new units that we wish to convert to; the numerator consists of the units that are the same as the value we wish to convert.—such that the initial units cancel, leaving the desired new units. Let's see how this works with an example. Suppose we desire to convert 134.5 mm to meters (m). Since 1 m is equal to 1000 mm (see table above), we write the conversion factor as the following fraction:

$$\frac{1\text{ m}}{1000\text{ mm}}$$

Notice that when we multiply our initial value by this conversion factor (or fraction), the result has the proper units: the units of millimeters (mm) associated with our initial value (134.5), cancels with millimeters in the denominator of our conversion factor, leaving the answer is units of meters (m):

$$134.5\text{ mm} \bullet \left[\frac{1\text{ m}}{1000\text{ mm}}\right] = 0.1345\text{ m}$$

Now, let's work this same conversion in reverse. Suppose we desire to convert 0.1345 m to mm. Since 1000 mm is equal to 1 m, we express the conversion factor as follows:

$$\frac{1000 \text{ mm}}{1 \text{ m}}$$

Notice, again, that we have placed in the numerator the units that we wish to convert to; in the denominator we have used the same units (m) of the value we desire to convert. So, when we multiply our initial value by this conversion factor, the units of meters (m) cancels in both the numerator and denominator, yielding a result with the proper units in millimeters (mm):

$$0.1345 \text{ m} \bullet \left[\frac{1000 \text{ mm}}{1 \text{ m}}\right] = 134.5 \text{ mm}$$

Sometimes, if we do not have a direct conversion factor relating the initial units to the desired final units, we must use several conversion factors in succession to achieve the conversion. Let's look at an example that requires two conversion factors. Suppose we wish to convert 9.5 inches to miles. Well, we can use the conversion factor that first relates feet (ft) to inches (in),

$$\frac{1 \text{ ft}}{12 \text{ in}}$$

Next, we can use the conversion factor that relates miles (mi) to feet (ft):

$$\frac{1 \text{ mi}}{5280 \text{ ft}}$$

So, now we are ready to proceed with the conversion of 9.5 inches to miles as follows:

$$9.5 \text{ in} \bullet \left[\frac{1 \text{ ft}}{12 \text{ in}}\right]\left[\frac{1 \text{ mi}}{5280 \text{ ft}}\right] = 0.00014\ 99368687 \text{ mi}$$

Notice that all units cancel except for miles, which is the desired units of the answer.

Let's now try a conversion from English to metric units. Suppose we wish to convert the English units of area, 288 in^2, to metric units of area, m^2.

From the conversion chart on p. 240, we cannot find a direct conversion from in^2 to m^2, so we will need to formulate several conversion factors as follows. First, we can convert from in^2 to cm^2, which at least accomplishes the conversion from English to metric units, using this conversion factor,

$$\frac{6.4516 \text{ cm}^2}{1 \text{ in}^2}$$

Next, we simply need a conversion factor to convert the metric units of cm^2 to m^2 which is found in the chart given on p. 240,

$$\frac{1 \text{ m}^2}{10000 \text{ cm}^2}$$

Applying both of these conversion factors, we have

$$288\ \text{in}^2 \bullet \left[\frac{6.4516\ \text{cm}^2}{1\ \text{in}^2}\right]\left[\frac{1\ \text{m}^2}{10000\ \text{cm}^2}\right] = 0.18580608\ \text{m}^2$$

Let's try one more. Suppose we wish to convert 3 quarts (qt) to liters. We cannot find a direct relationship between quarts and liters, so again we will need to formulate several conversion factors.

Note that we can convert quarts to gallons using the conversion factor:

$$\frac{1\ \text{gal}}{4\ \text{qt}}$$

Next, we can convert English units of gallons (gal) to metric units of liters (L) using this conversion factor from the table above,

$$\frac{3.78541\ \text{L}}{1\ \text{gal}}$$

Finally, applying the two conversion factors, we have

$$3\ \text{qt} \bullet \left[\frac{1\ \text{gal}}{4\ \text{qt}}\right]\left[\frac{3.78541\ \text{L}}{1\ \text{gal}}\right] = 2.8390575\ \text{L}$$

The process of formulating the proper conversion factor (or factors) such that the final answer has the correct units is known as **dimensional analysis**. Performing dimensional analysis properly will insure that not only is the numerical value of your answer correct, but that also the units are correct.

Exercise 8.9

All answers are approximate to 3 or more digits of accuracy. Answers are considered correct if nearly the same as those given in the Answer Key.

Convert the following lengths

1. 1236 mm to _____ m
2. 506 in to _____ cm
3. 5426 in to _____ mi
4. 342 mi to _____ ft
5. 85 cm to _____ m
6. 3.56 km to _____ m
7. 1.5 mi to _____ in
8. 800 ft to _____ km
9. 35.8 cm to _____ mm
10. 100 in to _____ km

Convert the following volumes:

11. 633 gal to _____ m^3
12. 572 qt to _____ liter
13. 178 in^3 to _____ ml
14. 11 ft^3 to _____ cm^3
15. 535 barrel to _____ ft^3
16. 31 cm^3 to _____ m^3
17. 934 m^3 to _____liter
18. 45 litter to _____barrel
19. 57 ml to _____ qt
20. 32 m^3 to _____ gal

Convert the following weights:

21. 517 mg to _____ lb
22. 911 g to _____ oz
23. 54 ton (English) to _____ metric ton
24. 95 metric ton to _____ grains
25. 79 lb to _____ kg
26. 14 grains to _____ metric ton
27. 33 oz to _____ g
28. 6 metric ton to _____ lb
29. 81 lb to _____ mg
30. 47 kg to _____ ton (English)

8.10 Volume, Mass, and Density

Now that you have just learned how to figure out and convert the capacity and weights in the metric system of measurement, you can learn how they relate. An interesting fact is when we realize that:

1 cubic centimeter (cc) = 1 milliliter (ml) = 1 gram of water

and

1000 cubic centimeters = 1000 milliliters (1 liter) = 1000 grams of water (1 kilogram)

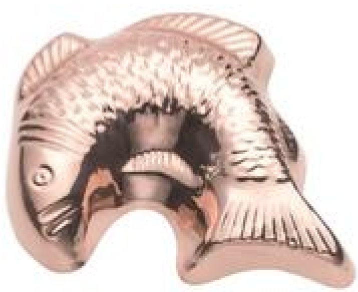
For example, if we had a container of juice that had a volume of 1 cubic centimeter, it would contain 1 milliliter of juice and would weigh approximately 1 gram. However if you have one cubic centimeter of lead, it would contain one milliliter of lead but would weigh 11.34 grams This introduces a new concept: different materials have different mass although they may occupy the same volume. The property that measures the mass per unit volume is called density and its formula is:

$$D = m/V$$

The substance used as reference for density is water because its density is 1 g/ml, which is a 1:1 relationship between mass (or weight) and volume. The knowledge of these facts comes handy when calculating volumes of irregular objects.

How to Find the Volume of a Container

If you wanted to find the amount and weight of water or liquid that an irregular container can hold, you would first need to find the volume of the container, and then use the density formula.



Imagine that you wanted to cast this fish container by filling it with liquid lead. What volume of lead is needed to fill the fish-shaped container?

The first step is to record the weight of the empty fish-shaped container. The weight is 200 g. We assign the letter C to describe that weight of the empty container. So $C = 200$ g.

Next, we fill the container with water (M_w) which is a liquid with known density. After filling the container with water, we proceed to weigh the filled container again. The total weight W will be:

$$W = C + M_w$$

The total weight of the container and water is $W = 580$ g. Thus, the weight of the mass of water is:

$$M_w = W - C$$

or

$$M_w = 580 \text{ g} - 200 \text{ g} = 380 \text{ g}$$

Finally, we apply the density formula, and associate the subscript w with each variable of the formula to specify that we are talking about the mass of water added to the container:

$$D_w = M_w/V_w$$

Since we know the density of the water is 1 g/ml and also we know the weight of water (M_w) that the filled container holds, the container volume can be calculated by solving the above equation for V_w.

$$V_w = M_w/D_w = \frac{(W - C)}{1\ \text{g/ml}} = \frac{380\ \text{g}}{1\ \text{g/ml}} = 380\ \text{ml}$$

Thus, the fish-shaped container can hold 380 ml or 0.38 liters of water.

Now, since the density of lead (at its melting point) is approximately 11 g/cc (or 11 g/ml) and we require a volume of 380 ml. Using the formula

$$M = DV$$

the mass of lead corresponding to a volume of 380 ml is approximately

$$M = (11\ \text{g/ml})(380\ \text{ml}) = 4180\ \text{g of lead}$$

Since 454 g = 1 pound (16 ounces), we can multiply 4180 g by the conversion factor $\frac{1\ \text{pound}}{454\ \text{g}}$ to obtain the equivalent weight of lead in pounds:

$$4180\ \text{g} \bullet \left[\frac{1\ \text{pound}}{454\ \text{g}}\right] = 9.2\ \text{pounds (to the nearest tenth of a pound)}$$

Summary of Density, Mass, and Volume formulas

Solve for	Formula
Density	$D = M/V$
Mass	$M = DV$
Volume	$V = M/D$

Exercise 8.10

Find the missing volume, density or mass to the nearest thousandth.

Exercise	Volume (mL)	Mass (g)	Density (g/mL)
1.	86	_____	0.357
2.	_____	69	0.914
3.	979	_____	0.735
4.	52	39	_____
5.	471	56	_____
6.	64	_____	0.989
7.	739	752	_____
8.	13	_____	0.916
9.	313	314	_____
10.		313	1.087
11.	87	25	_____
12.	53	_____	1.014
13.	339	856	_____
14.	75	819	_____
15.	_____	203	1.039
16.	_____	68	0.91
17.	448	78	_____
18.	28	_____	0.932
19.	_____	822	1.264
20.	227	_____	0.444

Answers to Chapter 8 Exercises

Exercise 8.1
1. triangle, A, 0; **2.** hexagon, A, 9; **3.** square (or quadrilateral or rhombus) A, 2; **4.** rectangle or quadrilateral, A, 2; **5.** pentagon, A, 5; **6.** dodecagon, B, 54; **7.** Parallelogram (or quadrilateral), A, 2; **8.** hexagon, B, 9; **9.** cube (or hexahedron), square, 6, 12, 8, 3; **10.** pyramid (or square pyramid); **11.** tetrahedron; triangle, 4,6,4,3; **12.** icosahedron, triangle, 20, 30, 12, 5

Exercise 8.2
1. A; **2.** C; **3.** B; **4.** C; **5.** A; **6.** C; **7.** B, F; **8.** A, E; **9.** C, F; **10.** C, F; **11.** C, E; **12.** A,D; **13.** irregular quadrilateral (or quadrilateral); **14.** parallelogram; **15.** square; **16.** square or rectangle; **17.** rhombus or square; **18.** trapezoid

Exercise 8.3
1. 26 cm, 32 cm^2; **2.** 28.8 m, 51.84 m^2; **3.** 22 cm, 26.25 cm^2; **4.** 19 cm; 18 cm^2; **5.** 19 cm, 18 cm^2; **6.** 15.5 cm, 11.25 cm^2; **7.** 24 yd, 30 yd^2; **8.** 38 in, 80 in^2; **9.** 18.45 cm, 19.425 cm^2; **10.** 40.4 mm, 102.01 mm^2

Exercise 8.4
1. 31.42 cm, 78.54 cm^2; **2.** 9.42 in, 7.07 in^2; **3.** 15.71 cm, 19.63 cm^2; **4.** 40.84 in, 132.73 in^2; **5.** 65.97 m, 346.36 m^2; **6.** 21.99 m, 38.48 m^2; **7.** 69.12 ft, 380.13 ft^2; **8.** 4.71 km, 1.77 km^2; **9.** 420.97 mm, 14102.61 mm^2; **10.** 24.82 cm, 49.02 cm^2; **11.** 13.1 cm; **12.** 10.7 in; **13.** 8.3 m; **14.** 4.2 km; **15.** 9.3 ft; **16.** $r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{49\pi}{\pi}} = \sqrt{49} = 7$ yd; **17.** 5.5 in; **18.** 3.9 cm; **19.** 10 m; **20.** 5.3 mm

Exercise 8.5
1. 43.01 cm^2, 387.1 cm^3; **2.** 1.73 cm^2, 10.4 cm^3; **3.** 1.72 cm^2, 10.3 cm^3; **4.** 391.10 cm^2, 782.2 cm^3; **5.** 35.07 cm^2, 105.2 cm^3; **6.** 25.00 cm^2, 50 cm^3; **7.** 395.64 cm^2, 1186.9 cm^3; **8.** 210.44 cm^2, 1052.2 cm^3; **9.** 84.30 cm^2, 674.4 cm^3; **10.** 130.82 cm^2, 523.3 cm^3; **11.** 75.40 cm^3; **12.** 254.47 cm^3; **13.** 904.78 cm^3; **14.** 251.33 cm^3; **15.** 78.54 cm^3; **16.** 201.06 cm^3; **17.** 62.83 cm^3; **18.** 791.68 cm^3; **19.** 307.88 cm^3; **20.** 1407.43 cm^3

Exercise 8.6
1. 81.00 cm^2; 27 cm^3; **2.** 6.88 cm^2, 9.2 cm^3; **3.** 27.71 cm^2, 83.1 cm^3; **4.** 130.82 cm^2, 174.4 cm^3; **5.** 90.85 cm^2, 90.8 cm^3; **6.** 110.11 cm^2, 293.6 cm^3; **7.** 1602.56 cm^2, 1602.6 cm^3; **8.** 120.71 cm^2, 281.7 cm^3; **9.** 154.55 cm^2, 412.1 cm^3; **10.** 127.31 cm^2, 297.0 cm^3 or 297.1 cm^3 (answer depends on if you use the actual base area or rounded base area); **11.** 230.91 cm^3; **12.** 97.49 cm^3; **13.** 18.85 cm^3; **14.** 536.17 cm^3; **15.** 4.19 cm^3

Exercise 8.7
1. 311.0 cm^2; **2.** 39.5 cm^2; **3.** 33.4 cm^2; **4.** 926.2 cm^2; **5.** 151.1 cm^2; **6.** 90.0 cm^2; **7.** 1007.3 cm^2; **8.** 690.9 cm^2; **9.** 448.6 cm^2; **10.** 429.6 cm^2; **11.** 100.5 cm^2; **12.** 226.2 cm^2; **13.** 527.8 cm^2; **14.** 226.2 cm^2; **15.** 188.5 cm^2; **16.** 201.1 cm^2; **17.** 88.0 cm^2; **18.** 490.1 cm^2; **19.** 395.8 cm^2; **20.** 754.0 cm^2

Exercise 8.8

1. 4.61 cm; 164.0 cm^2 **2.** 4.23 cm; 28.0 cm^2 **3.** 9.29 cm, 139.2 cm^2; **4.** 7.40 cm, 286.2 cm^2; **5.** 6.00 cm, 195.8 cm^2; **6.** 9.71 cm, 304.3 cm^2; **7.** 22.01 cm, 3220.3 cm^2; **8.** 9.24 cm, 305.5 cm^2; **9.** 10.54 cm, 391.7 cm^2; **10.** 9.26 cm, 321.8 cm^2; **11.** 8.32 cm, 336.9 cm^2; **12.** 7.25 cm, 313.4 cm^2; **13.** 3.61 cm, 62.3 cm^2; **14.** 11.31 cm, 485.3 cm^2; **15.** 2.24 cm, 26.6 cm^2

Exercise 8.9

1. 1236 mm • $\left[\frac{1\text{ m}}{1000\text{ mm}}\right]$ = 1.236 m; **2.** 506 in • $\left[\frac{2.54\text{ cm}}{1\text{ in}}\right]$ = 1285.24 cm;

3. 5426 in • $\left[\frac{1\text{ ft}}{12\text{ in}}\right]\left[\frac{1\text{ mi}}{5280\text{ ft}}\right]$ = 0.085638 mi; **4.** 342 mi • $\left[\frac{5280\text{ ft}}{1\text{ mi}}\right]$ = 1805760 ft;

5. 85 cm • $\left[\frac{1\text{ m}}{100\text{ cm}}\right]$ = 0.85 m; **6.** 3.56 km • $\left[\frac{1000\text{ m}}{1\text{ km}}\right]$ = 3560 m;

7. 1.5 mi • $\left[\frac{5280\text{ ft}}{1\text{ mi}}\right]\left[\frac{12\text{ in}}{1\text{ ft}}\right]$ = 95040 in;

8. 800 ft • $\left[\frac{12\text{ in}}{1\text{ ft}}\right]\left[\frac{2.54\text{ cm}}{1\text{ in}}\right]\left[\frac{1\text{ m}}{100\text{ cm}}\right]\left[\frac{1\text{ km}}{1000\text{ m}}\right]$ = 0.24384 km; **9.** 35.8 cm • $\left[\frac{10\text{ mm}}{1\text{ cm}}\right]$ = 358 mm;

10. 100 in • $\left[\frac{2.54\text{ cm}}{1\text{ in}}\right]\left[\frac{1\text{ m}}{100\text{ cm}}\right]\left[\frac{1\text{ km}}{1000\text{ m}}\right]$ = 0.00254 km; **11.** 633 gal • $\left[\frac{3{,}78541\text{ L}}{1\text{ gal}}\right]\left[\frac{1\text{ m}^3}{1000\text{ L}}\right]$ = 2.396 m^3;

12. 572 qt • $\left[\frac{1\text{ gal}}{4\text{ qt}}\right]\left[\frac{3.78541\text{ L}}{1\text{ gal}}\right]$ = 541.3 L; **13.** 178 in^3 • $\left[\frac{16.3870\text{ cm}^3}{1\text{ in}^3}\right]\left[\frac{1\text{ mL}}{1\text{ cm}^3}\right]$ = 2916.9 mL;

14. 11 ft^3 • $\left[\frac{1728\text{ in}^3}{1\text{ ft}^3}\right]\left[\frac{16.3870\text{ cm}^3}{1\text{ in}^3}\right]$ = 311484.1 cm^3;

15. 535 barrel • $\left[\frac{42\text{ gal}}{1\text{ barrel}}\right]\left[\frac{231\text{ in}^3}{1\text{ gal}}\right]\left[\frac{1\text{ ft}^3}{1728\text{ in}^3}\right]$ = 3003.8 ft^3;

16. 31 cm^3 • $\left[\frac{1\text{ mL}}{1\text{ cm}^3}\right]\left[\frac{1\text{ L}}{1000\text{ mL}}\right]\left[\frac{1\text{ m}^3}{1000\text{ L}}\right]$ = 0.000031 m^3; **17.** 934 m^3 • $\left[\frac{1000\text{ L}}{1\text{ m}^3}\right]$ = 934000 L;

18. 45 L • $\left[\frac{1\text{ gal}}{3.78541\text{ L}}\right]\left[\frac{1\text{ barrel}}{42\text{ gal}}\right]$ = 0.28304 barrel;

19. 57 mL • $\left[\frac{1\text{ L}}{1000\text{ mL}}\right]\left[\frac{1\text{ gal}}{3.78541\text{ L}}\right]\left[\frac{4\text{ qt}}{1\text{ gal}}\right]$ = 0.06023 qt; **20.** 32 m^3 • $\left[\frac{1000\text{ L}}{1\text{ m}^3}\right]\left[\frac{1\text{ gal}}{3.78541\text{ L}}\right]$ = 8453.5 gal;

21. 517 mg • $\left[\frac{1\text{ g}}{1000\text{ mg}}\right]\left[\frac{1\text{ lb}}{454\text{ g}}\right]$ = 0.0011388 lb; **22.** 911 g • $\left[\frac{1\text{ lb}}{454\text{ g}}\right]\left[\frac{16\text{ oz}}{1\text{ lb}}\right]$ = 32.1057 oz;

23. 54 ton • $\left[\frac{2000\text{ lb}}{1\text{ ton}}\right]\left[\frac{1\text{ kg}}{2.20462\text{ lb}}\right]\left[\frac{1\text{ metric ton}}{1000\text{ kg}}\right]$ = 48.988 metric ton;

24. 95 metric ton • $\left[\frac{1000\text{ kg}}{1\text{ metric ton}}\right]\left[\frac{2.20462\text{ lb}}{1\text{ kg}}\right]\left[\frac{16\text{ oz}}{1\text{ lb}}\right]\left[\frac{437.5\text{ grains}}{1\text{ oz}}\right]$ = 1466072300 grains;

25. 79 lb • $\left[\frac{1\text{ kg}}{2.20462\text{ lb}}\right]$ = 35.8338 kg;

26. 14 grains • $\left[\frac{1\text{ oz}}{437.5\text{ grains}}\right]\left[\frac{1\text{ lb}}{16\text{ oz}}\right]\left[\frac{1\text{ kg}}{2.20462\text{ lb}}\right]\left[\frac{1\text{ metric ton}}{1000\text{ kg}}\right]$ = 0.0000009072 metric ton;

27. 33 oz • $\left[\frac{1\text{ lb}}{16\text{ oz}}\right]\left[\frac{454\text{ g}}{1\text{ lb}}\right]$ = 936.375 g; **28.** 6 metric ton • $\left[\frac{1000\text{ kg}}{1\text{ metric ton}}\right]\left[\frac{2.20462\text{ lb}}{1\text{ kg}}\right]$ = 13227.72 lb;

29. 81 lb • $\left[\frac{454\text{ g}}{1\text{ lb}}\right]\left[\frac{1000\text{ mg}}{1\text{ g}}\right]$ = 36774000 mg; **30.** 47 kg • $\left[\frac{2.20462\text{ lb}}{1\text{ kg}}\right]\left[\frac{1\text{ ton}}{2000\text{ lb}}\right]$ = 0.05181 ton

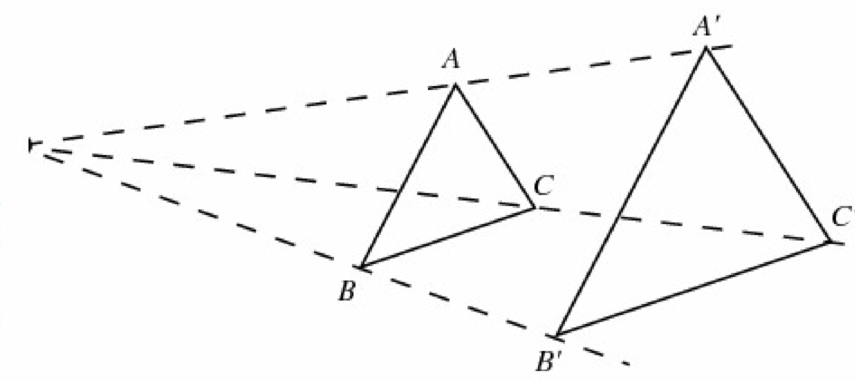
Exercise 8.10

1. 30.702 g; **2.** 75.492 ml; **3.** 719.565 g; **4.** 0.750 g/mL; **5.** 0.119 g/mL; **6.** 63.296 g; **7.** 1.018 g/mL; **8.** 11.908 g; **9.** 1.003 g/mL; **10.** 287.948 mL; **11.** 0.287 g/mL; **12.** 53.742 g; **13.** 2.525 g/mL; **14.** 10.920 g/mL; **15.** 195.380 mL; **16.** 74.725 mL; **17.** 0.174 g/mL; **18.** 26.096 g; **19.** 650.316 mL; **20.** 100.788 g

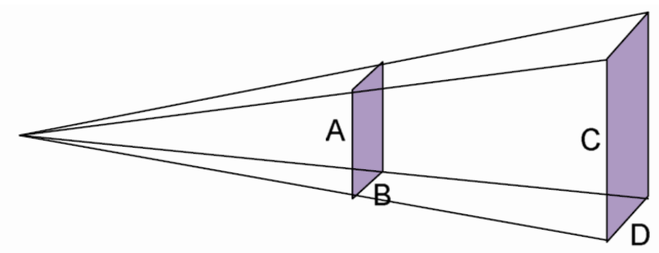
Chapter 9: Triangles, Trigonometry, and Transformations

9.1 Figures that are Similar

Two figures are **similar** if they have the same shape but are not necessarily the same size. The figure below shows two similar triangles. Triangle A'B'C' is a larger scale projection of triangle ABC.



Specifically, in similar figures, the angles are **congruent** or have the same measure, and the ratios of the lengths of corresponding sides are equal. Since this is the case, we can use the ratio (or proportion) of one shape in a pair of similar figures to find out information about the other shape. For example: suppose you knew the length and width measurements of the smaller rectangle shown below, and you knew the length, but not the width of the larger rectangle. What could you do?



Because similar shapes have congruent angles, the measurements of one shape are proportional to the measurements of the other similar shape. For example, if we have the length A and the width B of the small rectangle shown above, and we only have the length C of the larger, but similar shape rectangle, we can obtain the width D using a proportion of the measurements:

$$\frac{A}{B} = \frac{C}{D}$$

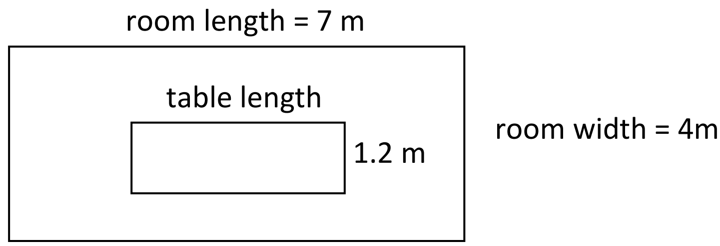
The ratio of side A to B equals the ratio of side C to D. Having three of four measurements allows us to solve the equation for the unknown width (D) of the larger rectangle. To solve for D, we first multiply both sides of the equation by D, yielding $D \bullet \frac{A}{B} = \frac{C}{D} \bullet D$. This simplifies to $D \bullet \frac{A}{B} = C$. Now multiplying both sides by $\frac{B}{A}$, we obtain

$$D = \frac{BC}{A}$$

How to Use Proportions to Solve a Similar-Figure Problem

Karla wants to put a table exactly in the middle of the room in a perfect, symmetrical way such that the table has the exact same proportions as the room. If the room is 7m x 4m, and Karla wants the table to have a width of 1.2 m width, what should be the length of the table?

First, let's draw a diagram of what is given in the problem:



Next, we must set the proportion of the room length and width equal to the proportion of the table length and width,

$$\frac{room\ width}{room\ length} = \frac{table\ width}{table\ length}$$

Now, substituting the known measurements, we have,

$$\frac{4\text{ m}}{7\text{ m}} = \frac{1.2\text{ m}}{table\ length}$$

Now, since *table length* is in the denominator; our goal is solve for the *table length* and so we can move this variable to the numerator by multiplying both sides of the equation by the *table length*, to obtain

$$table\ length \bullet \frac{4\text{ m}}{7\text{ m}} = 1.2\text{ m}$$

Now, since *table length* is multiplied by $\frac{4}{7}$, to find the variable (*table length*), we must multiply both sides of the equation by the reciprocal, $\frac{7}{4}$, to obtain

$$table\ length \bullet \frac{4\text{ m}}{7\text{ m}} \bullet \frac{\mathbf{7\ m}}{\mathbf{4\ m}} = 1.2\text{ m} \bullet \frac{\mathbf{7\ m}}{\mathbf{4\ m}}$$

Then simplifying, we obtain

$$table\ length = \frac{1.2\text{ m} \bullet 7\text{ m}}{4\text{m}}$$ and then finally, this simplifies to

$$table\ length = \frac{8.4\text{ m}^2}{4\text{ m}} = 2.1\text{ m}$$

Note: When setting up the proportions, we could have used $\frac{room\ length}{room\ width} = \frac{table\ length}{table\ width}$
It does not matter whether length is in the numerator or denominator, so long as both proportions are constructed consistently the same way.

Exercise 9.1

Solve for the unknown in each given proportion:

1. $\frac{9}{2} = \frac{G}{1}$

2. $\frac{8}{9} = \frac{3}{K}$

3. $\frac{89}{N} = \frac{2}{51}$

4. $\frac{8}{86} = \frac{M}{6}$

5. $\frac{9}{84} = \frac{65}{J}$

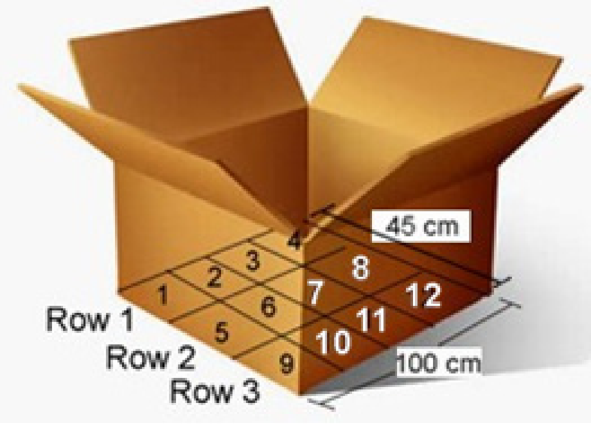
6. $\frac{P}{19} = \frac{5}{96}$

7. $\frac{23}{65} = \frac{S}{32}$

8. $\frac{35}{Z} = \frac{21}{75}$

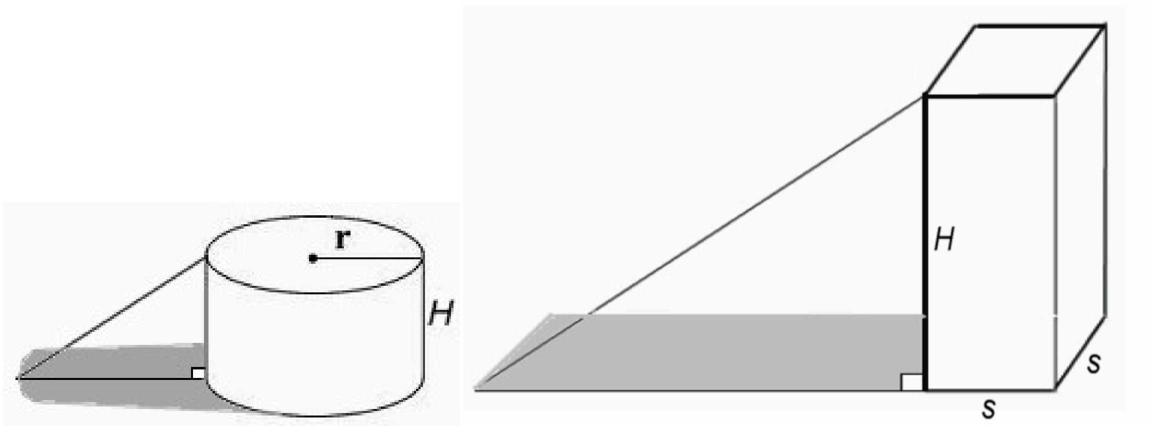
9. Anne wants to put a coffee table exactly in the middle of the living room The table must have the exact same proportions as the living room. If the room is 5m x 3m, and Anne wants the table to have an 0.8m width, what should be the length (longer dimension) of the table rounded to the nearest hundredths?

10. Peter is packing books in a box with a similar shape, he manages to fit 12 books perfectly in three rows of books on each layer (as shown below); if the base of the box is 45 cm x 100 cm, what are the dimensions of each book?



9.2 Right Triangles that are Similar

Previously, we saw how certain similar figures have the same interior angle measurements, but are of different sizes. The same holds true for triangles. A **right triangle** is a triangle that has a right angle (an angle of exactly 90° degrees). Right triangles can be used when we are trying to measure the height of an object.



If you have a cylinder that represents a silo and a square prism that represents a skyscraper at a given moment during daylight, the buildings are going to cast a shadow on the earth, and if you mark the point on the ground were the shadow of both shadows end, you will see that on each building a right triangle is formed. Both right triangles shown above have similar shapes and angles. What if you knew the length of the shadows of both figures and the height of the silo, but you did not know the height of the skyscraper. What could you do?

<u>How to Solve Right-Triangle Problems Using Proportion</u>

You could use what you learned previously to solve a similar-figure problem using proportions. A painting company is hired by the maintenance department of a skyscraper to paint the lateral area (or sides) of the building. The painting company knows that 1 liter of paint covers an area of 10 m^2. They also know that the building has the shape of a squared prism with a 30m x 30m base, but the height of the building is unknown. (Note: windows on a building will not be painted, so the company is going to consider that the area of the windows in the building is negligible.) Recently, however, they completed the painting of a nearby silo with a height of 35m. At 9:30 am the silo and the skyscraper cast a shadow on the ground of 25m and 75m, respectively. What is the height of the skyscraper? And how many liters of paint does the painting company need to paint the skyscraper if a side of the base of the skyscraper measures 50m?

Since the angle of the light and the projection of the shadow is the same for both buildings the right triangles formed on both structures are similar, so we can use proportions to solve this problem:

$$\frac{height\ of\ silo}{length\ of\ silo\ shadow} = \frac{height\ of\ skyscraper}{length\ of\ skyscraper\ shadow}$$

Let's substitute in the known measurements,

$$\frac{35\text{ m}}{25\text{ m}} = \frac{height\ of\ skyscraper}{75\text{ m}}$$

$$height\ of\ skyscraper = \frac{35\text{ m} \bullet 75\text{ m}}{25\text{ m}} = 105\text{ m}$$

Next, to determine the amount of paint, we first need to compute the lateral area of the skyscraper:

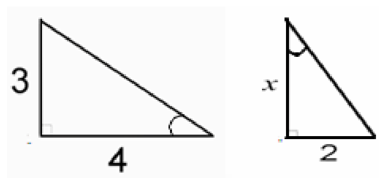
lateral area = perimeter of base • height = 4•(50 m)•(105 m) = 21000 m^2

Since the company can paint an area of 10 m^2 per liter of paint, the number of liters required is:

$$\frac{21000\text{ m}^2}{10\text{ m}^2/liter} = 2100\ liters$$

Notice that we divided the lateral area by the coverage area of the paint and that this yielded the proper units in the answer--liters of paint. As we discussed in an earlier chapter, this is also called dimensional analysis--that is, analyzing the units to insure we do the calculation properly.

Let's try one more problem that is "tricky" and demonstrates a caution that students should be aware of when given the two similar triangles shown on the right. To find the unknown value x, we might write the proportions as follows:



$$\frac{3}{4} = \frac{x}{2}$$

Solving for the unknown x by multiplying both sides of the equation by 2, we have

$$x = \frac{2 \bullet 3}{4} = \frac{6}{4} = 1.5$$

Please note that the above answer is completely INCORRECT. The orientation of the triangles might have fooled you because the rightmost triangle has been rotated (and mirrored). To use proportions properly, we must write similar proportions,

$$\frac{\boldsymbol{short\ side\ of\ 1st\ triangle}}{\boldsymbol{long\ side\ of\ 1st\ triangle}} = \frac{\boldsymbol{short\ side\ of\ 2nd\ triangle}}{\boldsymbol{long\ side\ of\ 2nd\ triangle}}$$

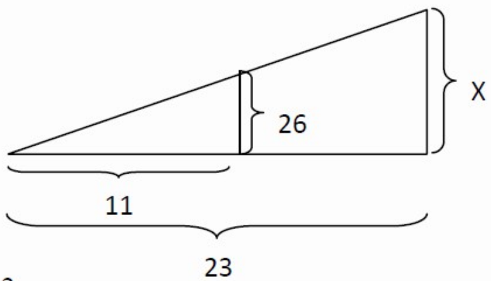
Thus, we should have correctly written the proportion as: $\frac{3}{4} = \frac{2}{x}$

To solve for x, we multiply both sides of the equation by x to obtain $\frac{3x}{4} = 2$, then since x is multiplied by $\frac{3}{4}$, we multiply by the reciprocal, $\frac{4}{3}$, to obtain $x = 2 \bullet \frac{4}{3} = \frac{8}{3}$ or $x = 2\frac{2}{3}$.

Exercise 9.2

Find the missing measurement (x) given the similar right-triangles (one smaller right triangle is contained within a larger right triangle).

1.



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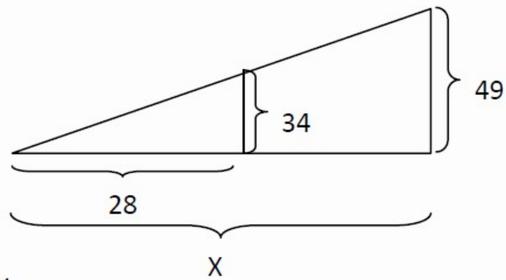

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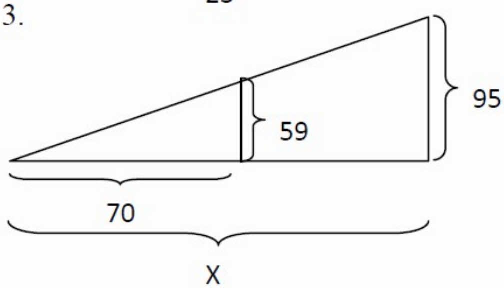

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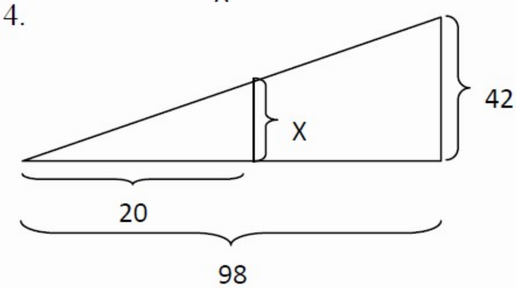

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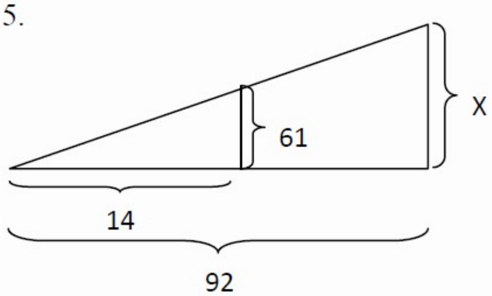

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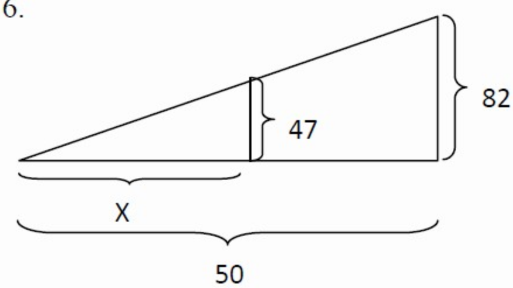

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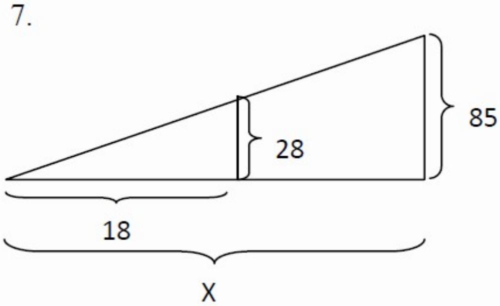

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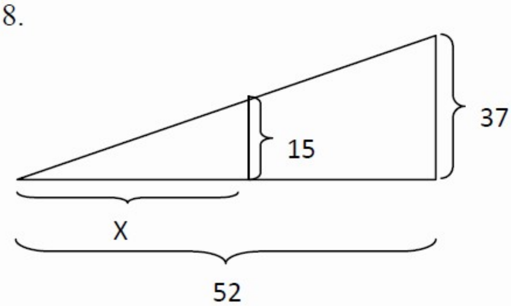


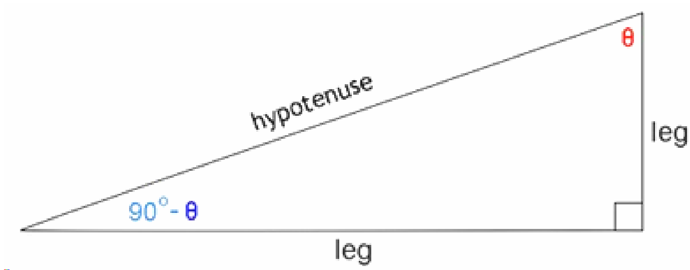
9. John’s height is 1.6m. At 5:30 pm John’s shadow is 2.5 m while his little brother’s shadow is 1.8m. How tall is John’s little brother?

10. At 4:45 pm a street lamp casts a 15m long shadow and an adjacent building casts a 25 m long shadow. If the street lamp’s height is 6 m, how tall is the building?

9.3 Sine and Cosine of an Angle

We've just been working with similar figures and similar right triangles and have learned how to find the length of sides by using relationships and ratios between two similar triangles. Now we are going to learn to use some t**rigonometric ratios** that can be useful for finding angle measurements and side lengths. To understand how we can apply trigonometric ratios, we first need to be familiar with some terminology describing the components of the right triangle.

In a right triangle the sides that form the right angle of the right triangle are called the **legs**. The side opposite the right angle, which is always the longest side, is called the **hypotenuse**.



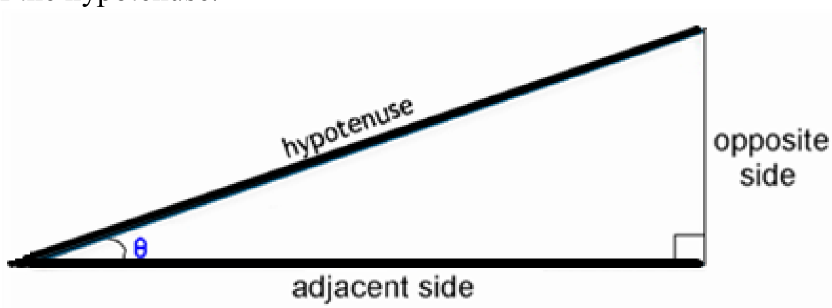
The sum of the inner angles of any triangle is 180°. In the case of a right triangle, one of the three inner angles is 90°, so you can deduce that the other two inner angles should add up 90° as well. If we know one of the interior angles of the right triangle is 30° (θ = 30°), for example, the other inner angle is 90° – θ (the Greek lowercase letter theta, pronounced *thā'tah*), or 90° – 30° = 60°. We could also use the fact that the *unknown angle* plus the sum of the right angle and known angle (30°) must total 180° or

$$\textit{unknown angle} + 90° + 30° = 180°$$

Solving this equation for the *unknown angle*, we have *unknown angle* = 60°.

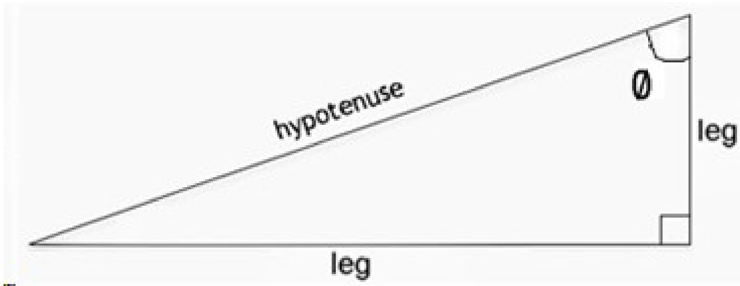
There are six right triangle relationships, called trigonometric ratios: sine, cosine, tangent, cotangent, secant and cosecant, and all of these can be defined in terms of the ratios of two of three sides of a right angle. In this section we are going to analyze the trigonometric ratios known as sine (pronounced *sign*) and cosine (pronounced *co-sign*).

The **sine of an angle** in a right triangle is the ratio of the length of the leg opposite an acute angle θ to the length of the hypotenuse.

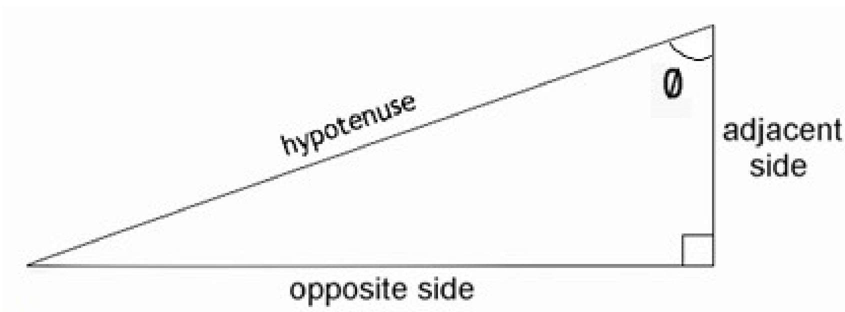


Notice that there are two sides (that are bold) that form the angle labeled θ in the triangle above—the hypotenuse and the **adjacent side**. The adjacent side is always the leg of the right triangle that forms the angle that we are interested in. The adjacent side and the hypotenuse meet at a vertex to form angle θ. The length of the adjacent side will always be less than the hypotenuse (which is the longest side of the right triangle). The **opposite side** of angle θ is the side that does not form the angle.

Suppose we are interested in a different angle ∅ (phi) as shown in the triangle below. Can you identify which legs represent the opposite and adjacent sides to that angle?



Since the vertical leg of the right triangle and the hypotenuse form the angle ∅ the vertical leg is the adjacent side of the angle. The horizontal leg of the right triangle is thus the opposite side to the angle ∅ shown above. Based on the labeling of angle ∅ on the above triangle, we now have identified the adjacent and opposite sides properly below:



So, again, we have the sine of angle ∅ (phi) [abbreviated as $sin(\emptyset)$] is defined as

$$sin(\emptyset) = \frac{opposite\ side}{hypotenuse}$$

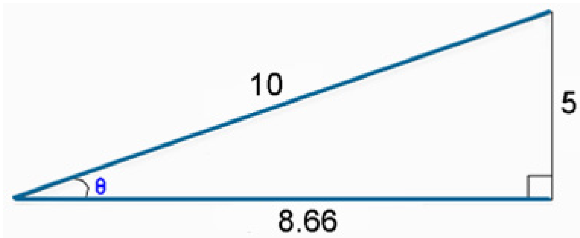
The **cosine of an angle** in a right triangle is the ratio of the length of the leg adjacent to an acute angle to the length of the hypotenuse. Thus, the cosine of angle ∅ [abbreviated as $cos(\emptyset)$] is defined as

$$cos(\emptyset) = \frac{adjacent\ side}{hypotenuse}$$

Enough theory! Now let's apply what we have just learned to some practical problems.

Finding the Sine and Cosine of an Angle

Consider the right triangle shown below:



The sine of angle θ is by definition:

$$sin(\theta) = \frac{opposite\ side}{hypotenuse}$$

Substituting the lengths of the opposite side (5) and hypotenuse (10), we have

$$sin(\theta) = \frac{5}{10} \qquad \text{or} \qquad sin(\theta) = 0.5$$

Now, in the table of trigonometric values (a complete table for angles ranging from 0° to 90° is given in Appendix I), scan down the column labeled "Sine" and find the value closest to 0.5. Notice the value 0.50000 (which happens to equal 0.5) corresponds to 30°.

Degrees	Sine	Cosine	Tangent
25	0.42262	0.90631	0.46631
26	0.43837	0.89879	0.48773
27	0.45399	0.89101	0.50953
28	0.46947	0.88295	0.53171
29	0.48481	0.87462	0.55431
30	**0.50000**	0.86603	0.57735
31	0.51504	0.85717	0.60086
32	0.52992	0.84805	0.62487
33	0.54464	0.83867	0.64941
34	0.55919	0.82904	0.67451
35	0.57358	0.81915	0.70021

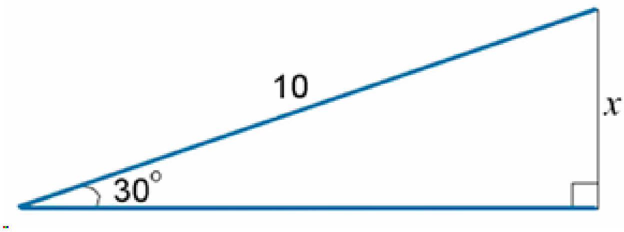
So, using trigonometry, more specifically the definition of sine, we have determined that angle θ = 30°. When we locate the angle (in degrees) that corresponds to the value for sin θ, we are performing what is called an inverse sine (or arc sine) function. On a "trig" calculator in degree mode (not radian mode), if we select the **inverse sine** function key and enter 0.5, this will yield an angle of 30°. In mathematical terms, $sin^{-1}(0.5) = 30°$ and in general the $sin^{-1}[sin(\theta)] = \theta$.

Alternatively, we could have used the definition of cosine, where $cos(\theta) = \frac{adj.\ side}{hypotenuse} = \frac{8.66}{10} =$ 0.866. Now if we go to Appendix I and scan down the 3rd column, we find that 0.86603 is the closest value to our 0.866 and this again corresponds to an angle θ = 30°. On a trig calculator we will find that we can use the inverse cosine function to yield $cos^{-1}(0.866) = 30.0029109312°$ or approximately 30°.

In geometry, lowercase Greek letters (besides θ) are also commonly used to represent angles such as ∅ (phi), α (alpha), β (beta), γ (gamma), δ (delta), etc. Just as the measurement of the length of a side of a triangle may be referred to as *x*, *y*, *a*, *b*, *c*, or *w* (or for that matter, any other variable name), so the measurement of a given angle of a triangle may be referred to or labeled with any one of the letters of the Greek alphabet.

Finding the Side of a Right Triangle using Sine and Cosine

Let's see another application of trigonometry to find the length x of a side of the following triangle:



First, notice that the side labeled x is the opposite side of the 30° angle. Also, we are given the length of the hypotenuse. The trigonometric ratio that involves the opposite side and hypotenuse is sine. So, again, we can write the following relationship:

$$sin(30°) = \frac{opposite\ side}{hypotenuse}$$

Now, let's substitute the given lengths:

$$sin(30°) = \frac{x}{10}$$

To solve for x, we multiply both sides of the equation by 10 to obtain

$$10 \bullet [sin(30°)] = x$$

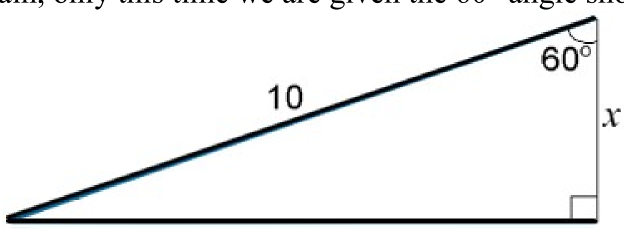
From the table of trigonometric values, we note that the value corresponding to the "Sine" of 30° is 0.50000. Performing this final substitution we have

$$10 \bullet (0.50000) = x$$

or

$$x = 5$$

Let's try the problem again, only this time we are given the 60° angle shown below:



First, notice that the side x is now the adjacent side to angle 60°. Since we are given the adjacent side and the hypotenuse, we can use the definition of cosine and write,

$$cos(60°) = \frac{adjacent\ side}{hypotenuse}$$

Substituting the values we know from the given triangle, we have

$$cos(60°) = \frac{x}{10}$$

Solving for x, we have

$$10•[cos(60°)] = x$$

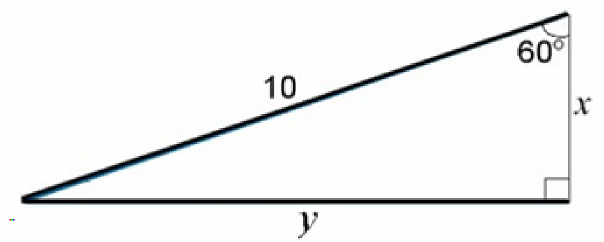
From Appendix I, we can find 60° in the table (in the leftmost column labeled "Degrees"), and then in the column labeled "Cosine", we find that the $cos(60°) = 0.50000$.

So, substituting the value of the $cos(60°)$ into the equation we have

$$10•(0.50000) = x \quad \text{or} \quad x = 5$$

Did you notice that the $cos(30°) = sin(60°) = 0.5$? In general $cos(θ) = sin\ (90° – θ)$.

Now, let's try one more example problem. Previously, we found the value of x (which was the adjacent side to the 60° angle). Suppose we want to find the value of y in the right triangle shown.



First, we ask ourselves what trigonometric function (*sine* or *cosine*) involves the opposite side (which is the side with measurement y) and the hypotenuse. You are correct if you said "sine". So we can write the definition that the sine of 60° is equal to the opposite side divided by the hypotenuse (10) as follows:

$$sin(60°) = \frac{y}{10}$$

Now, we solve for y by multiplying each side of the equation by 10, or $10 • [sin(60°)] = \frac{y}{10} • 10$. We obtain the value for the $sin(60°)$ from the trigonometric table in Appendix I by locating the "60" in the first column labeled "Degrees", and then in the second row labeled "Sine", we find the value 0.86603. Substituting this into the equation we have $10 • (0.86603) = y$ or $y = 8.6603$.

Summary of Right Triangle Sides and Relationships

Right Triangle	**Definition**
adjacent side	Leg of a right triangle that forms an acute angle.
opposite side	Leg of a right triangle that does not form the angle and is opposite the angle.
hypotenuse	Longest side of a right triangle ; side opposite the right angle.
$sin(θ)$	opposite side/hypotenuse
$cos(θ)$	adjacent side/hypotenuse

Exercise 9.3

(a) Find the value of *sin*(θ) for each triangle, then (b) using the trigonometric table in Appendix I, find the angle θ rounded to the nearest degree; next, (c) find the corresponding value of the *cos*(θ) from the table, then (d) utilize that value and the fact that the *cos*(θ) = $\frac{adjacent\ side}{hypotenuse}$ to solve for the measurement of the adjacent side rounded to the nearest tenths.

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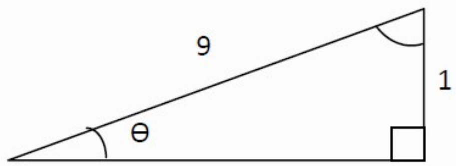

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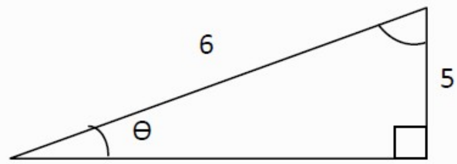

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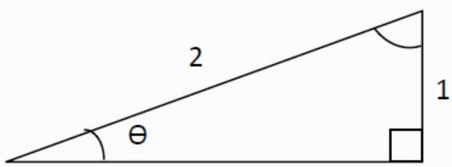

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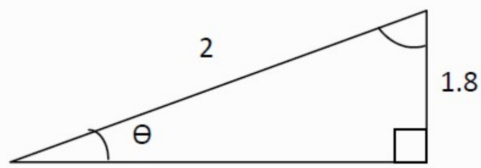

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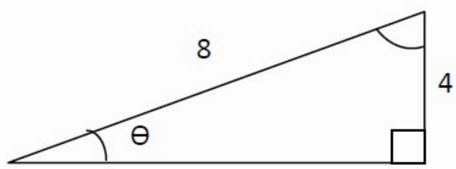

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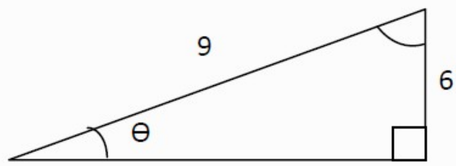

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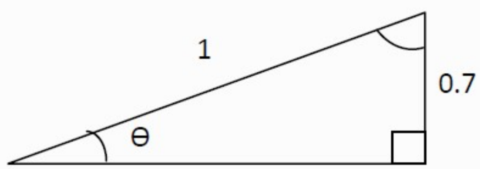

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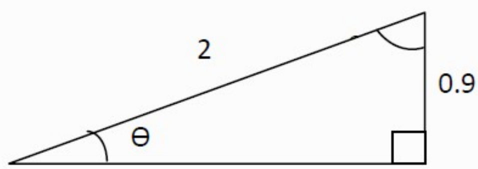

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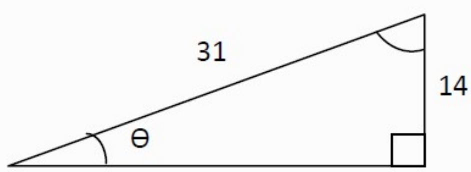

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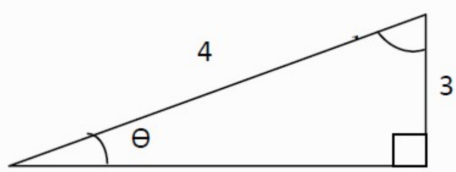
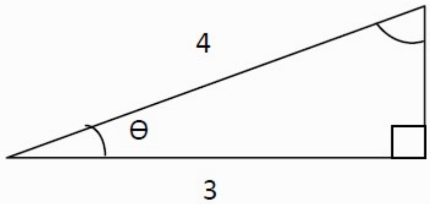


(a) Find the value of $cos(\theta)$ for each triangle, then (b) using the trigonometric table in Appendix I, find the angle θ rounded to the nearest degree; next, (c) find the corresponding value of $sin(\theta)$ from the table, then (d) utilize that value and the fact that the $sin(\theta) = \dfrac{opposite\ side}{hypotenuse}$ to solve for the measurement of the opposite side rounded to the nearest tenths.

11.



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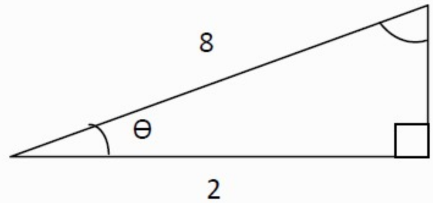
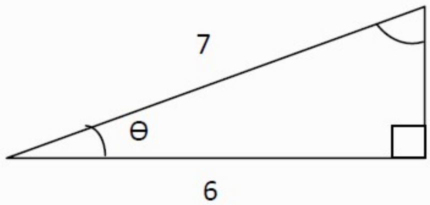


13.



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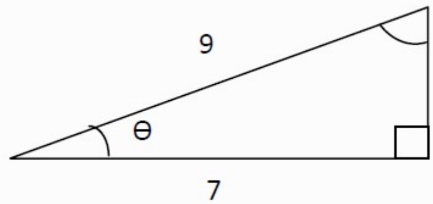

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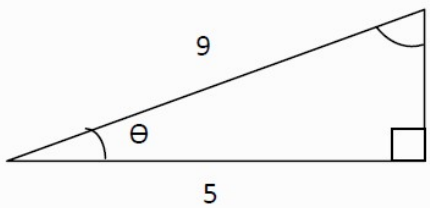

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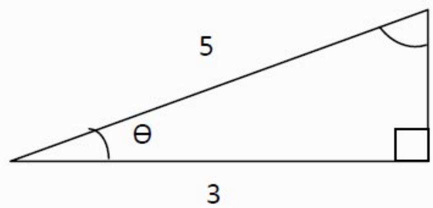

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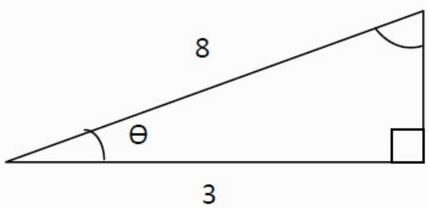

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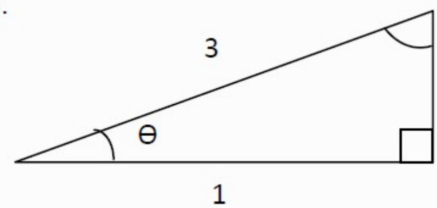

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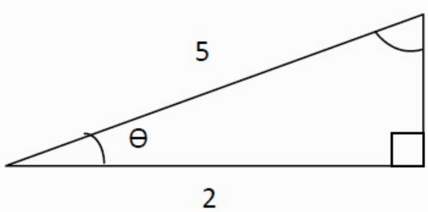

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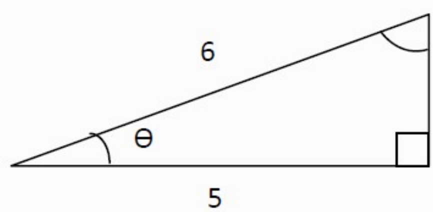
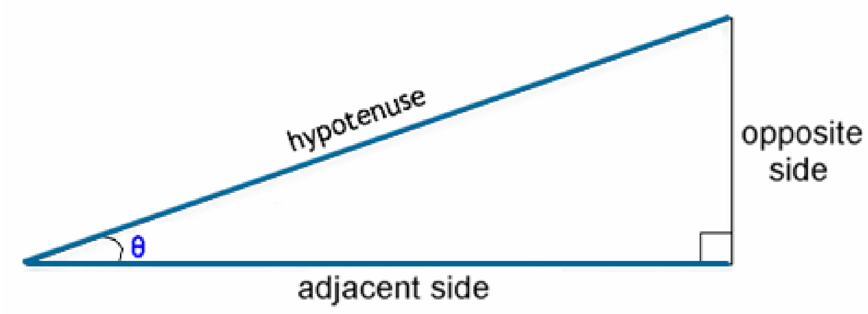


9.4 Tangent of an Angle

Previously, we learned the definitions of sine and cosine of an angle of a right triangle. Now, we are going to consider another trigonometric ratio known as the tangent.
The **tangent of an angle** in a right triangle is the ratio of the length of the side opposite an acute angle to the length of the side adjacent to (or that forms) the acute angle.



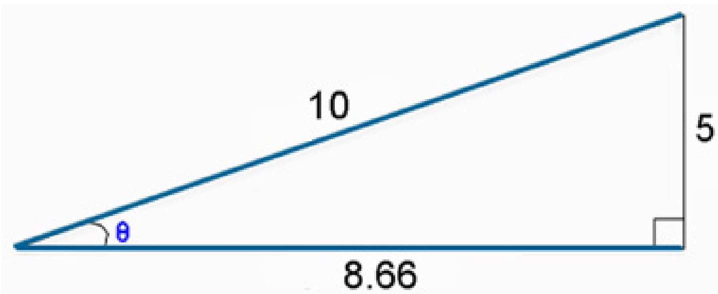
In mathematical terms, tangent is defined as follows:

$$tan(\theta) = \frac{opposite\ side}{adjacent\ side}$$

In just a moment, we will see how to apply the definition of tangent to find an interior angle of a right triangle or the length of one of the sides. But before proceeding, the remaining three trigonometric ratios—cotangent [$cot(\theta)$], secant [$sec(\theta)$, pronounced *see'cant*] and cosecant [$csc(\theta)$]—are simply the inverse (or reciprocal) of the tangent, cosine, and sine ratios:

$$cot(\theta) = \frac{1}{\tan(\theta)} = \frac{adjacent\ side}{opposite\ side} \qquad sec(\theta) = \frac{1}{cos(\theta)} = \frac{hypotenuse}{adj.\ side} \qquad csc(\theta) = \frac{1}{\sin(\theta)} = \frac{hypotenuse}{opp.side}$$

You need not be concerned with these last three trigonometric ratios as we will not be using them in pre-algebra.



Let's apply the definition of tangent and find the tangent of angle θ as shown in the right triangle above. Since the leg with measurement of 8.66 forms the angle, this is the adjacent side. The leg with measurement 5 is the opposite side to angle θ. So now we can substitute the proper values into the definition of tangent and write:

$$tan(\theta) = \frac{opposite\ side}{adjacent\ side} = \frac{5}{8.66} = 0.577367...$$

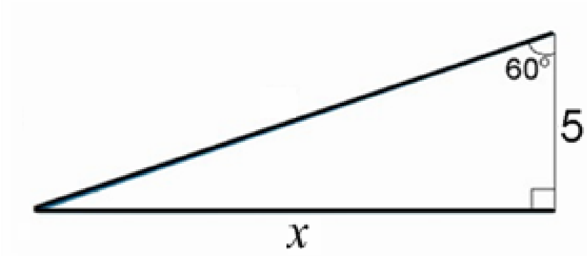
Now, in the trigonometric table given in Appendix I, (and abbreviated here), look down the column labeled "Tangent" until you come to the value closest to 0.57736 that we computed above. Notice that the value 0.57735 is very close and this corresponds to an angle of 30°. So, using trigonometry, more specifically the definition of tangent, we have determined the angle θ = 30°. If a tangent value, such as 0.576, lies between two entries in the table (0.55431 and 0.57735), find the closest value 0.57735 (since the difference 0.57735 – 0.576 = 0.00135 is less than 0.576 – 0.55431 = 0.02169) and then locate the corresponding angle.

Degrees	Sine	Cosine	Tangent
25	0.42262	0.90631	0.46631
26	0.43837	0.89879	0.48773
27	0.45399	0.89101	0.50953
28	0.46947	0.88295	0.53171
29	0.48481	0.87462	0.55431
30	0.50000	0.86603	**0.57735**
31	0.51504	0.85717	0.60086
32	0.52992	0.84805	0.62487
33	0.54464	0.83867	0.64941
34	0.55919	0.82904	0.67451
35	0.57358	0.81915	0.70021

On a trigonometric calculator, make certain the calculator is in degree (not radian) mode. Then select INV (for inverse), then tan^{-1}() and key in 0.57736 and press equal to obtain the corresponding degrees (30°). Some calculators use the key "atan" or "arc tan" instead of tan^{-1}.

Let's use the tangent ratio to find the measurement x of the leg of a right triangle shown below:

First, we identify the adjacent and opposite sides. Since the leg with measurement of 5 forms the 60° angle, that leg is the adjacent side to the angle. That leaves x as the opposite side to the 60° angle, since the longest side (which is not labeled) is always the hypotenuse.



Thus, applying the definition of tangent, we have, $tan(60°) = \frac{opposite\ side}{adjacent\ side} = \frac{x}{5}$

Solving for x, we multiply both sides of the equation by 5 to obtain,

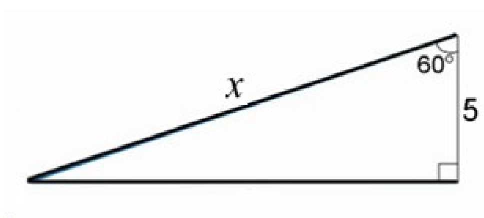
$$5 \bullet [tan(60°)] = x$$

Now, using the trigonometric table given in Appendix I, we find 60 in the first column labeled "Degrees" and then corresponding to that angle, we find 1.73205 in the fourth column labeled "Tangent", so that $tan(60°) = 1.73205$. Substituting the value for $tan(60°)$, we have,

$$5 \bullet (1.73205) = x$$

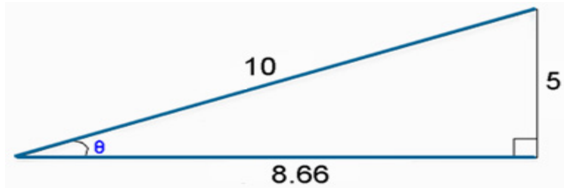
or

$$x = 8.6603 \text{ (to the nearest ten thousandth)}$$



Finally, let's now take the same triangle and this time let x be the length of the hypotenuse—so we will now find x. Since we have the length of the adjacent side (5) to the angle and the hypotenuse, we can write: $sin\ 60° = \frac{5}{x}$. Since the variable (x) is in the denominator, let's multiply both sides of the equation by x to obtain $x \bullet sin\ 60° = 5$.

Now that we have the variable in the numerator, let's divide both sides by *sin* 60° we have the solution: $x = \frac{5}{sin\ 60°} = \frac{5}{0.5} = 10.$



Note this important point about fractions:
Given this triangle on the right, you might be tempted to write *tan*(θ) = $\frac{opposite\ side}{adjacent\ side} = \frac{5}{8.66}$.
But it is unconventional to express a fraction with a numerator and/or denominator that is a decimal number. Thus, since 8.66 has two decimal digits, we can eliminate the decimal digits by multiplying by $\frac{100}{100}$ to obtain an equivalent fraction with whole numbers: $\frac{500}{866}$; next, this fraction can be reduced by dividing numerator and denominator by 2 to yield a more conventional fraction: $\frac{250}{433}$.

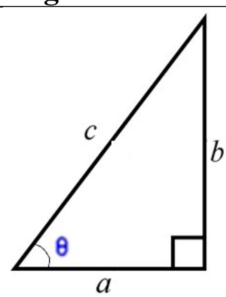
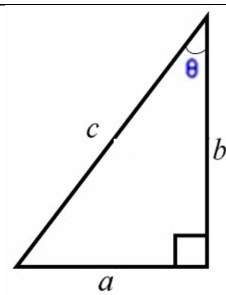
I (the author) can always recall the standard trigonometric definitions by remembering **SOH-CAH-TOA** (pronounced *soak-ă-toe-ă*), where **S**=sine, **C**=cosine, **T**=tangent, **O**=opposite side, and **A**=adjacent side and each three-letter combination is a mnemonic to assist the memory of these definitions:

SOH means ***S****in*(θ) = $\frac{\boldsymbol{O}pposite\ side}{\boldsymbol{H}ypotenuse}$

CAH means ***C****os*(θ) = $\frac{\boldsymbol{A}djacent\ side}{\boldsymbol{H}ypotenuse}$

TOA means ***T****an*(θ) = $\frac{\boldsymbol{O}pposite\ side}{\boldsymbol{A}djacent\ side}$

Summary of Sine, Cosine, and Tangent Definitions

Trigonometric function	**Definition**	**To find the angle θ on a calculator, use the inverse function**		
sin(θ)	$\frac{opposite\ side}{hypotenuse}$	$sin^{-1}(sin\ \theta)$	$\frac{b}{c}$	$\frac{a}{c}$
cot(θ)	$\frac{adjacent\ side}{hypotenuse}$	$cos^{-1}(cos\ \theta)$	$\frac{a}{c}$	$\frac{b}{c}$
tan(θ)	$\frac{opposite\ side}{adjacent\ side}$	$tan^{-1}(tan\ \theta)$	$\frac{b}{a}$	$\frac{a}{b}$

Exercise 9.4

For each triangle, (a) find the tangent of angle θ (tan θ) and reduce any fractions to lowest terms (with whole numbers in the fraction), then (b) find the measurement of the angle rounded to the nearest degree either using the trigonometric table in Appendix I or using the inverse tangent ($\tan^{-1}$ or atan or arctan) function on a calculator.

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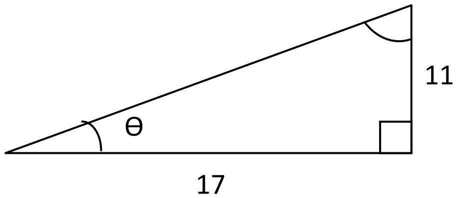

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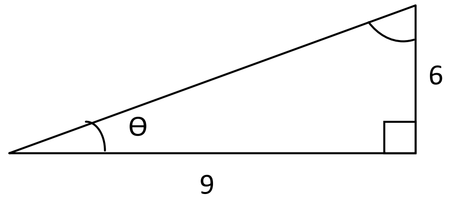

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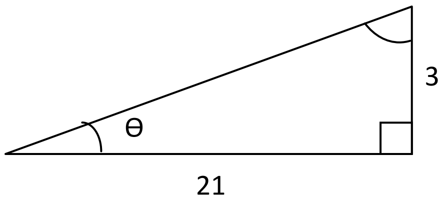

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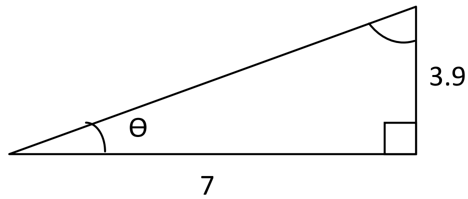

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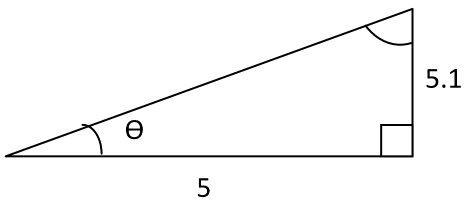

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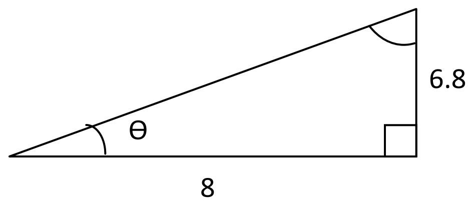

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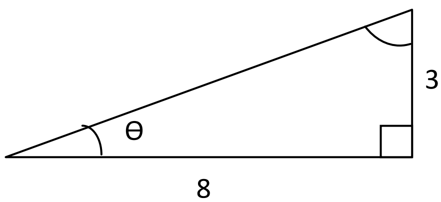

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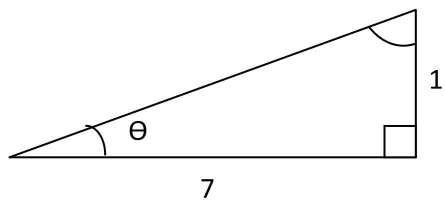

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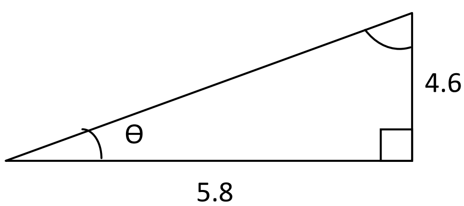

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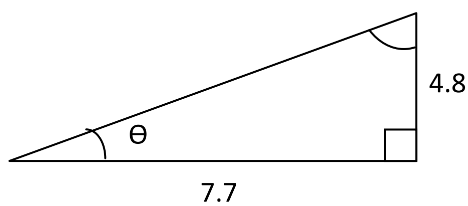

Use sine, cosine or tangent (as appropriate) to find the measurement of side *x* rounded to the nearest hundredth. (Note: Use Appendix I.)

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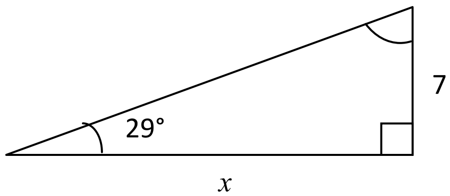

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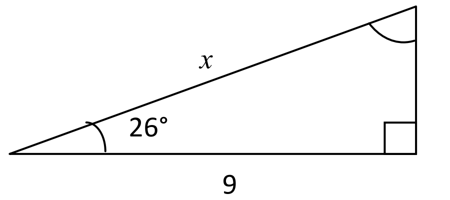

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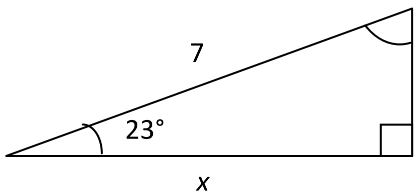

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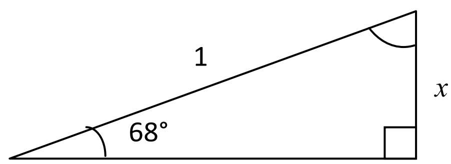

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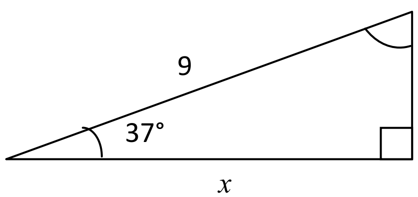

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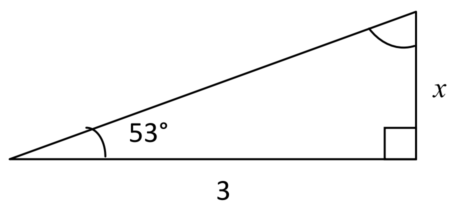

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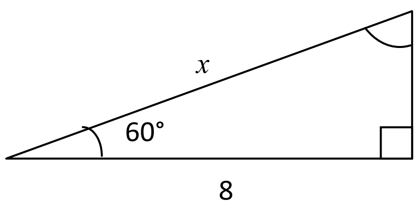

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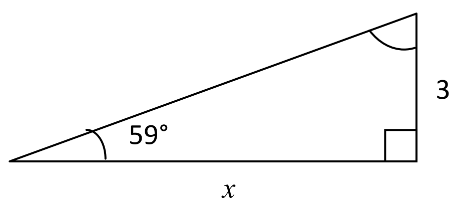

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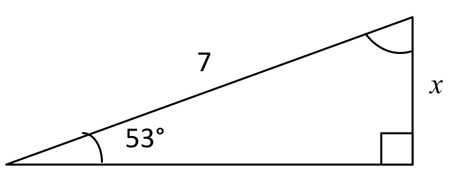

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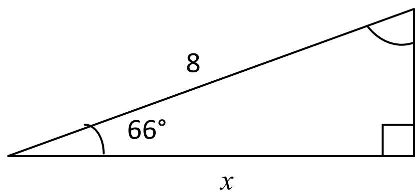

(a) Find the sine, cosine or tangent of the indicated angle (as appropriate), then (b) give the value of the indicated angle rounded to the nearest degree. (Note: Use Appendix I.)

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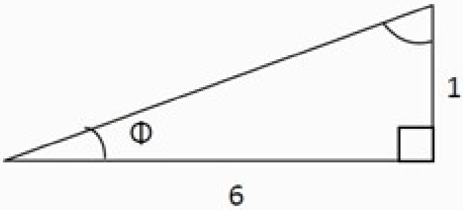

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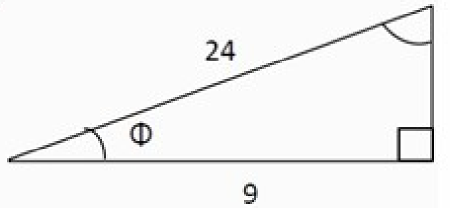

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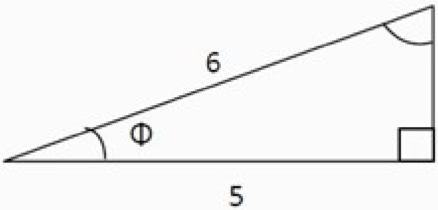

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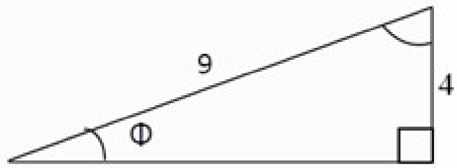

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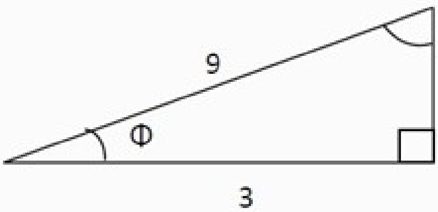

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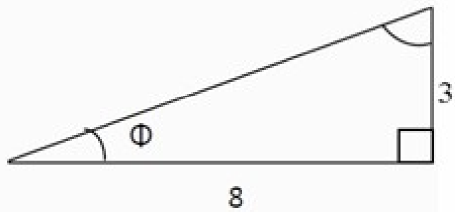

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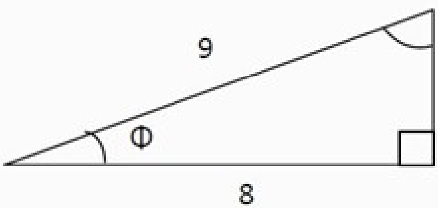

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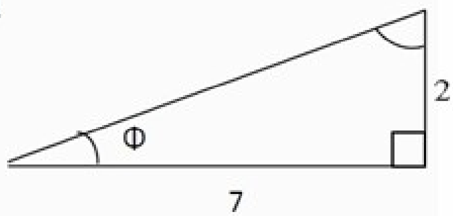

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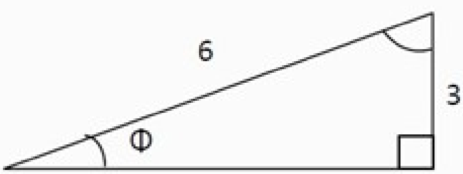

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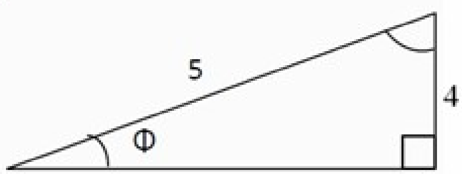


9.5 Square Roots

Square roots are often used in various areas of mathematics, and you have very likely seen them or worked with them before. In this lesson you will learn how to find square roots in a table (provided in Appendix II). In the next lesson, we will use square roots in connection with what is called the Pythagorean theorem.

<u>Finding the Square Root of a Number Using a Table</u>
In the following table are the squares roots of the numbers from 1 to 50. In this table, you can see that the square root of 4 is 2 (written as $\sqrt{4} = 2$) because $2^2 = 4$. It is easier to use a calculator to obtain the square root of a given number. To find the square root of 32, locate 32 and read the corresponding value of the square root, 5.657 because $5.657^2 = 32$. Technically, every positive number has two square roots, such that the square root of 4 is 2 and -2 [since $(-2)^2 = (-2)(-2) = 4$].

Number	Square root	Number	Square root
1	1	26	5.099
2	1.414	27	5.196
3	1.732	28	5.292
4	2	29	5.385
5	2.236	30	5.477
6	2.449	31	5.568
7	2.646	**<u>32</u>**	**<u>5.657</u>**
8	2.828	33	5.745
9	3	34	5.831
10	3.162	35	5.916
11	3.317	36	6
12	3.464	37	6.083
13	3.606	38	6.164
14	3.742	39	6.245
15	3.873	40	6.325
16	4	41	6.403
17	4.123	42	6.481
18	4.243	43	6.557
19	4.359	44	6.633
20	4.472	45	6.708
21	4.583	46	6.782
22	4.690	47	6.856
23	4.796	48	6.928
24	4.899	49	7
25	5	50	7.071

Please note that we can take the square root of a variable raised to a power such as c^2. The square root of c^2 (written as $\sqrt{c^2}$) is c, since $c \cdot c = c^2$.

Exercise 9.5

Using either the table of square roots given in Appendix II or a calculator, find each square root to the nearest tenths.

1. $\sqrt{15}$

2. $\sqrt{12}$

3. $\sqrt{79}$

4. $\sqrt{23}$

5. $\sqrt{19}$

6. $\sqrt{45}$

7. $\sqrt{13}$

8. $\sqrt{38}$

9. $\sqrt{14}$

10. $\sqrt{32}$

9.6 Calculating Square Roots (Optional)

Calculating a square root without the use of a calculator

Let's find $\sqrt{88613}$ to the nearest tenth.

Step #1. Starting at the decimal point, first group the numbers into pairs from right to left. Thus, using the number 88613, our first grouping is 886_**13**.0000. Our next group is 8_**86**_13.0000 and we have now grouped the number into pairs leaving a single digit (8) as the leftmost group. There will always be either one or two digits on the left. Each pair of numbers that we consider will end up contributing one number to our square root answer. Let's see precisely how this works.

Step 2. We now start with the 8 and ask what number whose square is less than or equal to 8? The answer is 2 (since $2^2 = 4$ and $3^2 = 9$ which is too high). So we place this answer (2) above the square root sign as follows:

```
 2
√8_86_13.0000
```

Now, we square the 2, giving 4, and write that underneath the 8, and subtract—bringing down the next pair of digits:

```
 2
√8_86_13.0000
-4
 486
```

Step 3. Next, we double the number above the square root sign (highlighted) and write it down off to the side of our calculation with an empty line next to it as shown

```
          2
         √8_86_13.0000
         -4
4_        486
```

Now we ask: what single digit number could go on the empty line so that forty-*something* times *that* single digit number would be less than or equal to 486.

4<u>**8**</u> x **8** = 384

4<u>**9**</u> x **9** = 441, so since 9 works, we write the 9 above the square root sign, perform 49 x 9 and write the result (441) under the 486 and subtract to obtain 45 and bring down the next two digits (13) as shown below:

```
          2  9
         √8_86_13.0000
         -4
49        486
         -441
           4513
```

Now, we simply repeat step 3 until we have the desired precision or accuracy of our answer. So, again, we are going to double the number above the square root sign (29) (highlighted) and write it down off to the side of our calculation with an empty line next to it as shown

```
              2  9
             √8_86_13.0000
              -4
       49      486
              -441
     58_        4513
```

Now we ask: what single digit number could go on the empty line so that five hundred and eighty-something times *that* single digit number would be less than or equal to 4513.
58**6** x **6** = 3516
58**7** x **7** = 4109
58**8** x **8** = 4704
Based on the computations above, 587 x 7 = 4109 is less than or equal to 4513, so since 7 works, we write the 7 above the square root sign, perform 587 x 7 and write the result (4109) under the 4513 and subtract to obtain 404 and bring down the next two digits (00 which are decimal digits) as shown below:

```
              2  9  7
             √8_86_13.0000
              -4
       49      486
              -441
     587        4513
              - 4109
                 40400
```

Let's continue to repeat step 3 by doubling the number above the square root sign (297) highlighted and write it down off to the side of our calculation with an empty line next to it as shown:

```
              2  9  7
             √8_86_13.0000
              -4
       49      486
              -441
     587        4513
              - 4109
     594_        40400
```

Now we ask: what single digit number could go on the empty line so that five thousand nine hundred and forty-something times *that* single digit number would be less than or equal to 440400.
594**6** x **6** = 35676
594**7** x **7** = 41629

5946 x 6 = 35676, so since 6 works, we write the 6 above the square root sign, perform 5946 x 6 and write the result (35676) under the 40400 and subtract to obtain 4724 and bring down the next two digits (00 which are again decimal digits) as shown below:

```
                    2   9   7. 6
                  √8_86_13.0000
                   -4
     49              486
                    -441
        587            4513
                     - 4109
            5946          40400
                       -  35676
                            472400
```

In order to compute the square root to the nearest tenth, we must repeat step 3 a final time.
So, again, we are going to double the number above the square root sign (2976) (highlighted) and write it down off to the side of our calculation with an empty line next to it as shown below:

```
                    2   9   7. 6
                  √8_86_13.0000
                   -4
     49              486
                    -441
        587            4513
                     - 4109
            5946          40400
                       -  35676
             5952_          472400
```

Now we ask: what single digit number could go on the empty line so that fifty-nine thousand five hundred and twenty-something times *that* single digit number would be less than or equal to 472400.
5952**5** x **5** = 297625
5952**6** x **6** = 357156
5952**7** x **7** = 416689
So, since 7 works, we write the 7 above the square root sign, perform 59527 x 7 and write the result (416689) under the 472400 and subtract to obtain 55711.

```
                 2   9   7. 6   7
                √8_86_13.0000
                 -4
49                486
                 -441
   587              4513
                  - 4109
       5946            40400
                    -  35676
          59527          472400
                       - 416689
                            55711
```

We complete our initial problem by rounding to the nearest tenth to obtain $\sqrt{88613} = 297.7$

<u>Calculating the square root using a calculator having only basic functions</u>

A quick method (or algorithm) to calculate a square root is called the **Babylonian method** which consists of making a guess, performing a divide and then computing an average. Let's demonstrate the Babylonian method by using the same example problem we used previously: find $\sqrt{88613}$ to the nearest tenth and start with a guess of 300, since 300^2 is 90000 (which is close to the actual number 88613).

Step #1. We make an initial guess: 300 and then divide our given number 88613 by our guess to obtain

$$88613/300 = 295.377$$

Next we average our guess (300) and the result of the division above (295.377) to obtain (300+ 295.377)/2 = 297.685, which when rounded to the nearest tenth is 297.7.

That was certainly quick. Let's do the problem one more time, but this time we will select a poorer initial guess of 100 (instead of 300).

Step #1. We make an initial guess: 100 and then divide our given number 88613 by our guess to obtain

$$88613/100 = 886.13$$

Next, we average our guess (100) and the result of the division above (886.13) to obtain

$$(100 + 886.13)/2 = 493.065$$

We now repeat step #1 using the averaged result of 493.065 as the updated guess. So again we divide the given number by our updated guess result to obtain

$$88613/493.065 = 179.719$$

Next, we average our guess and the result of our division to obtain a newly updated guess:

$$(493.065 + 179.719)/2 = 286.392$$

We again repeat step #1 using the guess of 286.392. So again we divide the given number by our guess to obtain

$$88613/286.392 = 309.412$$

Averaging our guess and the result of our division, we obtain another updated guess,

$$(286.392 + 309.412)/2 = 297.902$$

We repeat the procedure one more time using the guess 297.902. So dividing the given number by our guess, we obtain

$$88613/297.902 = 297.457$$

Averaging our guess and the result of our division, we obtain

$$(297.902 + 297.457)/2 = 297.67$$

Rounding to the nearest tenth we have $\sqrt{88613} = 297.7$

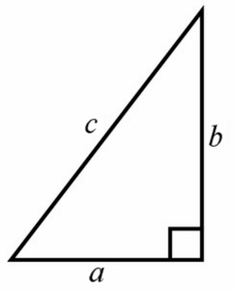
When using the Babylonian method, the closer that the original approximation is to the true value of the square root, the faster the convergence to the true value. Most modern computers utilize this method internally when the user initiates the request to find the square root of a number.

9.7 The Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. In other words:

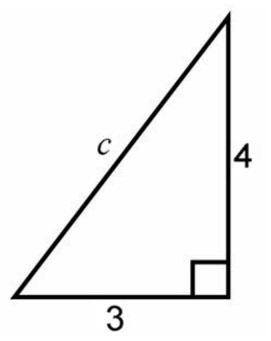
$$a^2 + b^2 = c^2$$

This is known as the **Pythagorean theorem**, where *a* and *b* are the lengths of the legs and *c* is the length of the hypotenuse of a right triangle.



How to Use the Pythagorean Theorem

Let's consider a word problem example. A certain farmer has a triangular field that he wants to fence. The field has the shape of a right triangle, with the length of the legs 3 m and 4 m, respectively. What is the length of the fencing needed for the perimeter of the field?



We apply the Pythagorean Theorem to find the measurement of the hypotenuse, c.

$$a^2 + b^2 = c^2$$

Substituting the measurements of the legs *a* and *b* we have,

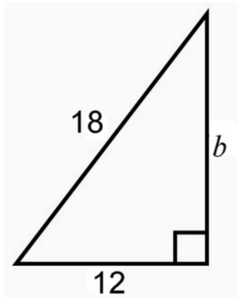
$$3^2 + 4^2 = c^2$$

Simplifying

$$c^2 = 25$$

Now, we take the square root of each side so that $\sqrt{c^2} = \sqrt{25}$ or $c = 5$. Now we can sum all the side lengths to determine the total amount of fencing needed: 3 m + 4 m + 5 m = 12 m.

Let's do one more example where we know the measurement of one leg of a right triangle and the hypotenuse and want to determine the measurement of the other leg.



Again, we apply the Pythagorean Theorem and write,

$$12^2 + b^2 = 18^2$$

or

$$144 + b^2 = 324$$

Solving for b^2, we have,

$$b^2 = 324 - 144$$

Simplifying we have

$$b^2 = 180$$

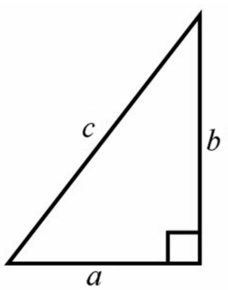
Now, we take the square root of each side to obtain

$$b = \sqrt{180}$$

Using the square root key on a calculator, we find

$$b = 13.42 \text{ (to the nearest hundredth)}$$

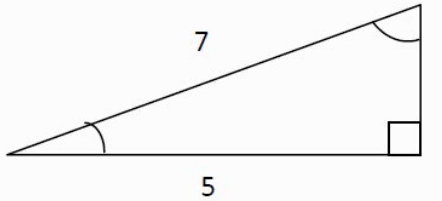
Summary of the Pythagorean Theorem

 Given a right triangle with legs a and b and hypotenuse c	$c^2 = a^2 + b^2$ or $c = \sqrt{a^2 + b^2}$ Also, $a^2 = c^2 - b^2$ $b^2 = c^2 - a^2$

Exercise 9.7

Use the Pythagorean Theorem to find the length of the missing side of each triangle to the nearest hundredths.

1.


2.


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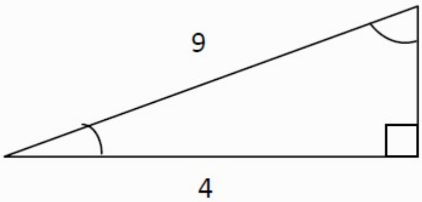

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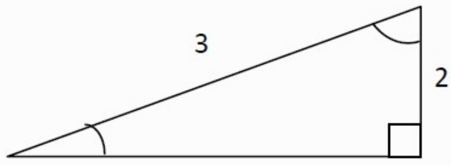

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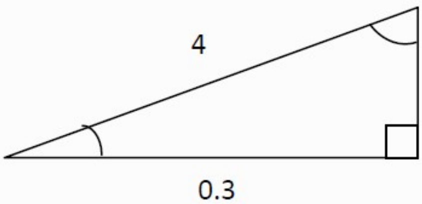

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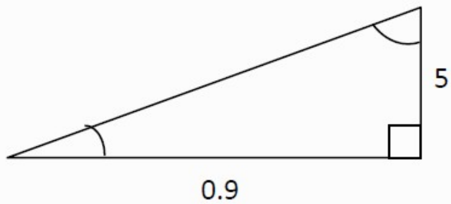

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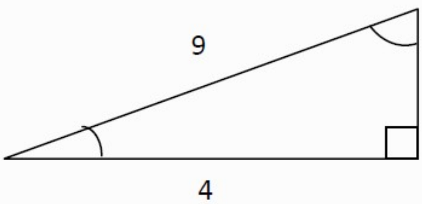

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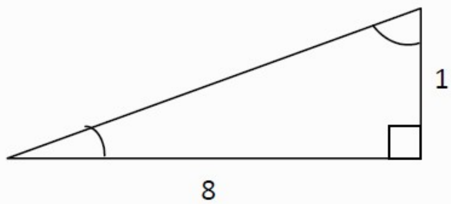

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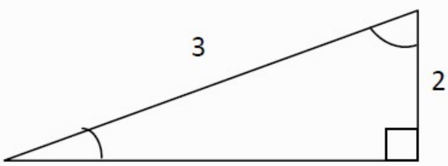

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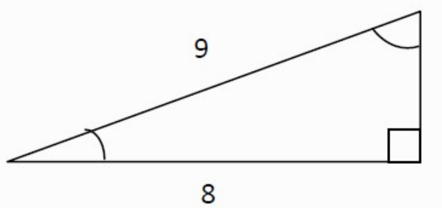
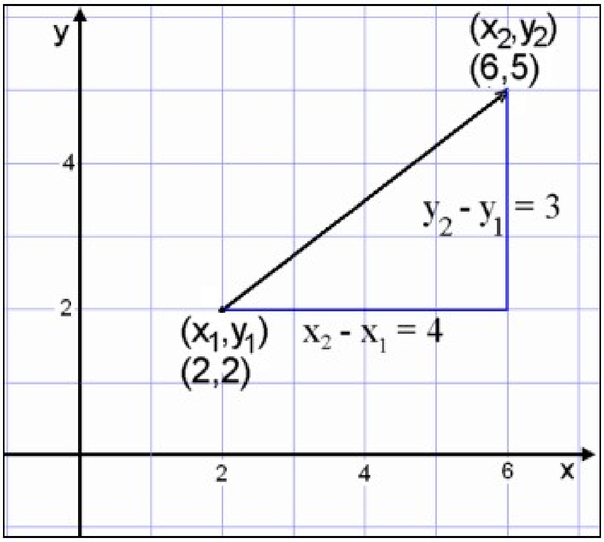


9.8 Finding the Distance Between Two Points

The graph below shows two points that have been placed at coordinates (2,2) and (6,5). Notice that the second coordinate (6,5) is located 4 horizontal units and 3 vertical units from the first coordinate (2,2).



The distance, d, between two points (x_1, y_1) and (x_2, y_2) is given by the **distance formula**,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that the distance formula is simply an application of the Pythagorean Theorem where the distance between two points can be modeled as a right triangle having legs with measurements $(x_2 - x_1)$ and $(y_2 - y_1)$. So, then, the distance between the two points, d, is simply the measurement of the hypotenuse. Substituting the coordinates of our two sample points, (2,2) and (6,5) into the equation we have

$$d = \sqrt{(6 - 2)^2 + (5 - 2)^2}$$

Simplifying further, we have

$$d = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25}$$

Now since 25 is a perfect square (5^2), we have

$$d = 5$$

Let's try one more example—finding the distance between the two points (-3, -4) and (-5, -6). Using the distance formula we have, $x_2 = -5$, $x_1 = -3$, $y_2 = -6$, and $y_1 = -4$. Thus, substituting these values into the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we have

$$d = \sqrt{(-5-(-3))^2 + (-6-(-4))^2} = \sqrt{(-5+3)^2 + (-6+4)^2}$$

Simplifying further, we have

$$d = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8}$$

The $\sqrt{8}$ can be further simplified by recognizing that 8 is the product of 4 and 2, so

$$\sqrt{8} = \sqrt{4 \bullet 2}$$

But since 4 is a perfect square, we can move the 4 from underneath the square root sign and write it as 2 outside the square root sign. Thus,

$$\sqrt{8} = 2\sqrt{2}$$

Alternatively, using Appendix II, $\sqrt{8} = 2.83$ to the nearest hundredths.

Summary of the Distance Formula

Given two coordinate points: **(x_1, y_1) and (x_2, y_2)**	**Distance, d =** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Exercise 9.8

Find the distance between the two given points. If the answer is not a perfect square, round the answer to the nearest hundredths.

1. (4, 26) and (-8, 10)

2. (28, 11) and (8, -10)

3. (-5, 6) and (0, 18)

4. (5, -6) and (-7, -22)

5. (-9, -1) and (-10, -2)

6. (-9, 1) and (-10, -2)

7. (-4, 1) and (-10, 2)

8. (-4, -1) and (10, -2)

9. (-3, -1) and (10, 2)

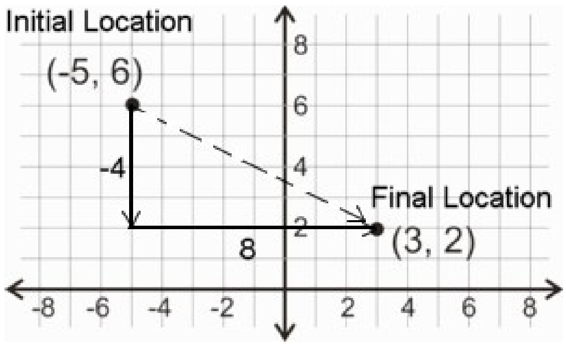
10. (-2, 1) and (10, 2)

9.9 Translations

In geometry, when a figure or image is changed to a different location or size, this is called a transformation. One type of transformation is called a **translation**--where each point of an original figure is moved the same distance and in the same direction to create a new congruent image.

How to Describe a Translation

Observe the movement of a point in a rectangular coordinate system. The translation shows movement of the point from an initial location of (-5, 6) to a final location of (3, 2).



The change in the x-position is 3 – (-5) = 8 and the change in the y-position is 2 – 6 = -4. Thus, we can describe this translation as follows:

$$(x,y) \rightarrow (x + 8, y - 4)$$

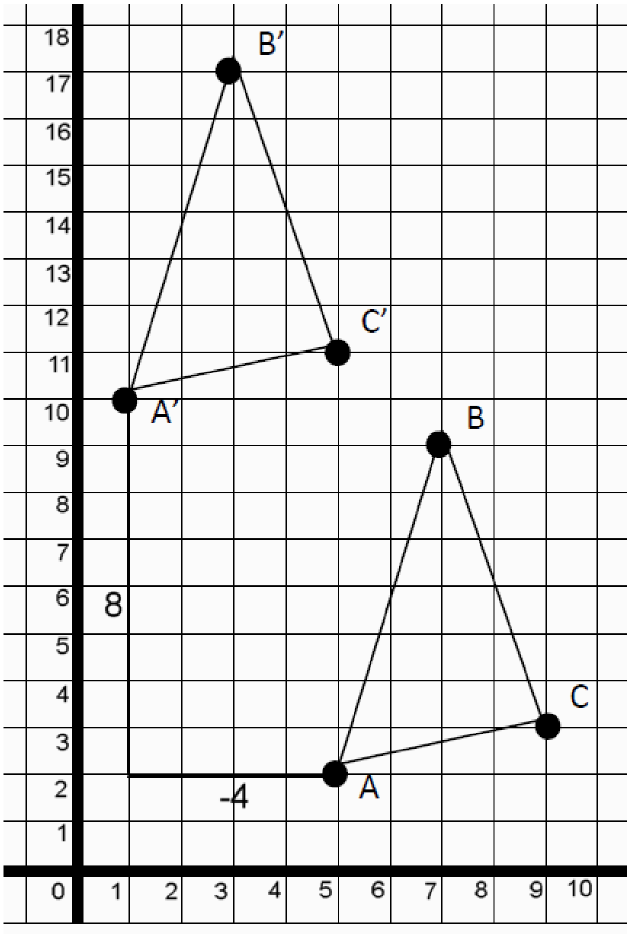
Notice that when we start with the coordinate (-5, 6), this is translated to (-5 + 8, 6 – 4) or (3, 2).

How to Translate a Figure

Consider ΔABC with vertices A(5,2), B(7,9) and C(9,3) shown on the next page. We can call also call these original vertex coordinates the original or starting coordinates Now let's find the translation formula needed to obtain the positions of the final coordinates of ΔA'B'C'. Note from the figure that the vertices of the image after the translation are given by $(x,y) \rightarrow (x - 4, y + 8)$, We simply apply this translation formula to each ordered pair. In the case of this particular triangle, the new vertices will be A', B' and C' as given in the table below.

Original Vertex	Translation	Translated Vertex
$A(5, 2)$	$(x,y) \rightarrow (x - 4, y + 8) = (5 - 4, 2 + 8)$	$A'(1, 10)$
$B(7, 9)$	$(x,y) \rightarrow (x - 4, y + 8) = (7 - 4, 9 + 8)$	$B'(3, 17)$
$C(9, 3)$	$(x,y) \rightarrow (x - 4, y + 8) = (9 - 4, 3 + 8)$	$C'(5, 11)$

The original and translated figures are shown on the graph below.



In summary, to find the translation formula take the final location for x-value and subtract it from the initial location for x and do the same for the initial location and final location for y-value. Next, take the original location of a point P (x,y) and apply the translation formula for both x and y to obtain your final coordinates of the transformed point P'.

Exercise 9.9

Translate each figure according to the translation given.

1. $A(4,6)$, $B(6,4)$, $C(-8,7)$, $D(4,0)$ $(x,y) \rightarrow (x - 3, y + 8)$
2. $A(-9,8)$, $B(5,9)$, $C(0,-9)$, $D(-1,6)$ $(x,y) \rightarrow (x - 9, y - 3)$
3. $A(3,3)$, $B(9,0)$, $C(8,5)$, $D(-8,-6)$ $(x,y) \rightarrow (x + 1, y + 7)$
4. $A(-3,3)$, $B(1,8)$, $C(-6,-3)$, $D(6,1)$ $(x,y) \rightarrow (x - 8, y + 2)$
5. $A(5,7)$, $B(1,0)$, $C(-4, 4)$, $D(-2,3)$ $(x,y) \rightarrow (x + 3, y - 2)$
6. $A(-1,5)$, $B(9,9)$, $C(0,-6)$, $D(8,-3)$ $(x,y) \rightarrow (x + 4, y - 5)$
7. $A(-6,0)$, $B(6,5)$, $C(-9,5)$, $D(4,5)$ $(x,y) \rightarrow (x - 1, y - 9)$
8. $A(8,2)$, $B(5,3)$, $C(4,-3)$, $D(6,-3)$ $(x,y) \rightarrow (x + 8, y - 1)$
9. $A(3,1)$, $B(2,7)$, $C(-8,-3)$, $D(5,1)$ $(x,y) \rightarrow (x - 2, y + 6)$
10. $A(-9,2)$, $B(0,5)$, $C(1,-6)$, $D(7,-1)$ $(x,y) \rightarrow (x + 6, y + 5)$

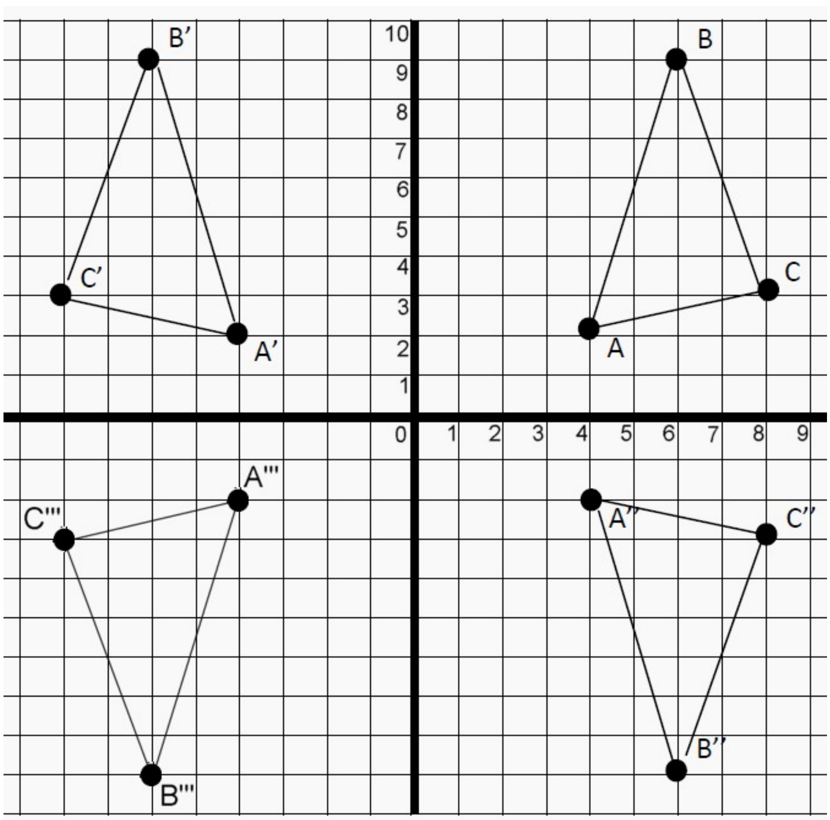
9.10 Reflections, Rotations and Symmetry

Previously, we learned about transformations, and more specifically translations. Now we are going to consider two other transformations, namely, reflections and rotations. Finally, we will conclude with the topic of symmetry, with which you might already have some familiarity.

How to Reflect a Figure

Have you ever looked in a mirror? What you are seeing is what is referred to as, literally, a **reflection**. In mathematics a reflection is a transformation where a figure is reflected or flipped about a line, known as the **line of reflection**.

Consider the figure ABC shown in the first quadrant (top-right) of the coordinate system with vertices $A(4,2)$, $B(6,9)$ and $C(8,3)$. Each (x,y) coordinate has been reflected into the second quadrant (top-left) by multiplying the x-coordinate by -1. This effectively reflected (or "flipped") the image about the y-axis to yield the figure labeled A'B'C' transformed by $(x,y) \rightarrow (-x, y)$ with the vertices $A'(-4,2)$, $B'(-6,9)$ and $C'(-8,3)$. The figure shown in the fourth quadrant (bottom-right), A''B''C'', was transformed by multiplying each y-coordinate in the first quadrant by -1 which effectively reflected (or "flipped") the initial image about the x-axis with vertices $A''(4,-2)$, $B''(6,-9)$ and $C''(8,-3)$. Finally, A'''B'''C''' (shown in the third quadrant) was transformed by multiplying both the x and y-coordinates of the original figure by -1, thus using $(x,y) \rightarrow (-x,-y)$.



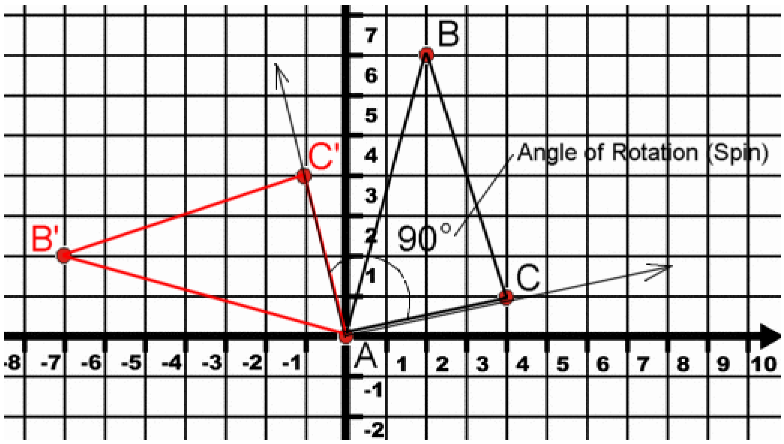
Summary of Reflection Transformations

Type of Reflection	Transformations
Reflection about (or over or across) x-axis	$(x, y) \rightarrow (x, -y)$
Reflection about (or over or across) y-axis	$(x, y) \rightarrow (-x, y)$
Reflection about (or over or across) a 45° line drawn through the origin	$(x, y) \rightarrow (y, x)$

How to Rotate (or Spin) a Figure

When a figure is turned about at a fixed point, it is called a **rotation**. Rotations can be either in a clockwise or counter clockwise direction. The fixed point about which the figure is rotated is called the **center of rotation**. Rays can be drawn from the center of the rotation through corresponding points on an original figure and its image; these form **angle(s) of rotation**.

Take for example triangle ABC with vertices $A(0,0)$, $B(2,7)$ and $C(4,1)$. This triangle is shown rotated 90° counter clockwise about vertex A which served as the center of rotation to create the new triangle $AB'C'$ with vertices $A(0,0)$, $B'(-7,2)$ and $C'(-1,4)$.



Notice that when rotating a figure about the origin (0,0), the transformation for a 90° counter-clockwise rotation is specified simply as follows $(x,y) \rightarrow (-y,x)$

In general, to rotate a point (P_x,P_y) in a figure about a specified point (or center of rotation) (R_x,R_y) the transformations are given in the table below.

Summary of Rotational Transformations

Type of Rotation	Transformation
90° counter-clockwise rotation	$(P_x,P_y) \rightarrow (R_x + R_y - P_y, -R_x + R_y + P_x)$
180° rotation	$(P_x,P_y) \rightarrow (2R_x - P_x, 2R_y - P_y)$
90° clockwise rotation	$(P_x,P_y) \rightarrow (R_x - R_y + P_y, R_x + R_y - P_x)$

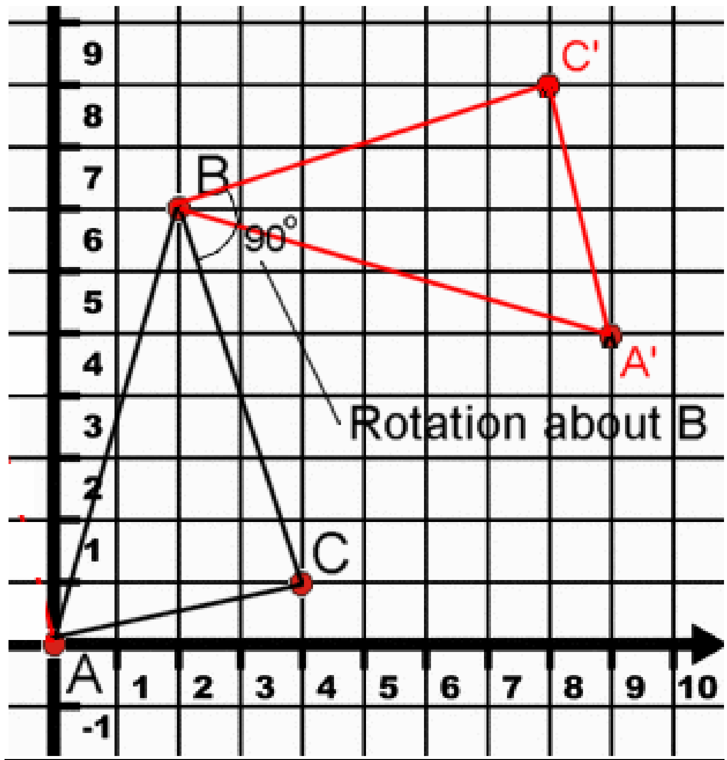
Let's do one example. Suppose we have the same triangle ABC above, but desire to rotate the triangle 90° counter-clockwise about vertex B. Thus, the specified point of rotation is point B with coordinates (2,7). So, we have $R_x = 2$ and $R_y = 7$. The new coordinates for C(4,1) can be determined by applying the transformation above with $P_x = 4$ and $P_y = 1$. Substituting the values for $(P_x,P_y) \rightarrow (R_x + R_y - P_y, -R_x + R_y + P_x)$, we have

$$C(4,1) \rightarrow C'(2 + 7 - 1, -2 + 7 + 4)$$

Simplifying, we have

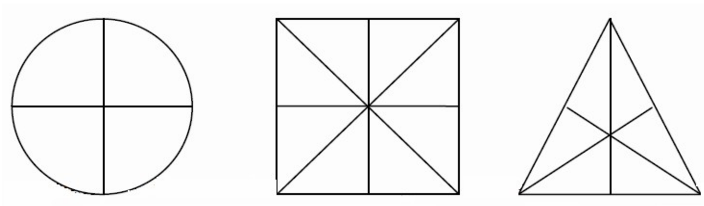
$$C(4,1) \rightarrow C'(8, 9)$$

We repeat this process for point *A* to obtain *A*(0,0) → *A*'(9,5). Since we are rotating about point *B*(2,7), which is also referred to as the center point of rotation, its coordinates remain unchanged. So the vertices of the rotated triangle become *A*'(9,5), *B*(2,7), *C*'(8,9) as shown below

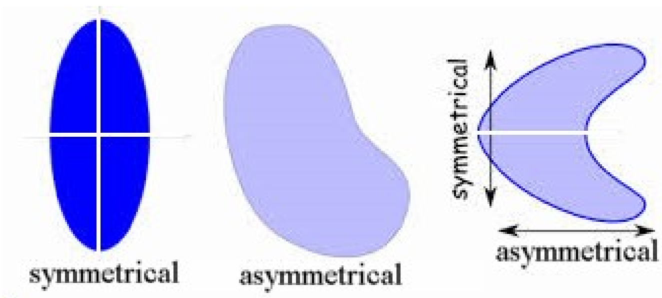


Lines of Symmetry

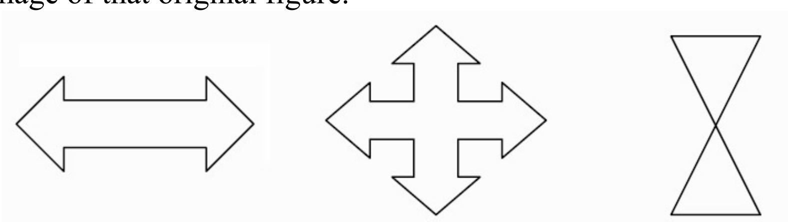
Symmetry refers to a figure or image's ability to be "cut" in half, and be a "mirror image" of itself on either side. For example, a circle is symmetrical, since if we were to slice it in half, each half would look exactly alike. Similarly the square and the triangle shown each have lines of symmetry shown; in other words, the image on each side of the line is identical or a mirror image.



Of the three objects shown below, the first is symmetrical about the x and y-axes (that is, it is symmetrical about horizontal and vertical lines that divide the object into half). The second object is asymmetrical—meaning there is no line that can be drawn through the image such that the image becomes a mirror image on both sides of the line. The third object demonstrates the possibility of an object being both symmetrical and asymmetrical, depending how it is divided. The third object appears symmetrical about a horizontal line that divides the object into two mirrored images; however, no matter what vertical line is drawn, the shapes on the right and left side of the vertical line are never mirrored images.



An object is said to have **rotational symmetry** is if a rotation of 180° or less clockwise or counter clockwise produces an image that fits exactly on the original figure, or in other words, is a perfect mirror image of that original figure.



Exercise 9.10

Reflect each figure using the x-axis as line of reflection.

1. $A(4,6)$, $B(6,4)$, $C(-8,7)$, $D(4,0)$
2. $A(-9,8)$, $B(5,9)$, $C(0,-9)$, $D(-1,6)$
3. $A(3,3)$, $B(9,0)$, $C(8,5)$, $D(-8,-6)$
4. $A(-3,3)$, $B(1,8)$, $C(-6,-3)$, $D(6,1)$
5. $A(5,7)$, $B(1,0)$, $C(-4, 4)$, $D(-2,3)$

Reflect each figure using the y-axis as line of reflection.

6. $A(-1,5)$, $B(9,9)$, $C(0,-6)$, $D(8,-3)$
7. $A(-6,0)$, $B(6,5)$, $C(-9,5)$, $D(4,5)$
8. $A(8,2)$, $B(5,3)$, $C(4,-3)$, $D(6,-3)$
9. $A(3,1)$, $B(2,7)$, $C(-8,-3)$, $D(5,1)$
10. $A(-9,2)$, $B(0,5)$, $C(1,-6)$, $D(7,-1)$

Rotate each figure 90° counter-clockwise about vertex A.

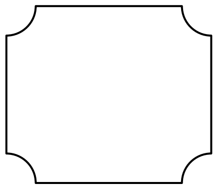
11. $A(4,6)$, $B(6,4)$, $C(-8,7)$, $D(4,0)$
12. $A(-9,8)$, $B(5,9)$, $C(0,-9)$, $D(-1,6)$
13. $A(3,3)$, $B(9,0)$, $C(8,5)$, $D(-8,-6)$
14. $A(-3,3)$, $B(1,8)$, $C(-6,-3)$, $D(6,1)$
15. $A(5,7)$, $B(1,0)$, $C(-4, 4)$, $D(-2,3)$

Rotate each figure 180° about vertex A.

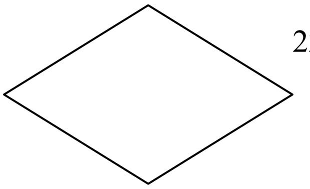
16. $A(-1,5)$, $B(9,9)$, $C(0,-6)$, $D(8,-3)$
17. $A(-6,0)$, $B(6,5)$, $C(-9,5)$, $D(4,5)$
18. $A(8,2)$, $B(5,3)$, $C(4,-3)$, $D(6,-3)$
19. $A(3,1)$, $B(2,7)$, $C(-8,-3)$, $D(5,1)$
20. $A(-9,2)$, $B(0,5)$, $C(1,-6)$, $D(7,-1)$

State if the following figures are
A. symmetrical
B. rotationally symmetrical
C. asymmetrical (not symmetrical)

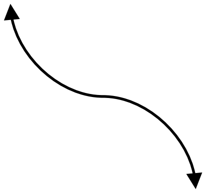
21.



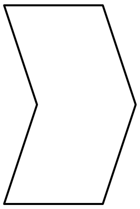
22.



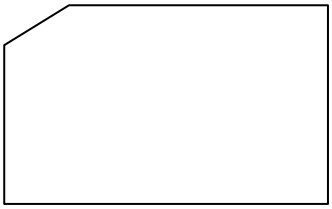
23.



24.



25.



Answers to Chapter 9 Exercises

Exercise 9.1

1. $G = 4.5$; **2.** $K = 3.375$ (or $3\frac{3}{8}$); **3.** $N = 2269.5$ (or $2269\frac{1}{2}$); **4.** $M = \frac{24}{43}$; **5.** $J = 606\frac{2}{3}$; **6.** $P = \frac{95}{96}$;

7. $S = 11\frac{21}{65}$; **8.** $Z = 125$; **9.** 1.33m; **10.** 15cmx25cm

Exercise 9.2

1. $54\frac{4}{11}$; **2.** $40\frac{6}{17}$; **3.** $112\frac{42}{59}$; **4.** $8\frac{4}{7}$; **5.** $400\frac{6}{7}$; **6.** $28\frac{27}{41}$; **7.** $54\frac{9}{14}$; **8.** $21\frac{3}{37}$; **9.** 1.152 m; **10.** 10 m

Exercise 9.3

1. $\frac{1}{9}$ (or $0.\overline{1}$), 6°, 0.99452, 9.0; **2.** $\frac{5}{6}$ (or $0.8\overline{3}$), 56°, 0.55919, 3.4; **3.** $\frac{1}{2}$ (or 0.5), 30°, 0.86603,1.7;

4. $\frac{9}{10}$ (or 0.9), 64°,0.43837, 0.9; **5.** $\frac{1}{2}$ (or 0.5), 30°, 0.86605, 6.9; **6.** $\frac{2}{3}$ (or $0.\overline{6}$), 42°, 0.74314, 6.7;

7. 0.7 (or $\frac{7}{10}$), 44°, 0.71934, 0.7; **8.** 0.45 (or 9/20), 27°, 0.89101, 1.8;

9. $\frac{14}{31}$ (or 0.45161…), 27°, 0.89101, 27.6; **10.** $\frac{3}{4}$ (or 0.75), 49°, 0.65606, 2.6;

11. $\frac{3}{4}$ (or 0.75), 41°, 0.65606, 2.6; **12.** $\frac{1}{4}$ (or 0.25), 76°, 0.97030, 7.8;

13. $\frac{6}{7}$ (or 0.85714…), 31°, 0.51504, 3.6; **14.** $\frac{7}{9}$ (or $0.\overline{7}$), 39°, 0.62932, 5.7;

15. $\frac{5}{9}$ (or $0.\overline{5}$), 56°, 0.82904; 7.5 **16.** $\frac{3}{5}$ (or 0.6), 53°, 0.79864, 4.0; **17.** $\frac{3}{8}$ (or 0.375), 68°, 0.92718, 7.4;

18. $\frac{1}{3}$ (or $0.\overline{3}$), 71°, 0.94552, 2.8; **19.** $\frac{2}{5}$ (or 0.4), 66°, 0.91355, 4.6;

20. $\frac{5}{6}$ (or $0.8\overline{3}$), 34°, 0.55919, 3.4

Exercise 9.4

1. $\frac{11}{17}$ (or 0.64705…), 33°; **2.** $\frac{2}{3}$ (or $0.\overline{6}$), 34°; **3.** $\frac{1}{7}$ (or 0.14285…), 8°; **4.** $\frac{39}{70}$ (or 0.55714…), 29°;

5. $\frac{51}{50}$ (or 1.02), 46°; **6.** $\frac{17}{20}$ (or 0.85), 40°; **7.** $\frac{3}{8}$ (or 0.375) 21°; **8.** $\frac{1}{7}$ (or 0.14285…), 8°;

9. $\frac{23}{29}$ (or 0.79310…), 38°; **10.** $\frac{48}{77}$ (or 0.62337…), 32°; **11.** 12.63; **12.** 10.01; **13.** 6.44; **14.** 0.93;

15. 7.19; **16.** 3.98; **17.** 16.00; **18.** 1.80; **19.** 5.59; **20.** 3.25; **21.** $\tan \phi = \frac{1}{6}$ (or $0.1\overline{6}$), 9°;

22. cos φ = $\frac{3}{8}$ (or 0.375), 68°; **23.** cos φ = $\frac{5}{6}$ (or $0.8\overline{3}$), 34°; 24. sin φ = $\frac{4}{9}$ (or $0.\overline{4}$), 26°;

25. cos φ = $\frac{1}{3}$ (or $0.\overline{3}$), 71°; **26.** tan ϕ = $\frac{3}{8}$ (or 0.375), 21°; **27.** cos φ = $\frac{8}{9}$ (or $0.\overline{8}$), 27°;

28. tan ϕ = $\frac{2}{7}$ (or 0.28571…), 16°; **29.** sin φ = $\frac{1}{2}$ (or 0.5), 30°; **30.** sin φ = $\frac{4}{5}$ (or 0.8), 53°

Exercise 9.5
1. 3.9; **2.** 3.5; **3.** 8.9; **4.** 4.8; **5.** 4.4; **6.** 6.7; **7.** 3.6; **8.** 6.2; **9.** 3.7; **10.** 5.7

Exercise 9.6
no exercises

Exercise 9.7
1. 10.63; **2.** 4.90; **3.** 8.06; **4.** 2.24; **5.** 3.99; **6.** 5.08; **7.** 8.06; **8.** 8.06; **9.** 2.24; **10.** 4.12

Exercise 9.8
1. 20.00; **2.** 29.00; **3.** 13.00; **4.** 20.00; **5.** 1.41; **6.** 3.16; **7.** 6.08; **8.** 14.03; **9.** 13.34; **10.** 12.04

Exercise 9.9
1. *A'*(1,14), *B'*(3,12), *C'*(-11,15), *D'*(1,8); **2.** *A'*(-18,5), *B'*(-4,6), *C'*(-9,-12), *D'*(-10,3);
3. *A'*(4,10), *B'*(10,7), *C'*(9,12), *D'*(-7,1); **4.** *A'*(-11,5), *B'*(-7,10), *C'*(-14,-1), *D'*(-2,3);
5. *A'*(8,5), *B'*(4,-2), *C'*(-1,2), *D'*(1,1); **6.** *A'*(3,0), *B'*(13,4), *C'*(4,-11), *D'*(12,-8);
7. *A'*(-7,-9), *B'*(5,-4), *C'*(-10,-4), *D'*(3,-4); **8.** *A'*(16,1), *B'*(13,2), *C'*(12,-4), *D'*(14,-4);
9. *A'*(1,7), *B'*(0,13), *C'*(-10,3), *D'*(3,7); **10.** *A'*(-3,7), *B'*(6,10), *C'*(7,-1), *D'*(13,4);

Exercise 9.10
1. *A*(4,-6), *B*(6,-4), *C*(-8,-7), *D*(4,0); **2.** *A*(-9,-8), *B*(5,-9), *C*(0,9), *D*(-1,-6);
3. *A*(3,-3), *B*(9,0), *C*(8,-5), *D*(-8,6); **4.** *A*(-3,-3), *B*(1,-8), *C*(-6,3), *D*(6,-1);
5. *A*(5,-7), *B*(1,0), *C*(-4,-4), *D*(-2,-3); **6.** *A*(1,5), *B*(-9,9), *C*(0,-6), *D*(-8,-3);
7. *A*(6,0), *B*(-6,5), *C*(9,5), *D*(-4,5); **8.** *A*(-8,2), *B*(-5,3), *C*(-4,-3), *D*(-6,-3);
9. *A*(-3,1), *B*(-2,7), *C*(8,-3), *D*(-5,1); **10.** *A*(9,2), *B*(0,5), *C*(-1,-6), *D*(-7,-1);
11. *A*(4,6), *B*(6,8), *C*(3,-6), *D*(10,6);
12. *A*(-9,8), *B*(-10,22), *C*(8,17), *D*(-7,16); **13.** *A*(3,3), *B*(6,9), *C*(1,8), *D*(12,-8);
14. *A*(-3,3), *B*(-8,7), *C*(3,0), *D*(-1,12); **15.** *A*(5,7), *B*(12,3), *C*(8,-2), *D*(9,0);
16. *A*(-1,5), *B*(-11,1), *C*(-2,16), *D*(-10,13); **17.** *A*(-6,0), *B*(-18,-5), *C*(-3,-5), *D*(-16,-5);
18. *A*(8,2), *B*(11,1), *C*(12,7), *D*(10,7); **19.** *A*(3,1), *B*(4,-5), *C*(14,5), *D*(1,1);
20. *A*(-9,2), *B*(-18,-1), *C*(-19,10), *D*(-25,5); **21.** A; **22.** A; **23.** B; **24.** A; **25.** C

Chapter 10: Radicals

10.1 Introduction to Radicals

Previously in Section 9.7 we encountered square roots in connection with the Pythagorean Theorem which states for a right triangle, the hypotenuse (*c*) squared is equal to the sum of the square of the sides, or

$$c^2 = a^2 + b^2$$

To find the measure of the hypotenuse c, we took the square root of each side of the equation to obtain

$$\sqrt{c^2} = \sqrt{a^2 + b^2}$$

which simplified to

$$c = \sqrt{a^2 + b^2}$$

In Section 9.8 the square root was again used in connection with the distance formula, where

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and (x_1, y_1) and (x_2, y_2) were the coordinates of two points. Both the hypotenuse of a right triangle and the distance between two points are positive measurements of lengths or distances. A negative length or distance is impossible and does not make sense. However, in general, when we take the **square root** of a number *n* (written as $\sqrt{n}$), there are actually two results—a positive number and a negative number. So, in general, the square root of a number *n* is given by

$$\sqrt{n} = \pm b$$

The plus-or-minus sign (±) indicates that we really have two results as follows,

$$\sqrt{n} = +b \quad \text{or} \quad \sqrt{n} = -b$$

where b^2 or $(-b)^2$ is equal to *n*. In other words, the square root of a number *n*, is a number *b* (or a number -b) such that if you square the number *b* you obtain the original number *n*, In this chapter, you will learn how to simplify, multiply, divide, and solve equations that contain square roots, or, as they are commonly referred to, **radicals**.

Let's consider a few examples. The square root of 16 (written $\sqrt{16}$) is 4 (or -4) because 4^2 is 16. The square root of 81 is 9 because 9^2 is 81. The square root of 81 is also -9 because $(-9)^2 = 81$. The square root of the fraction $\frac{1}{4}$ is $\frac{1}{2}$ (or $-\frac{1}{2}$) because $\left(\frac{1}{2}\right)^2$ is $\frac{1}{4}$.

The symbol used to represent a square root is known as a **radical** or **radical sign ($\sqrt{}$)**. The expression within or "under" the radical sign is known as the **radicand**. An expression containing a radical is known as a **radical expression**. Thus, in the expression $\sqrt{a}$, *a* is the radicand. Interestingly, $\sqrt{a}$ can be written as $a^{1/2}$ (or *a* raised to the ½ power).

Square Roots of Negative and Irrational Numbers

The square root of a negative number ($\sqrt{-9}$) is not a real number, since when we square 3 or -3, this yields a positive 9. It is impossible (using real numbers) to square a number and obtain a negative result.

When evaluating square roots, some radicands are what are called **perfect squares**. That is, when you take their square root, the answer is a whole number. For example:

$$\sqrt{25} = \pm 5$$
$$\sqrt{9} = \pm 3$$
$$\sqrt{81} = \pm 9$$
$$\sqrt{144} = \pm 12$$

Some (most) radicands, however, will not be perfect squares. For example, $\sqrt{5}$ is not a rational or whole number. Rather, the result is a whole number accompanied by decimal digits. Therefore, we would refer to $\sqrt{5}$ as an **irrational number**. The following examples show square roots that yield an irrational number:

$$\sqrt{3} = \pm 1.73205... \quad \sqrt{5} = \pm 2.23606... \quad \sqrt{6} = \pm 2.44948... \quad \sqrt{7} = \pm 2.64575...$$

When the result of a square root is an irrational number, it is often common to use an **approximation** for the decimals. So, if we wanted to state the value of $\sqrt{3}$ to two decimals places (or to the nearest hundredths), we would write,

$$\sqrt{3} = \pm 1.73$$

Exercise 10.1

Classify each square root as either
A. a perfect square
B. an irrational number.

1. $\sqrt{16}$
2. $\sqrt{221}$
3. $\sqrt{94}$
4. $\sqrt{196}$
5. $\sqrt{17}$
6. $\sqrt{91}$
7. $\sqrt{225}$
8. $\sqrt{324}$
9. $\sqrt{38}$
10. $\sqrt{178}$
11. $\sqrt{114}$
12. $\sqrt{636}$
13. $\sqrt{169}$
14. $\sqrt{639}$
15. $\sqrt{361}$
16. $\sqrt{70}$
17. $\sqrt{625}$
18. $\sqrt{916}$
19. $\sqrt{25}$
20. $\sqrt{576}$

10.2 Simplifying Radicals

A radical is simplified when the radicand contains no perfect square factors other than 1. For example, let's consider simplifying $\sqrt{18}$. First, we notice that we can rewrite 18 as a product, 9•2, where 9 is a perfect square (since $3^2 = 9$). We did not choose 3•6 because neither 3 nor 6 are perfect squares. So, we have the equivalent terms $\sqrt{18} = \sqrt{9 \cdot 2}$

Now, let's apply a general rule known as the **product rule.** This rule states that the square root of a product is equal to the product of the square roots. That is:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \quad \text{where } a \geq 0 \text{ and } b \geq 0$$

Applying this rule with $a = 9$ and $b = 2$, we have $\sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2}$. Now, since 9 is a perfect square (or 3^2), the $\sqrt{9} = 3$ so that we can simplify the expression to obtain $\sqrt{18} = 3\sqrt{2}$.

As another example, consider simplifying $\sqrt{216}$. If we recognize that 216 is the product of the perfect square 36 and 6 we can rewrite the problem using the product rule: $\sqrt{216} = \sqrt{36 \cdot 6}$. We can now take the perfect square 36 from under the square root sign and rewrite it as 6 outside the square root sign to obtain $6\sqrt{6}$. But what if we did not initially select or choose the largest perfect square factor (which was 36) of 216? Suppose instead we rewrote 216 as the product of the perfect square 9 and 24, such that $\boldsymbol{\sqrt{216} = \sqrt{9 \cdot 24}}$. We can now take the perfect square **9** from under the square root sign and rewrite it as 3 outside the square root sign to obtain $\mathbf{3}\sqrt{24}$. This answer is partially simplified at this point; but we can further simplify by recognizing that 24 is the product of a perfect square 4 and 6, thus $3\sqrt{\mathbf{24}} = 3\sqrt{\mathbf{4 \cdot 6}}$. Again, we can take the perfect square **4** from under the square root sign and rewrite it as **2** outside the square root sign to obtain $3\sqrt{\mathbf{4} \cdot 6} = 3 \cdot \mathbf{2}\sqrt{6}$. Finally, this simplifies to our final answer of $6\sqrt{6}$.

Another useful rule is the **quotient rule** for square roots, which states:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{where } a \geq 0 \text{ and } b \geq 0$$

Remembering these rules will be helpful when simplifying or factoring square roots.

<u>How to Simplify Radicals</u>

Let's try an example. Let's simplify $\sqrt{0.27}$, where the radicand is the decimal number 0.27. First, we always convert the decimal to a fraction to obtain

$$\sqrt{0.27} = \sqrt{\frac{27}{100}}.$$

Next, using the quotient rule: $\sqrt{\frac{27}{100}} = \frac{\sqrt{27}}{\sqrt{100}}$

Next, we observe that 27 (in the numerator) can be written as the product 9•3, where 9 is a perfect square. Also, we note that $\sqrt{100}$ is 10, so that we have $\frac{\sqrt{27}}{\sqrt{100}} = \frac{\sqrt{9 \cdot 3}}{10}$

Using the product rule, we can write $\frac{\sqrt{9\cdot 3}}{10} = \frac{\sqrt{9}\cdot\sqrt{3}}{10}$

Now, since $\sqrt{9} = 3$, we can simplify our result to obtain $\frac{3}{10}\sqrt{3}$ or $0.3\sqrt{3}$

As another example, let's consider simplifying: $\sqrt{1.69}$

First, we can rewrite the radicand as a mixed number, since $1.69 = 1\frac{69}{100}$. Next, we can rewrite the mixed number as an improper fraction, $\frac{169}{100}$, and then our problem becomes

$$\sqrt{\frac{169}{100}} = \frac{\sqrt{169}}{\sqrt{100}} = \frac{13}{10} = 1\frac{3}{10} \text{ or } 1.3$$

As another example, let's simplify: $\sqrt{2.32}$

Again, we rewrite the decimal number in the radicand as a mixed number, so that $2.32 = 2\,\frac{32}{100}$. Now, we write the mixed number as an improper fraction, $\frac{232}{100}$, and then we can simplify as follows:

$$\sqrt{\frac{232}{100}} = \frac{\sqrt{232}}{\sqrt{100}} = \frac{2\sqrt{58}}{10} = \frac{\sqrt{58}}{5} \text{ or } 0.2\sqrt{58}$$

Let's consider a different example: $\sqrt{\frac{2}{37}}$ which we can rewrite as $\frac{\sqrt{2}}{\sqrt{37}}$. Anytime there is a radical (that is, a square root term) in the denominator of a fraction, that fraction is ***not*** considered to be in simplest form. So, then, what must we do to get rid of the $\sqrt{37}$ in the denominator? Well, since $\sqrt{37} \bullet \sqrt{37} = 37$, we can obtain an equivalent fraction without a radical in the denominator by multiplying by $\frac{\sqrt{37}}{\sqrt{37}}$ (or 1 which does not change the fraction) to obtain,

$$\frac{\sqrt{2}}{\sqrt{37}} \bullet \frac{\sqrt{37}}{\sqrt{37}} = \frac{\sqrt{2}\bullet\sqrt{37}}{37} = \frac{\sqrt{74}}{37}$$

Thus, $\sqrt{\frac{2}{37}}$ has been simplified to $\frac{\sqrt{74}}{37}$ (and notice there is no radical in the denominator).

As a final example, to simplify $\frac{1}{3\sqrt{2}}$, we would multiply by $\frac{\sqrt{2}}{\sqrt{2}}$ to obtain

$$\frac{1}{3\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{6}.$$

Exercise 10.2

Simplify each radical by factoring the radicand (finding perfect squares) and remember to use either the product or the quotient rules.

1. $\sqrt{180}$
2. $\sqrt{160}$
3. $\sqrt{80}$
4. $\sqrt{200}$
5. $\sqrt{24}$
6. $\sqrt{25}$
7. $\sqrt{50}$
8. $\sqrt{0.18}$
9. $\sqrt{0.98}$
10. $\sqrt{0.02}$
11. $\sqrt{0.32}$
12. $\sqrt{0.96}$
13. $\sqrt{0.54}$
14. $\sqrt{0.36}$
15. $\sqrt{0.80}$
16. $\sqrt{\frac{48}{64}}$
17. $\sqrt{\frac{20}{144}}$
18. $\sqrt{\frac{50}{200}}$
19. $\sqrt{\frac{99}{36}}$
20. $\sqrt{\frac{20}{45}}$
21. $\sqrt{4.84}$
22. $\sqrt{4.86}$
23. $\sqrt{4.90}$
24. $\sqrt{5.29}$
25. $\sqrt{2.89}$

10.3 Adding and Subtracting Radicals

To add or subtract radicals, you must first identify **like radicals** or radicals that are alike (or the same).

Identifying Like Radicals

Radical expressions that have the same radicand are said to be **like radicals**. For example, the following (simplified) radicals are like radicals:

$$3\sqrt{7} \text{ and } 2\sqrt{7} \qquad 2\sqrt{6} \text{ and } 8\sqrt{6} \qquad 5\sqrt{10} \text{ and } 12\sqrt{10} \qquad 7\sqrt{2} \text{ and } 4\sqrt{2}$$

In other words, like radicals exist when the radicands (the number under the square root symbol) are the same. Just like with fractions that can be added or subtracted only if their denominators are the same, only like radicals can be added together or subtracted from each other.

How to Add Like Radicals

To add like radicals, we simply add the numbers IN FRONT of the square root sign, and keep the radicand (the number UNDER the square root sign) the same. Often, you will need to first simplify the radical expressions in order to be able to add. Let's consider an example of how to simplify

$$\sqrt{75} + \sqrt{27}$$

First, we simplify the terms by factoring the radicands into products that contain a perfect square. Since 75 = 3•5•5 and 27 = 3•3•3 we can rewrite our problem as

$$\sqrt{5^2 \bullet 3} + \sqrt{3^2 \bullet 3}$$

Applying the product property which effectively moves the squared term out from under the square root sign, we have

$$5\sqrt{3} + 3\sqrt{3}$$

Using the distributive property by noticing that $\sqrt{3}$ is common in both terms,

$$5\sqrt{3} + 3\sqrt{3} = (5+3)\sqrt{3} = 8\sqrt{3}$$

Let's consider another example by simplifying

$$\sqrt{98} + \sqrt{8}$$

First, we rewrite the radicands as products involving perfect squares,

$$\sqrt{7^2 \bullet 2} + \sqrt{2^2 \bullet 2}$$

Simplifying, we have

$$7\sqrt{2} + 2\sqrt{2}$$

Using the distributive property by noting $\sqrt{2}$ is common to each term, we have

$$7\sqrt{2} + 2\sqrt{2} = (7 + 2)\sqrt{2}$$

Thus, we have the result,

$$\sqrt{98} + \sqrt{8} = 9\sqrt{2}$$

Of course, if the radicand is a prime number (i.e., the only products are the number itself and 1), the radical expression cannot be further simplified. Some examples of radicals that cannot be further reduced because the radicand is a prime number (which contains no factors that are perfect squares) are as follows:

$$\sqrt{2} \quad \sqrt{151} \quad \sqrt{443} \quad \sqrt{991}$$

Appendix III contains a list of prime numbers that you may find useful to determine if a radicand can possibly be simplified. While the radicand is not a prime number, in the examples that follow, the radical expression still cannot be simplified since the radicand contains no factors that are perfect squares:

$$\sqrt{30} \quad \sqrt{70} \quad \sqrt{302} \quad \sqrt{453}$$

How to Subtract Like Radicals

To subtract like radicals, simply subtract the numbers IN FRONT of the square root sign, and keep the radicand (the number UNDER the square root sign) the same. Often, you will need to first simplify the radical expressions in order to be able to subtract.

Let's consider simplifying $\sqrt{486} - \sqrt{6}$.

We note that one of the radicands (486) can be factored where one of the factors is a perfect square (81),

$$486 = 81 \bullet 6$$

If you did not readily recognize 486 was comprised of a perfect square factor, you can perform a prime factorization of 486 = 2•3•3•3•3•3 and notice this is equivalent to $2 \bullet 3^2 \bullet 3^2 \bullet 2$. Then take the product of all the factors that are squared: $3^2 \bullet 3^2$ to derive the highest perfect square of 81. If you do not recall how to perform prime factorization of a number, please review Section 4.2.

Next, we apply the product and/or quotient property to simplify the radical to obtain

$$\sqrt{(9^2)6} - \sqrt{6} = 9\sqrt{6} - \sqrt{6}$$

Finally we use the distributive property and combine the like terms:

$$(9 - 1)\sqrt{6} = 8\sqrt{6}$$

Thus, we have

$$\sqrt{486} - \sqrt{6} = 8\sqrt{6}$$

Let's try one more example and simplify

$$\sqrt{2} - \sqrt{32}$$

We note that 32 is the product of 16•2, so we can take the 16 from under the square root sign and write 4 outside the square root sign, to obtain

$$\sqrt{2} - 4\sqrt{2}$$

Finally, using the distributive property by recognizing $\sqrt{2}$ is common to both terms, we have

$$1\sqrt{2} - 4\sqrt{2} = (1 - 4)\sqrt{2} = -3\sqrt{2}$$

Important: Let's say we want to (try to) simplify the expression $6 + 5\sqrt{2}$. It is **COMPLETELY INCORRECT** to add the 6 and 5 to obtain $11\sqrt{2}$. Now, if we were adding $6\sqrt{2} + 5\sqrt{2}$, then we would obtain the answer $11\sqrt{2}$ because we can factor out the common $\sqrt{2}$ as follows: $\sqrt{2} \cdot (6 + 5)$. But since we have $6 + 5\sqrt{2}$, this expression cannot be further reduced. This particular point of instruction should be kept in mind especially in Chapter 11, Section 11.3.

Exercise 10.3

Add the radicals by simplifying first if needed.

1. $\sqrt{6} + \sqrt{486}$
2. $\sqrt{216} + \sqrt{96}$
3. $\sqrt{196} + \sqrt{324}$
4. $\sqrt{180} + \sqrt{80}$
5. $\sqrt{50} + \sqrt{2}$
6. $\sqrt{100} + \sqrt{16}$
7. $\sqrt{180} + \sqrt{320}$
8. $\sqrt{567} + \sqrt{28}$
9. $\sqrt{36} + \sqrt{324}$
10. $\sqrt{48} + \sqrt{50}$

Subtract the radicals by simplifying first if needed.

11. $\sqrt{576} - \sqrt{324}$
12. $\sqrt{24} - \sqrt{54}$
13. $\sqrt{648} - \sqrt{128}$
14. $\sqrt{576} - \sqrt{225}$
15. $\sqrt{512} - \sqrt{648}$
16. $\sqrt{144} - \sqrt{36}$
17. $\sqrt{542} - \sqrt{216}$
18. $\sqrt{192} - \sqrt{75}$
19. $\sqrt{175} - \sqrt{63}$
20. $\sqrt{54} - \sqrt{216}$

10.4 Multiplying and Dividing Radicals

In Section 10.2, you learned the Product and Quotient properties for radicals, and used these to simplify radicals. Now you are going to use those two same properties to simplify the products and the quotients of the radicals.

How to Multiply Radicals

Recall that the product property for radicals is

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{or} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \quad \text{where } a \geq 0 \text{ and } b \geq 0$$

Here are some examples that demonstrate the application of this property:

$$\sqrt{11} \cdot \sqrt{5} = \sqrt{11 \cdot 5} = \sqrt{55}$$

$$\sqrt{8} \cdot \sqrt{3} = \sqrt{8 \cdot 3} = \sqrt{24}$$

Therefore, to multiply a radical, simply multiply the radicands, and keep the square root sign. Notice that in the above example problem: $\sqrt{8} \cdot \sqrt{3}$, we could have simplified $\sqrt{8}$ since

$$\sqrt{8} = \sqrt{4 \cdot 2}$$

Since 4 is a perfect square, we can take the 4 from under the square root sign and write 2 outside the square root sign:

$$\sqrt{8} = 2\sqrt{2}$$

Now, let's do the multiplication substituting $2\sqrt{2}$ in place of $\sqrt{8}$ to obtain

$$2\sqrt{2} \cdot \sqrt{3} = 2\sqrt{6}$$

How to Divide Radicals

Recall that the quotient property for radicals is

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \quad \text{where } a \geq 0 \text{ and } b \geq 0$$

Therefore, to divide a radical by another radical, simply divide the radicands, and keep the square root sign.

Let's consider the following example,

$$\frac{\sqrt{648}}{\sqrt{9}}$$

First, we apply the quotient property to obtain $\frac{\sqrt{648}}{\sqrt{9}} = \sqrt{\frac{648}{9}}$

Since the fraction under the radical (square root sign) $\frac{648}{9}$ reduces to 72, we now have

$$\sqrt{\frac{648}{9}} = \sqrt{72}$$

Recognizing that 72 = 36•2 and 36 is a perfect square (6^2), we can remove the 36 from under the square root sign and write 6 outside the square root sign to obtain the simplified result,

$$\sqrt{36 \bullet 2} = 6\sqrt{2}$$

Now, here is a different example problem where the radical is only in the denominator,

$$\frac{8}{\sqrt{8}}$$

To eliminate the radical ($\sqrt{8}$) in the denominator, we multiple the numerator and denominator of the fraction by $\frac{\sqrt{8}}{\sqrt{8}}$, (which is equal to 1, so we do not change the value of the fraction) to obtain,

$$\frac{8}{\sqrt{8}} \bullet \frac{\sqrt{8}}{\sqrt{8}} = \frac{8\sqrt{8}}{8}$$

Notice that $\sqrt{8} \bullet \sqrt{8} = 8$, thus effectively removing the radical in the denominator. Now we simplify the expression since $\frac{8}{8} = 1$, we have $1\sqrt{8}$ which is simply $\sqrt{8}$.

When a radical appears in the denominator of a fraction, the fraction is not considered to be reduced to lowest terms and is not in simplest form. You must always multiply a denominator that contains a radical, by that same radical as we did in the above example to eliminate the square root in the denominator.

Reducing Radicals to their Simplest Form

Rule for Simplest Form	**Example of Rule**
1. Factor out the highest perfect square.	$\sqrt{8} = \sqrt{4 \bullet 2} = 2\sqrt{4}$ or $\sqrt{980} = \sqrt{196 \bullet 5} = 14\sqrt{5}$
2. There should be no fractions under the square root (radical) sign.	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$
3. A square root should not appear in the denominator of any fraction.	$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \bullet \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$
4. Reduce all fractions to simplest form.	$\frac{3}{\sqrt{27}} = \frac{3\sqrt{27}}{27} = \frac{3\sqrt{9 \bullet 3}}{27} = \frac{3 \bullet 3\sqrt{3}}{27} = \frac{9\sqrt{3}}{27} = \frac{\sqrt{3}}{3}$ or $\frac{3}{\sqrt{27}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Exercise 10.4

If possible, simplify each radical expression.

1. $3\sqrt{6}$
2. $6\sqrt{9}$
3. $\frac{\sqrt{196}}{7}$
4. $\frac{4}{\sqrt{4}}$
5. $7\sqrt{9}$
6. $9\sqrt{3}$
7. $\frac{\sqrt{243}}{9}$
8. $\frac{2}{\sqrt{2}}$
9. $6\sqrt{4}$
10. $\frac{3\sqrt{6}}{\sqrt{2}}$
11. $\frac{\sqrt{9}}{3}$
12. $\frac{2}{\sqrt{4}}$
13. $5\sqrt{4}$
14. $6\sqrt{7}$
15. $\frac{\sqrt{36}}{2}$
16. $\frac{6}{\sqrt{2}}$
17. $2\sqrt{8}$
18. $5\sqrt{2}$
19. $\frac{\sqrt{16}}{4}$
20. $\frac{6}{\sqrt{4}}$
21. $\frac{1}{\sqrt{3}}$
22. $\frac{9}{\sqrt{27}}$
23. $\frac{3}{\sqrt{27}}$
24. $\frac{27}{\sqrt{27}}$
25. $\frac{\sqrt{3}}{\sqrt{50}}$

10.5 Simplifying Radical Expressions with Variables (Optional)

In this pre-algebra course, we are not going to consider the topic of radical expressions that involve variables. However, when variables are involved in the terms, the procedures that you learned above, are again helpful to reduce radical expressions. For example, let's consider the problem of $\sqrt{4x^5}$. Since 4 is a perfect square, when can rewrite it as a 2 outside the square root sign,

$$2\sqrt{x^5}$$

Now, since $x^5 = x{\bullet}x{\bullet}x{\bullet}x{\bullet}x$, we can regroup the x's as follows $x^5 = (x{\bullet}x)\,(x{\bullet}x)x$. Notice this gives us several squared terms,

$$x^5 = x^2x^2{\bullet}x$$

Substituting for x^5, we now have

$$2\sqrt{x^5} = 2\sqrt{x^2x^2 \bullet x}$$

Now, we can remove each x^2 from under the square root sign and write x outside the square root sign,

$$2x{\bullet}x\sqrt{x} = 2x^2\sqrt{x}$$

Thus, we have

$$\sqrt{4x^5} = 2x^2\sqrt{x}$$

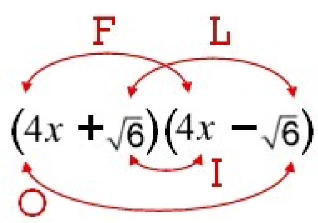
Let's try one more example as follows

$$(4x + \sqrt{6})(4x - \sqrt{6})$$

To solve the above multiplication, many students will often use the 4-step "FOIL" method: (1) multiply the First term in each set of parenthesis, or $4x{\bullet}4x$, to obtain $16x^2$; (2) multiply the two Outside terms, or $4x{\bullet}(-\sqrt{6}) = -4x\sqrt{6}$ (or $-4\sqrt{6}x$ using the commutative property of multiplicaton); (3) multiply by the two Inside terms, or $\sqrt{6} \bullet 4x = 4x\sqrt{6}$ (or $4\sqrt{6}x$); and (4) multiply the Last term in each set of parenthesis, or $\sqrt{6} \bullet (-\sqrt{6}) = -6$. Now we simply add these four terms together,

$$(4x + \sqrt{6})(4x - \sqrt{6}) = 16x^2 - 4x\sqrt{6} + 4x\sqrt{6} - 6 = 16x^2 - 6$$

Thus, the sequence is summarized as follows



I personally like to use the distributive property and rewrite problems such as $(4x + \sqrt{6})(4x - \sqrt{6})$ in this equivalent form:

$$4x(4x - \sqrt{6}) + \sqrt{6}(4x - \sqrt{6})$$

Next, using the distributive problem, we expand the terms to obtain the following,

$$16x^2 - 4\sqrt{6}x + 4\sqrt{6}x - 6$$

Finally, combining like terms we have

$$(4x + \sqrt{6})(4x - \sqrt{6}) = 16x^2 - 6$$

Did you notice that when multiplying binomial terms of the form $(a + b)(a - b)$, the inner terms (shown in red) cancel

$$a^2 - ab + ab - b^2$$

so that only the first and last terms of the FOIL method remain. In general, however, you will need to include the inner terms of the foil method to insure a correct answer when multiplying binomial terms together.

Let's try one more example problem:

$$(-4x + \sqrt{6})(x - \sqrt{6})$$

You do the problem on your own. Did you get $-4x^2 - 6$ as your answer? If so, you are INCORRECT, making a mistake that is very common—you only multiplied the first terms of each binomial, $-4x \bullet x$, and the outer terms, $\sqrt{6} \bullet (-\sqrt{6})$. Instead, you must use the FOIL method which yields the follower four terms:

$$\underset{\text{First}}{-4x \bullet x} \; \underset{\text{Outer}}{- \sqrt{6} \bullet \sqrt{6}} \; \underset{\text{Inner}}{+ x\sqrt{6}} \; \underset{\text{Last}}{+ 4x\sqrt{6}}$$

Simplifying (and combining like terms), we have

$$-4x^2 + 5x\sqrt{6} - 6$$

10.6 Solving Equations Containing Radicals

Let's put all our knowledge of radials together now and learn how to solve equations that contain radicals. More specifically, you are going to learn what is known as the squaring property.

The Squaring Property of Equality

The squaring property of equality states: *If $a = b$, then $a^2 = b^2$*

So, if one value is equal to another value, the square of the values will also be equal. We can use this rule to solve equations that contain radicals.

Solving a Radical Equation

Step 1. Isolate the radical. Arrange the terms so that one radical is all by itself on one side of the equation.

Thus, given $\sqrt{10x-2} - \sqrt{x} = 0$, we first add $\sqrt{x}$ to each side of the equation to obtain,

$$\sqrt{10x-2} = \sqrt{x}$$

Step 2. Square both sides of the equation.

$$(\sqrt{10x-2})^2 = (\sqrt{x})^2$$

Step 3. Simplify both sides of the equation. When you square an expression that consists of a square root, the result is simply what was under the square root sign.

$$10x - 2 = x$$

Step 4. If it is the case that the equation still has a radical, repeat steps one through three.

Step 5. Solve the equation.

Using our example, $10x - 2 = x$, we group the variables on one side (the left) of the equation by subtracting x from each side to yield $9x - 2 = 0$. Then, to isolate the variable x, we add 2 to each side, to yield $9x = 2$. Finally, we divide both sides by 9 to obtain the solution, $x = \frac{2}{9}$

Step 6. Check all solutions in the original equation for **extraneous solutions, s**ince squaring the equation can result in reversing the sign of the answer. It is important to remember that the answer could be positive OR negative. Double check to make sure your answer is correct by substituting your answer into the original equation and solving. Using our example problem, let's substitute $x = \frac{2}{9}$ back into the original example problem,

$$\sqrt{10x-2} = \sqrt{x}$$

$$\sqrt{10\left(\frac{2}{9}\right) - 2} = \sqrt{\frac{2}{9}}$$

$$\sqrt{\frac{20}{9} - 2} = \sqrt{\frac{2}{9}}$$

$$\sqrt{\frac{20}{9} - \frac{18}{9}} = \sqrt{\frac{2}{9}}$$

$$\left(\sqrt{\frac{2}{9}}\right)^2 = \left(\sqrt{\frac{2}{9}}\right)^2$$

$\frac{2}{9} = \frac{2}{9}$ Correct.

Let's do one more example problem: $3x = \sqrt{4x^2 + 3}$

First, we square each side to obtain: $9x^2 = 4x^2 + 3$

Now, we combine like terms by subtracting $4x^2$ from each side of the equation to obtain:

$$5x^2 = 3$$

Now we have two methods to solve the above equation. The first method involves dividing both sides of the equation by 5 to obtain $x^2 = \frac{3}{5}$. Then, we take the square root of each side to obtain the answer

$$x = \pm\sqrt{\frac{3}{5}} = \pm\frac{\sqrt{3}}{\sqrt{5}} = \pm\frac{\sqrt{3}}{\sqrt{5}} \bullet \frac{\sqrt{5}}{\sqrt{5}} = \pm\frac{\sqrt{15}}{5}$$

Since only the positive answer works in the original equation ($3x = \sqrt{4x^2 + 3}$), the final answer is

$$x = \frac{\sqrt{15}}{5}$$

A second method that we will soon learn in Chapter 11 (and which is optional at this time) is introduced on page 319. If we write $5x^2 = 3$ as a quadratic equation in this form $5x^2 - 3 = 0$, we can apply what is called the **quadratic formula** with $a = 5$, $b = 0$, and $c = -3$ to obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{60}}{10} = \frac{\pm 2\sqrt{15}}{10} = \pm\frac{\sqrt{15}}{5}$$

Again, since only the positive answer satisfies the original equation, we derive the exact same result $x = \frac{\sqrt{15}}{5}$ as the only solution.

Exercise 10.6

Solve each equation containing radicals and check your answer.

1. $3 = \sqrt{2x + 2}$

2. $8 = \sqrt{9x + 2}$

3. $x = \sqrt{2x^2 - 8}$

4. $4x = \sqrt{5x^2 + 4}$

5. $2x = \sqrt{7x^2 - 3}$

6. $2x = \sqrt{3x^2 + 7}$

7. $4x = \sqrt{5x^2 + 30}$

8. $9x = \sqrt{6x^2 + 2}$

9. $x = \sqrt{3x^2 - 7}$

10. $x = \sqrt{7x^2 - 1}$

11. $x = \sqrt{8x^2 - 8}$

12. $3x = \sqrt{2x^2 + 9}$

13. $5x = \sqrt{7x^2 + 1}$

Answers to Chapter 10 Exercises

Exercise 10.1

1. A; **2.** B; **3.** B; **4.** A **5.** B; **6.** B; **7.** A; **8.** A; **9.** B; **10.** B; **11.** B; **12.** B; **13.** A; **14.** B; **15.** A; **16.** B; **17.** A; **18.** B; **19.**A; **20.** A

Exercise 10.2

1. $6\sqrt{5}$; **2.** $4\sqrt{10}$; **3.** $4\sqrt{5}$; **4.** $10\sqrt{2}$; **5.** $2\sqrt{6}$; **6.** 5; **7.** $5\sqrt{2}$; **8.** $\frac{3\sqrt{2}}{10}$ or $0.3\sqrt{2}$; **9.** $\frac{7\sqrt{2}}{10}$ or $0.7\sqrt{2}$;

10. $\frac{\sqrt{2}}{10}$ or $0.1\sqrt{2}$; **11.** $\frac{2\sqrt{2}}{5}$ or $0.4\sqrt{2}$; **12.** $\frac{2\sqrt{6}}{5}$ or $0.4\sqrt{6}$; **13.** $\frac{3\sqrt{6}}{10}$ or $0.3\sqrt{6}$; **14.** 5 or 0.6;

15. $\frac{2\sqrt{5}}{5}$ or $0.4\sqrt{5}$; **16.** $\frac{\sqrt{3}}{2}$; **17.** $\frac{\sqrt{5}}{6}$; **18.** $\frac{1}{2}$; **19.** $\frac{\sqrt{11}}{2}$; **20.** $\frac{2}{3}$; **21.** $2\frac{1}{5}$ or 2.2; **22.** $\frac{9\sqrt{6}}{10}$ or $0.9\sqrt{6}$;

23. $\frac{7\sqrt{10}}{10}$ or $0.7\sqrt{10}$; **24.** $2\frac{3}{10}$ or 2.3; **25.** $1\frac{7}{10}$ or 1.7

Exercise 10.3

1. $10\sqrt{6}$; **2.** $10\sqrt{6}$; **3.** 32; **4.** $10\sqrt{5}$; **5.** $6\sqrt{2}$; **6.** 14; **7.** $14\sqrt{5}$; **8.** $11\sqrt{7}$; **9.** 24; **10.** $4\sqrt{3} + 5\sqrt{2}$; **11.** 6; **12.** $-\sqrt{6}$; **13.** $10\sqrt{2}$; **14.** 9; **15.** $-2\sqrt{2}$; **16.** 6; **17.** $\sqrt{542} - 6\sqrt{6}$; **18.** $3\sqrt{3}$; **19.** $2\sqrt{7}$; **20.** $-3\sqrt{6}$

Exercise 10.4

1. $3\sqrt{6}$; **2.** 18; **3.** 2; **4.** 2; **5.** 21; **6.** $9\sqrt{3}$; **7.** $\sqrt{3}$; **8.** $\sqrt{2}$; **9.** 12; **10.** $3\sqrt{3}$; **11.** 1; **12.** 1; **13.** 10;

14. $6\sqrt{7}$; **15.** 3; **16.** $3\sqrt{2}$; **17.** $4\sqrt{2}$; **18.** $5\sqrt{2}$; **19.** 1; **20.** 3; **21.** $\frac{\sqrt{3}}{3}$; **22.** $\sqrt{3}$; **23.** $\frac{\sqrt{3}}{3}$; **24.** $3\sqrt{3}$; **25.** $\frac{\sqrt{6}}{5}$

Exercise 10.5

no answers

Exercise 10.6

1. $\frac{7}{2}$ or 3½ ; **2.** $6\frac{8}{9}$; **3.** $2\sqrt{2}$; **4.** $\sqrt{\frac{4}{11}} = \frac{2}{\sqrt{11}} = \frac{2\sqrt{11}}{11}$; **5.**1; **6.** $\sqrt{7}$; **7.** $\sqrt{\frac{30}{11}} = \frac{\sqrt{330}}{11}$; **8.** $\frac{\sqrt{6}}{15}$;

9. $\frac{\sqrt{14}}{2}$; **10.** $\frac{\sqrt{6}}{6}$; **11.** $\frac{2\sqrt{14}}{7}$; **12.** $\frac{3\sqrt{7}}{7}$; **13.** $\frac{\sqrt{2}}{6}$

Chapter 11: Quadratic Equations

Previously, in Section 6.7, you learned to solve quadratic equations by means of factoring, and then setting each factor equal to zero, you solved for x. Factoring quadratic equations using the manner that was taught earlier works conveniently only on selected problems where the coefficients and constant are whole numbers (and the equation is easy to factor). In this chapter we will learn several new techniques that will enable you to solve virtually any quadratic equation using a straightforward approach. The approach that we will take will consist of first showing our prior method of factoring, then the new method.

<u>What Are Quadratic Equations</u>

Recall that quadratic equations are equations that are in the form:

$ax^2 + bx + c = 0$ where a and b are coefficients and c is a constant.

11.1 Solving Quadratic Equations Using the Square Root Method

The square root method (similar to the square property of equality discussed in Section 10.6) states:

$$\textit{If } a^2 = b^2\textit{, then } \sqrt{a^2} = \sqrt{b^2} \text{ or } a = b$$

Let's first consider a special case of quadratic equations where there is no bx term, in other words, where $b = 0$, thus

$$ax^2 + c = 0$$

To solve for x, we first isolate the term with the variable by subtracting c from each side of the equation,

$$ax^2 = -c$$

Since x is multiplied by a, we do the inverse and divide each side of the equation by a,

$$x^2 = \frac{-c}{a}$$

Next, taking the square root of each side (i.e., this is using the "square root method"), we have

$\sqrt{x^2} = \pm\sqrt{\frac{-c}{a}}$ which simplifies to $x = \pm\sqrt{\frac{-c}{a}}$. This is equivalent to $x = \pm\frac{\sqrt{-c}}{\sqrt{a}}$. Now, since there is a radical in the denominator, we can multiply by $\frac{\sqrt{a}}{\sqrt{a}}$ to obtain $x = \pm\frac{\sqrt{-ac}}{a}$

Thus, because of the plus-or-minus sign (±), we have two distinct solutions:

$$x = +\frac{\sqrt{-ac}}{a} \quad \text{and} \quad x = -\frac{\sqrt{-ac}}{a}$$

Now, let's use this method on the following example problem,

$$x^2 - 16 = 0$$

Previously, we factored the above quadratic equation to obtain

$$(\boldsymbol{x + 4})(\boldsymbol{x - 4}) = 0$$

Then, setting each factor to 0, we have $\boldsymbol{x + 4} = 0$ or $x = -4$ and $\boldsymbol{x - 4} = 0$ or $x = 4$, which can be expressed more concisely (or compactly) as $x = \pm 4$.

But now using a different approach, to apply the square root method, first we must rewrite the equation with the x-squared term isolated on one side of the equation (by adding 16 to each side),

$$x^2 = 16$$

Next, we use the square root method which simply involves taking the square root of each side to yield,

$$x = \pm\sqrt{16} = \pm 4$$

It is extremely important to remember that when you take the square root of each side, this step yields a plus and minus sign or actually two separate equations $x = \sqrt{16}$ and $x = -\sqrt{16}$. Let's try one more example,

$$4x^2 - 9 = 0$$

Using our prior factoring method, we would have had to determine the factors and write

$$(\boldsymbol{2x + 3})(\boldsymbol{2x - 3}) = 0$$

Then setting each factor equal to zero $\boldsymbol{2x + 3} = 0$ and $\boldsymbol{2x - 3} = 0$ and solving for x, we obtain

$$x = \pm\frac{3}{2}$$

To use the square root method, we must first rewrite the initial equation with the x-squared term isolated on one side of the equation (by adding 9 to each side of the equation),

$$4x^2 = 9$$

Next, we isolate the x by dividing each side of the equation by 4 to obtain,

$$x^2 = \frac{9}{4}$$

Next, we take the square root of each side to obtain

$$x = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2} \text{ (or } \pm 1\frac{1}{2} \text{ or } \pm 1.5\text{)}$$

Let's try one more example as follows:

$$2.42x^2 - 18 = 0$$

We will not even attempt to use our old method of factoring this expression. Rather, we will use our square root method which requires that we rewrite the equation with the *x*-squared term isolated on one side of the equation,

$$2.42x^2 = 18$$

Next, we divide both sides of the equation by 2.42 to obtain

$$x^2 = \frac{18}{2.42}$$

When we perform the division on the right side, we obtain

$$\frac{18}{2.42} = 7\frac{53}{121}$$

Next, we convert the mixed number to an improper fraction (by taking 121•7 + 53 to obtain the numerator of 900):

$$7\frac{53}{121} = \frac{900}{121}$$

Now, substituting our improper fraction for $\frac{18}{2.42}$, we can write

$$x^2 = \frac{900}{121}$$

Finally, we take the square root of each side of the equation to obtain the solution,

$$x = \pm\sqrt{\frac{900}{121}} = \pm\frac{30}{11} = \pm 2\frac{9}{11}$$

Important Note: In this book and on the chapter tests, when working problems that contain radicals in the solution, the usual rule is not to evaluate or compute the square root, unless the square root is a perfect square. Thus, answers, for example, should be stated as $2\sqrt{3}$ instead of computing an approximate value for $\sqrt{3} \approx 1.73205080757$ and performing the calculation 2•1.73205080757 to obtain an approximate answer of 2.464. Answers such as $\sqrt{80}$ should be simplified to $4\sqrt{5}$ since 80 can be factored as 16•5 and 16 is a perfect square. Thus, $\sqrt{80} = \sqrt{16 \bullet 5} = 4\sqrt{5}$ in simplest form.

Exercise 11.1

Solve each of the following quadratic equations using the square root method.

1. $x^2 - 225 = 0$
2. $x^2 - 50 = 0$
3. $2x^2 - 50 = 0$
4. $5x^2 - 50 = 0$
5. $25x^2 - 50 = 0$
6. $2x^2 = 5$
7. $7x^2 = 35$
8. $0.4x^2 = 2.1$
9. $1.8x^2 = 81$
10. $1.8x^2 = 80$
11. $1.8x^2 = 79$
12. $1.8x^2 = 78$
13. $1.8x^2 = 76$
14. $3.52x^2 = 8$
15. $3.52x^2 = 6$

11.2 Solving Quadratic Equations by Completing the Square

"Completing the square" is another method that can be used in a limited way to solve quadratic equations that are in the form

$$x^2 + bx = c$$

Notice that the coefficient of the x^2 term is implied to be 1. So, if this is not the case and there is some other coefficient used on the x^2 term, this method will not be suitable to find a solution. In the next section (11.3) we will discuss the quadratic formula which will work even if the coefficient of the x^2 term is a value other that 1.

Here is how the method works. First, we note the value of b (the coefficient of the x-term) and compute half the b value (as the initial step in completing the square):

$$\frac{b}{2}$$

Then, we factor $x^2 + bx$ using this expression:

$$\left(x + \frac{b}{2}\right)^2$$

This particular step is called "completing the square."

Let's actually work with a real example problem at this point. So, let's complete the square on this problem

$$x^2 + 20x = 44$$

First, we note that $b = 20$ in this problem; so that $\frac{b}{2} = 10$, so that we complete the square by writing

$$(x + 10)^2$$

At this point, notice that when we expand $(x + 10)^2 = (x + 10)(x + 10)$ using the foil method (described on pages 151-152), we obtain $x^2 + 20x + 100$. So, by completing the square, we have an extra 100 that we did not previously have in the original problem (where the expression on the left side of the equal sign was $x^2 + 20x$). Because we have introduced an extra "100" on the left side of the equal sign, we must add this value to the right side of the equation to obtain

$$x^2 + 20x + \mathbf{100} = 44 + \mathbf{100}$$

By performing this step, we have created a quadratic expression on the left side of the equation that can always be factored. Continuing with our example, we now substitute $(x+10)^2$ for $x^2 + 20x + 100$, and rewrite the problem as

$$(x+10)^2 = 144$$

Now, we are in a position to take the square root of each side (using the square root method discussed previously in Section 11.1) to obtain

$$x + 10 = \pm\sqrt{144}$$

Don't forget that when we take the square root, it yields both a positive and negative result, hence we write $\pm\sqrt{144}$ and not simply $\sqrt{144}$. Since the radicand is a perfect square (144), the right side of the equation simplifies to

$$x + 10 = \pm 12$$

Finally, we subtract 10 from each side of the equation to obtain the solution

$$x = -10 \pm 12$$

So, our two solutions are

$$x = -10 + 12 \quad \text{and} \quad x = -10 - 12$$

or

$$x = 2 \quad \text{and} \quad x = -22$$

Let's do another example problem,

$$x^2 + x = 5$$

In this example, $b = 1$ so that we complete the square by writing $(x + ½)^2$. But, when we complete the square, we obtain $(x + ½)^2 = (x + ½)(x + ½) = x^2 + x + ¼$. After completing the square, we have an additional ¼ that was not in the initial expression $x^2 + x$, so we must add ¼ to the right side of the equation to obtain

$$x^2 + x + \frac{1}{4} = 5 + \frac{1}{4}$$

By "completing the square", we are now able to factor the left side of the equation and write

$$\left(x + \frac{1}{2}\right)^2 = 5\frac{1}{4}$$

Perhaps you had forgotten how to factor $x^2 + x + \frac{1}{4}$, but this topic was addressed in Section 6.6 (pages 157-158) of this book. For purposes of review, we know that the factoring must be of the form

$$(x + c)(x + d)$$

since the x in the first factor times the x in the second factor yields x^2. The "c" and "d" represent constants such that when the two numbers (or constants) are multiplied together, they equal ¼; however, when they are added together, they equal the coefficient of the middle x term (which is 1 in this case). Notice that when c and d are both equal to ½, we have the multiplication of the constants yielding ½ • ½ = ¼ and the addition of these constants yielding ½ + ½ = 1. Thus, we have determined that $x^2 + x + \frac{1}{4}$ can be factored as follows:

$$(x + ½)(x + ½) = (x+ ½)^2$$

Writing $5\frac{1}{4}$ as an improper fraction, we have

$$\left(x + \frac{1}{2}\right)^2 = \frac{21}{4}$$

Taking the square root of each side of the equation (and remembering that we obtain both a positive and negative result on the right side), we obtain

$$\mathrm{x} + \frac{1}{2} = \pm\sqrt{\frac{21}{4}}$$

Simplifying the radical on the right side of the equation we have

$$x + \frac{1}{2} = \pm\frac{\sqrt{21}}{2}.$$

To solve for x, we subtract $\frac{1}{2}$ from each side of the equation to obtain

$$x = -\frac{1}{2} \pm \frac{\sqrt{21}}{2}$$

So, the two solutions are thus

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{2} \quad \text{and} \quad x = -\frac{1}{2} - \frac{\sqrt{21}}{2}$$

Note: As a reminder, please recall that if, for example, our solution was $x = 2 \pm 3\sqrt{7}$, you can ***NOT*** combine the "2" with the radical term $3\sqrt{7}$, to obtain $\mathbf{5\sqrt{7}}$ and $\mathbf{-\sqrt{7}}$. This was previously discussed on page 300 at the bottom of the page in a paragraph labeled **Important**.

Exercise 11.2

Solve each quadratic equation by completing the square.

1. $x^2 + 7x = 0$
2. $x^2 + 4x = 96$
3. $x^2 + 6x = 16$
4. $x^2 + 18x = -81$
5. $x^2 + 5x = 10$
6. $x^2 + x = 0$
7. $x^2 - 4x = 59$
8. $x^2 - 5x = \frac{3}{4}$
9. $x^2 + 3x = 4$
10. $x^2 - 14x = 1$
11. $x^2 - 28x = 1$
12. $x^2 + 38x = 39$
13. $x^2 + 11x = \frac{1}{3}$
14. $x^2 + 13x = -25$
15. $x^2 - \sqrt{6}x = 0$

11.3 Solving Quadratic Equations Using the Quadratic Formula

In sections 11.1 and 11.2 we considered solving special cases of quadratic equations using a square root method and the method of completing the squares. Now we are going to learn an approach or method for solving any quadratic equation having real (not imaginary) roots (or solutions).

Previously in Section 6.7, we considered solving the following quadratic equation by the method of factoring:

$$x^2 + 10x + 10 = -11$$

Recall that there were two solutions for x. First we rewrote the equation in the standard form $ax^2 + bx + c = 0$, where a is the coefficient of the x^2 term, b is the coefficient of the x term, and c is always the constant. So, we added 11 to each side of the equation to obtain the standard form,

$$x^2 + 10x + 21 = 0$$

In this example, $a = 1$, $b = 10$, and $c = 21$.

Next, we mentally noted that 3 • 7 = 21 and 3 + 7 = 10, so that we instantly factored the expression on the left side of the equation

$$(x + 3)\ (x + 7) = 0$$

Now, since the two factors, $(x + 3)$ and $(x + 7)$ are equal to zero, if any one of these factors is zero, then the product will automatically be zero. So the next step was to set each factor equal to zero and solve for x. Thus we wrote,

$$x + 3 = 0 \quad \text{and} \quad x + 7 = 0$$

Solving each equation, we found that $x = -3$ or $x = -7$ satisfied the original equation.

Note that if we form factors using the opposite value of the solutions for x, we can derive the original quadratic equation. Thus, given the two solutions, $x = -3$ and $x = -7$, we can write the factors as $(x + 3)(x+7) = x^2 + 10x +21$--our original equation.

Using the quadratic ***formula*** to solve a quadratic equation can actually be easier, especially in those cases where it is not readily apparent how to factor the quadratic equation. Given a quadratic equation of the form

$$ax^2 + bx + c = 0$$

the quadratic formula is given as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's return to our previous example where the equation in standard form was $x^2 + 10x + 21 = 0$, where $a = 1$, $b = 10$, and $c = 21$. We simply substitute the values for a, b, and c into the quadratic formula as follows:

$$x = \frac{-10 \pm \sqrt{10^2 - 4 \bullet 1 \bullet 21}}{2 \bullet 1}$$

Simplifying, we have

$$x = \frac{-10 \pm \sqrt{100 - 84}}{2}$$

or

$$x = \frac{-10 \pm \sqrt{16}}{2}$$

Since 16 is a perfect square (4^2), we can evaluate $\sqrt{16} = 4$, so that the above solution simplifies to

$$x = \frac{-10 \pm 4}{2}$$

Now, because the above equation has a plus-or-minus sign ($\pm$), we must break the above statement into two different equations—one that adds the 4 and one that subtracts the 4, as follows:

$$x = \frac{-10 + 4}{2} \text{ and } x = \frac{-10 - 4}{2}$$

Thus, our final answers are

$$x = -3 \text{ and } x = -7$$

Notice that the results using the quadratic formula are the same as when solving the equation by factoring.

Let's try another example of using the quadratic formula to solve this quadratic equation:

$$x^2 + 36x + 6 = 0$$

Here, a = 1, b = 36, and c = 6 (since the standard form of the quadratic equation is $ax^2 + bx + c = 0$), so substituting these values into the quadratic formula we have,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-36 \pm \sqrt{36^2 - 4(1)(6)}}{2(1)}$$

Simplifying further we have,

$$x = \frac{-36 \pm \sqrt{1296 - 24}}{2} = \frac{-36 \pm \sqrt{1272}}{2} = \frac{-36 \pm 2\sqrt{318}}{2} = \frac{-36}{2} \pm \frac{2\sqrt{318}}{2}$$

So our final solution is

$$x = -18 \pm \sqrt{318}$$

If we evaluate that $\sqrt{318}$ is approximately 17.83255, then the two solutions for x to the nearest ten thousandths is:

$$x = -0.1675 \quad \text{and} \quad x = -35.8326$$

Notice that if we place the opposite of these solutions into the factors and set them to 0,

$$(x + 0.1675)(x + 35.8326) = 0$$

we obtain the original equation when we carry out the above multiplication (using the FOIL method): $x^2 + 36x + 6=0$. I don't know about you, but I am sure glad we did not have to try to use the factoring technique given in Section 6.7 to solve that quadratic equation. As we discovered previously when using the factoring method of solving quadratic equations, multiplication of the constant factors, or 0.1675•35.8326, gives us the constant term 6; the addition of the two factors, 0.1675 + 35.8326, gives us the coefficient of x—or 36.

Number of solutions when using the Quadratic Formula

Notice that the solution to quadratic equations of the form $ax^2 + bx + c = 0$ is given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Since we can only take the square root of zero and positive numbers (otherwise the result will be an imaginary number), we have the restriction that the expression under the square root sign, $b^2 - 4ac$, called the **discriminant**, must be greater than or equal to zero.

$$b^2 - 4ac \geq 0 \quad \text{or} \quad b^2 \geq 4ac$$

When $b^2 = 4ac$, then the discriminant, $b^2 - 4ac$, evaluates to zero, yielding only one real solution to the quadratic equation since ($\sqrt{b^2 - 4ac} = 0$): $x = \frac{-b}{2a}$

When $b^2 > 4ac$, there will be two real solutions to the quadratic equation as given by the quadratic formula. When $b^2 < 4ac$, there will be no real solutions to the quadratic equation.

Quadratic equation example	a	b	c	Value of the Discriminant	No. of real solutions	Real Solutions for x
$x^2 + x - 2 = 0$	1	1	-2	$b^2 - 4ac < 0$	0	No Real Solutions
$x^2 + 2x + 1 = 0$	1	2	1	$b^2 - 4ac = 0$	1	$x = -1$
$-x^2 - 4x - 3 = 0$	-1	-4	-3	$b^2 - 4ac > 0$	2	$x = -1$ and $x = -3$

Important Note: When solving problems that involve a quadratic equation that describes a measurement (such a length or area), or perhaps an age or speed, keep in mind that while two different (both positive and negative) solutions may exist, you should only select the positive solution—since negative values do not make sense in these cases. Also, if you have a quadratic equation without the **b**x term, such as $2x^2 - 5 = 0$, then simply use **a** = 2, **b** = 0 **c** = -5; notice that the value of **b** is zero since there is no x-term in the equation.

Remember from Section 10.3 concerning the adding and subtraction of radicals that, for example, $3 + 8\sqrt{5}$ does ***NOT*** equal $11\sqrt{5}$.

Exercise 11.3

Solve each quadratic equation for x using the quadratic formula. If there is no solution, then type NO

1. $x^2 + x - 1 = 0$

2. $x^2 + 2x - 3 = 0$

3. $x^2 + 3x - 3 = 0$

4. $x^2 - 3 = 0$

5. $-x^2 + 4x + 5 = 0$

6. $-2x^2 + 4x = 0$

7. $-2x^2 + 4x + 4 = 0$

8. $-2x^2 + 8x - 8 = 0$

9. $-3x^2 + 8x - 8 = 0$

10. $x^2 - 11 = 0$

11. $3x^2 + 4x + 1 = 0$

12. $3x^2 + 4x - 1 = 0$

13. $-5x^2 - 20 = 0$

14. Jill is 1 year older than her friend Jack. The product of Jill and Jack's ages is 21. What is Jill's age?

15. The area of a rectangle of width = $2x$ and length = $x + 6$ is 20 m^2. What is the width of the rectangle?

11.4 Derivation of the Quadratic Formula (Optional)

A quadratic equation has solutions given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's see how this solution was derived given the general quadratic equation

$$ax^2 + bx + c = 0$$

First, we subtract *c* from each side of the equation to obtain

$$ax^2 + bx = -c$$

Next, multiply both sides of the equation by 4*a*:

$$4a(ax^2 + bx) = 4a(-c)$$

Simplifying by using the distributive property, we have

$$4a^2x^2 + 4abx = -4ac$$

We can "complete the square" by adding b^2 to both sides of the equation:

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

This permits us to now factor the left side of the equation:

$$(2ax + b)^2 = b^2 - 4ac$$

We take the square root of both sides of the equation:

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

Finally, we subtract *b* from each side of the equation and then divide by 2*a* to obtain the simplified result

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It follows from the quadratic formula that

$$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

11.5 Graphing Quadratic Functions

We have been working with quadratic equations of the form $ax^2 + bx + c = 0$. Instead of setting the equation to zero, let's write $y = ax^2 + bx + c$, where y is a function of (or depends on) the variable x). To graph such quadratic functions, we simply substitute various values for x into the equation and then compute the corresponding values for y and plot these as (x,y) point coordinates on a Cartesian coordinate system (or graph with x- and y- axes).

Let's see precisely how to graph a quadratic function using the following example,

$$y = x^2 - 10x - 24$$

Let's substitute $x = 0$, since that is an easy calculation, to obtain the corresponding y-value

$$y = 0^2 - 10 \cdot 0 - 24 = -24$$

We have computed our first x-y coordinate point: (0, -24)

Next, let's substitute $x = -1$ into the equation to obtain the corresponding y-value:

$$y = (-1)^2 - 10 \cdot (-1) - 24 = 1 + 10 - 24 = -13$$

So, now we have another point on the graph, (-1,-13). Let's continue and now substitute $x = 1$ into the equation,

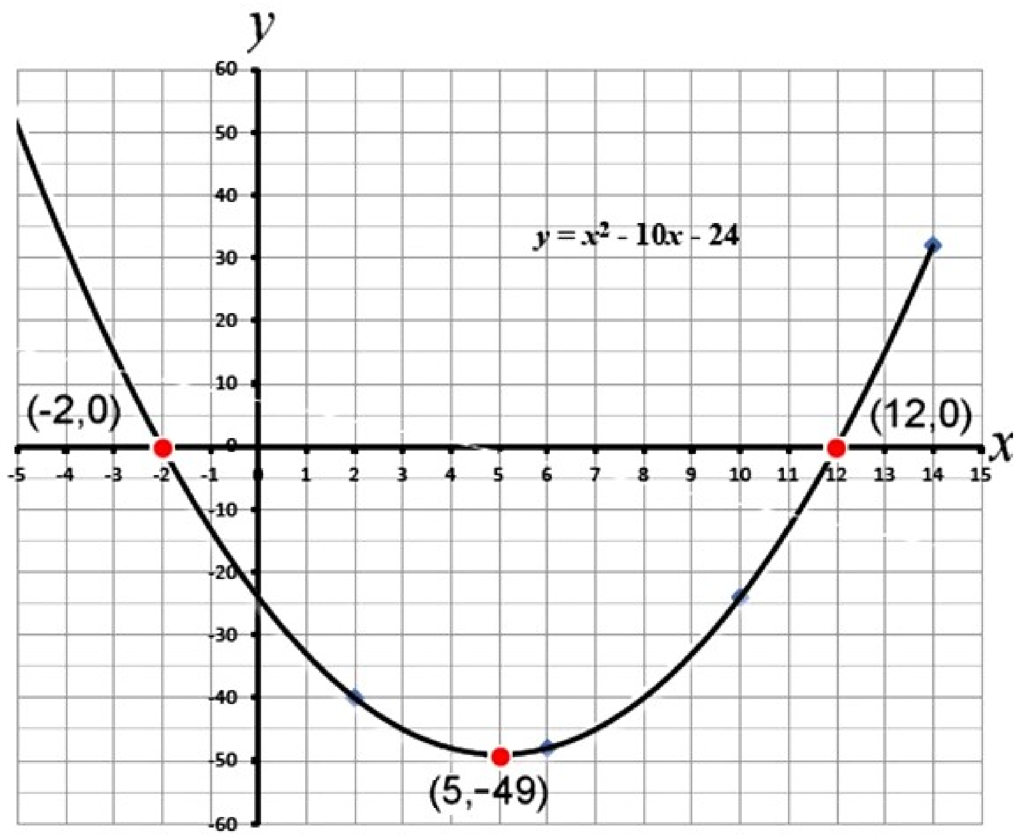
$$y = 1^2 - 10 \cdot 1 - 24 = 1 - 10 - 24 = -33$$

So, now we have still another point on the graph (1, -33). We have placed the y-values and their corresponding x-values in the table below along with additional pairs of y-values that correspond to others selected x-values.

Table of x and y values for $y = x^2 - 10x - 24$

***x*-coordinate value**	***y*-coordinate value**
-5	51
-2	0
-1	-13
0	-24
1	-33
2	-40
5	-49
6	-48
10	-24
12	0

Each (x,y) coordinate pair from the table above has been plotted in the graph shown below to derive the curve drawn through the points. The curve created by a quadratic function is called a **parabola**. We can call this method of plotting the parabola the "table" method and we can only approximate the shape of the parabola because we do not know precisely the the minimum or maximum point on the parabola.



Let's now consider another method of plotting the same quadratic equation (or parabola)—but this time, we will not have to complete a table of x and y coordinate pairs. Notice the three large points shown on the graph. First, we see two points of the quadratic function that intersect the x-axis. These points are located at (-2, 0) and (12, 0). These points can be found using algebra (without graphing) by setting the function equal to zero—in other words, by setting $y = 0$ and then solving the quadratic equation for the x-value solutions (just as we have done in the previous sections of this chapter). Thus, we have $0 = x^2 - 10x - 24$

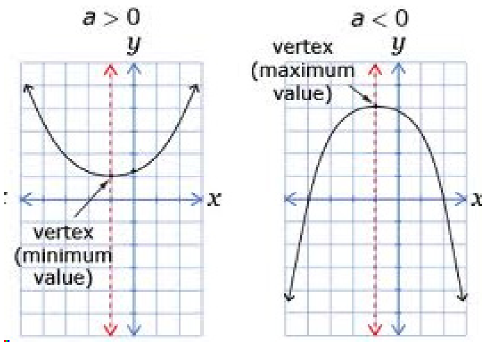
To obtain the solution, we can either use the factoring method $x^2 - 10x - 24 = (x+2)(x-12)$ and then set each factor to 0, or the quadratic equation $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{10 \pm \sqrt{(-10)^2 - 4 \bullet 1 \bullet (-24)}}{2 \bullet 1} = \frac{10 \pm \sqrt{196}}{2} = \frac{10 \pm 14}{2} = 5 \pm 7$ to obtain the two solutions: $x = -2$ and $x = 12$

So, when y is zero, we have the x-intercepts of -2 and 12, or the coordinates (-2,0) and (12,0). Now we must perform one additional step to locate either the minimum or maximum vertex point (the lowest or highest point) of the parabola. The x-coordinate of the minimum or maximum point is computed by using the "a" and "b" coefficients of the original quadratic equation (i.e., since we have $x^2 - 10x - 24$, a = 1 and b = -10) and is given by this formula:

$$x = \frac{-\mathrm{b}}{2\mathrm{a}}$$

The x-coordinate will be the lowest or minimum value (referred to as x_{min}) of the parabola when a > 0; or it will be the highest or maximum value (referred to as x_{max}) of the parabola when a < 0. Now, by observation, we see that the quadratic function reaches a minimum value when $x = 5$. To find the corresponding y-value, we simply substitute $x = 5$ into our function and solve for y:

$$y = x^2 - 10x - 24 = 5^2 - 10 \bullet 5 - 24 = 25 - 50 - 24 = -49$$



But we can actually calculate the x and y coordinates of the minimum (or maximum) point using our formula, $x = \frac{-b}{2a}$. Notice, that in our example problem, since a = 1 and b = -10, using the above formula we confirm that $x_{min} = \frac{-(-10)}{2 \bullet 1} = 5$. When a>0 you will always be finding the minimum point; when a<0 you will always be finding the maximum point using the same formula $\frac{-b}{2a}$. In this example, since a = 1 and "a" is positive, we are finding the x-value corresponding to the minimum point (or vertex) of the parabola. Next, we simply place our x-value back into the original equation ($y = x^2 - 10x - 24$) and solve for the corresponding y-value or y_{min}.

How the minimum and maximum point on the parabola is derived (Optional)

In general, given the quadratic function $y = ax^2 + bx + c$, the derivative, called y' (or y-prime) is $y' = 2ax + b$. The derivative tells us the slope of the line tangent to any given point on the quadratic function. When the derivative is set equal to 0, or the slope is zero, this indicates a horizontal line that will be tangent to the minimum (or maximum) point of the quadratic function. Returning to our example equation, $y = x^2 - 10x - 24$, since a = 1 and b = -10, we take the derivative to obtain $y' = 2 \bullet 1 \bullet x + (-10)$ or $y' = 2x - 10$

Next, we set the derivative equal: $0 = 2x - 10$ and solve for x by adding 10 to each side of the equation: $10 = 2x$. Thus, we have the solution, $x = 5$. Once we have the x-coordinate of the minimum (or maximum) value of the quadratic function, we simply substitute this value of x into the original function to obtain the corresponding y-value, $y = x^2 - 10x - 24 = 5^2 - 10 \bullet 5 - 24 = 25 - 50 - 24 = -49$.

Thus, we have determined the coordinate of the minimum value is (5,-49). In general, since the derivative is $y' = 2ax + b$, if we set the derivative to zero and solve for x, we find that the minimum x-value is given by $x_{min} = \frac{-b}{2a}$.

Determining the Bending of the Quadratic Equation **(Optional)**

By taking the first derivative of $y = ax^2 + bx + c$ and setting it equal to zero, we were able to solve for the x-coordinate of the minimum or maximum point of the quadratic equation (also called a parabola). By way of review, the first derivative is given by

$$y' = 2ax + b$$

By inspecting the sign of the second derivative, this indicates whether the coordinate we found (through use of the first derivative) is really a minimum (i.e., the second derivative will be positive) or a maximum (the second derivate will be negative).

The second derivative is given by $y'' = 2a$.

In conclusion, all that you really have to remember is this:

1. If the coefficient "a" of the squared variable of a quadratic function is positive, then the quadratic function (or parabola) will bend upward from a minimum point. Thus, the point you find using the first derivative will represent (x_{min}, y_{min}).

2. If the coefficient "a" of the squared variable of a quadratic function is negative, then the quadratic function (or parabola) will bend downward from a maximum point. Thus, the point you find using the first derivate will represent (x_{max}, y_{max}).

Returning to our example function, $y = x^2 - 10x - 24$, we see that a = 1, which is positive. Therefore, we know that our quadratic function bends upward from a minimum point.

Graphing Quadratic Functions

To find the x-intercept	Set the quadratic function equal to 0 and solve for x	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; if no solution, the quadratic equation does not intersect the x-axis.
To find the minimum or maximum x-value	Compute x_{min} or $x_{max} = \frac{-b}{2a}$	Then substitute this x-value into the original equation to find the corresponding y_{min} or y_{max}.
To determine if the curve bends upward (i.e., you have a minimum) or downward (i.e., you have a maximum)	Inspect the sign of coefficient "a" (to determine the bending direction)	If a>0 then you have (x_{min}, y_{min}); if a<0 then you have (x_{max}, y_{max})
To graph the function	Find the x-intercept points, and the minimum or maximum point, and bending direction	Select several points surround the x-intercept points and plot these along with the minimum or maximum value and draw a parabola through the points.

Exercise 11.5

Find the (x,y) coordinates of the minimum or maximum vertex of each quadratic function and indicate whether this point on the parabola is a

A. minimum value
B. maximum value

1. $y = x^2 + x - 1$
2. $y = x^2 + 2x - 3$
3. $y = x^2 + 3x - 3$
4. $y = x^2 - 3$
5. $y = -x^2 + 4x + 5$
6. $y = -2x^2 + 4x$
7. $y = -2x^2 + 4x + 4$
8. $y = -2x^2 + 8x - 8$
9. $y = x^2 - 11$
10. $y = 3x^2 + 4x + 1$

Answers to Chapter 11 Exercises

Exercise 11.1

1. ± 15; **2.** $\pm 5\sqrt{2}$; **3.** **± 5;** **4.** $\pm\sqrt{10}$; **5.** $\pm\sqrt{2}$; **6.** $\pm\frac{\sqrt{10}}{2}$**;** **7.** $\pm\sqrt{5}$; 8. $\pm\frac{\sqrt{21}}{2}$; **9.** $\pm 3\sqrt{5}$; **10.** $\pm 6\frac{2}{3}$;

11. $\pm\frac{\sqrt{395}}{3}$; **12.** $\pm\frac{\sqrt{390}}{3}$; **13.** $\pm\frac{2\sqrt{95}}{3}$; **14.** $\pm\frac{5\sqrt{11}}{11}$; **15.** $\pm\frac{5\sqrt{33}}{22}$

Exercise 11.2

1. -7, 0; **2.** -12, 8; **3.** -8, 2; **4.** -9; **5.** $-\frac{5}{2}\pm\frac{\sqrt{65}}{2}$ or $-2.5 \pm 0.5\sqrt{65}$; **6.** $-\frac{1}{2}\pm\frac{1}{2}$ or -1, 0 ; **7.** $2\pm 3\sqrt{7}$;

8. $\frac{5}{2}\pm\sqrt{7}$; **9.** -4, 1; **10.** $7\pm 5\sqrt{2}$; **11.** $14\pm\sqrt{197}$; **12.** 1, -39; **13.** $-\frac{11}{2}\pm\frac{\sqrt{1101}}{6}$ or $-5.5\pm\frac{1}{6}\sqrt{1101}$;

14. $-\frac{13}{2}\pm\frac{\sqrt{69}}{2}$; **15.** $\frac{\sqrt{6}}{2}\pm\frac{\sqrt{6}}{2}$ or $0, \sqrt{6}$

Exercise 11.3

1. $\frac{-1\pm\sqrt{5}}{2}$; **2.** -3, 1; **3.** $\frac{-3\pm\sqrt{21}}{2}$; **4.** $\pm\sqrt{3}$; **5.** -1, 5; **6.** 0, 2; **7.** $\frac{-4\pm 4\sqrt{3}}{-4} = \frac{-4}{-4}\pm\frac{4\sqrt{3}}{-4} = 1\pm\sqrt{3}$; **8.** 2;

9. NO; **10.** $\pm\sqrt{11}$; **11.** $-\frac{1}{3}$, -1; **12.** $-\frac{2}{3}\pm\frac{\sqrt{7}}{3} = \frac{-2\pm\sqrt{7}}{3}$; **13.** NO;

14. Let x = Jill's age; then $x - 1$ = Jack's age; $x(x-1) = 21$ or $x^2 - x - 21 = 0$; $x = \frac{1+\sqrt{85}}{2}$ (age requires only the positive result) ≈ 5.1 years (or 5 years and 1 months);

15. $2x(x + 6) = 20$ or $2x^2 + 12x - 20 = 0$; $x = -3 + \sqrt{19}$ (length requires only the positive result), so width = $2x$ or $-6 + 2\sqrt{19} \approx 2.72$ m

Exercise 11.4
no problems

Exercise 11.5

1. $(-\frac{1}{2}, -1\frac{1}{4})$, A; **2.** (-1, -4), A; **3.** $(-1\frac{1}{2}, -5\frac{1}{4})$, A; **4.** (0, -3), A; **5.** (2, 9), B; **6.** (1, 2), B; **7.** (1, 6), B;

8. (2, 0), B; **9.** (0, -11) A; **10.** $(\frac{2}{3}, -\frac{1}{3})$

Chapter 12: Statistics and Probability

12.1 Permutations

Probability is the measure of the expectation that an event will occur—how LIKELY it is that something will happen. **Statistics** is the study of the collection, organization, analysis, interpretation, and presentation of data, data that often includes or deals with probability. In this chapter, you will be learning about various aspects of probability and statistics: how to understand these topics, how to interpret data and how to find probabilities. First, it is important to start with the very basics of probability, permutations. There are two basic formulas for permutations depending on whether we are interested in events or items that can occur only once (without replacement) or events or items that can occur again and again.

<u>Permutations Without Replacement</u>

A **permutation** is a possible arrangement of things where precisely how the events or items are ordered matters. A permutation **without replacement** (or without repetition) is when outcomes are arranged in a definite order, however once an event occurs, it cannot occur again. For example, if Ted, Josh, and Andy were to stand in a line, each perhaps waiting to buy a theater ticket, they could all stand in different positions. Ted might be first in line, then Andy, and next Josh. Or, Josh might be first in line, then Andy, with Ted third. The number of distinct ways these three individuals could stand in line is called permutations. Notice that there are three choices for the first person in line: Ted, Josh, and Andy. For the second person in line we now have two choices remaining. Finally, only one person remains to occupy the third position in the line. So the number of permutations or different possible arrangements is 3•2•1 or 3 "factorial" (written as 3!) given by the multiplication of a series of descending integers:

$$3 \bullet 2 \bullet 1 = 6$$

In general, *n* **factorial** is given by

$$n! = n(n-1)(n-2)...3 \bullet 2 \bullet 1$$

Let's consider another example of permutations without replacement. Consider that there are six runners in a race. In how many different permutations or arrangements can the runners finish?

Since there are 6 runners that could potentially be the winner, then 5 runners remaining for 2^{nd} place, then 4 runners remaining for 3^{rd} place, etc., the number of possible outcomes is given by 6! (or 6 factorial) which is given by the multiplication of a the series of descending integers shown:

$$6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 720$$

In a permutation without replacement, every time one option takes place in a given arrangement, that option is no longer available. This condition fits our race scenario since once one runner has achieved first place, that runner is no longer in the competition to achieve second place (or any other place), only one of the five remaining runners can achieve second place, and so on.

Now say we want to see how many possible arrangements exist for the first two places in the race mentioned in the first example. To derive the answer, let's assign a letter to each runner **a**, **b**, **c**, **d**, **e,** and **f**, and make a listing of all the possible outcomes for the first two places:

ab	**ba**	**ca**	**da**	**ea**	**fa**
ac	**bc**	**cb**	**db**	**eb**	**fb**
ad	**bd**	**cd**	**dc**	**ec**	**fc**
ae	**be**	**ce**	**de**	**ed**	**fd**
af	**bf**	**cf**	**df**	**ef**	**fe**

There are 30 possible ways in which the first and second place could be filled by six different runners. This can be calculated using a formula for permutations without repetition given by

$$\frac{n!}{(n-r)!}$$

where n is the number of runners (n = 6) and r is the number of places (r = 2), so that we have

$$\frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \bullet 5 \bullet 4!}{4!} = 6 \bullet 5 = 30$$

But what if we decided to know how the first, second, and third place could be filled? We have added one of the remaining letters to each of the possible outcomes displayed above to obtain a list of 120 possible outcomes:

abc	**bac**	**cab**	**dab**	**eab**	**fab**	**abd**	**bad**	**cad**	**dac**	**eac**	**fac**
acb	**bca**	**cba**	**dba**	**eba**	**fba**	**acd**	**bcd**	**cbd**	**dbc**	**ebc**	**fbc**
adb	**bda**	**cda**	**dca**	**eca**	**fca**	**adc**	**bdc**	**cdb**	**dcb**	**ecb**	**fcb**
aeb	**bea**	**cea**	**dea**	**eda**	**fda**	**aec**	**bec**	**ceb**	**deb**	**edb**	**fdb**
afb	**bfa**	**cfa**	**dfa**	**efa**	**fea**	**afc**	**bfc**	**cfb**	**dfb**	**efb**	**feb**
abe	**bae**	**cae**	**dae**	**ead**	**fad**	**abf**	**baf**	**caf**	**daf**	**eaf**	**fae**
ace	**bce**	**cbe**	**dbe**	**ebd**	**fbd**	**acf**	**bcf**	**cbf**	**dbf**	**ebf**	**fbe**
ade	**bde**	**cde**	**dce**	**ecd**	**fcd**	**adf**	**bdf**	**cdf**	**dcf**	**ecf**	**fce**
aed	**bed**	**ced**	**dec**	**edc**	**fdc**	**aef**	**bef**	**cef**	**def**	**edf**	**fde**
afd	**bfd**	**cfd**	**dfc**	**efc**	**fec**	**afe**	**bfe**	**cfe**	**dfe**	**efd**	**fed**

Is there an easier way to calculate this? Yes--using the formula above with n = 6 (number of runners) and r = 3 (number of outcomes—first, second, and third):

$$\frac{n!}{(n-r)!} = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6 \bullet 5 \bullet 4 = 120$$

Notice that if we are concerned with the number of arrangements that are possible for all six runners, we have n = 6 and r = 6 and the formula still works since 0! is defined as 1 as shown below:

$$\frac{n!}{(n-r)!} = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{1} = 720$$

The formula for permutations without repetition, $\frac{n!}{(n-r)!}$, can be represented in two different ways: $_nP_r$ or $P(n, r)$.

Permutations With Replacement (Repetion)

For example, if Danny went to *Thrifty's* to get a triple (3-scoop) ice cream cone and there are 3 different flavors to choose from, what is the number of different permutations that Danny could select?

Let's assign the letters **a**, **b**, and **c** to the three different flavors of ice cream (**a**—apple-flavored, **b**—berry, and **c**—cherry). The possible ways Danny can order his triple cone are listed below:

aaa	**aba**	**aca**	**baa**	**bba**	**bca**	**caa**	**cba**	**cca**
aab	**abb**	**acb**	**bab**	**bbb**	**bcb**	**cab**	**cbb**	**ccb**
aac	**abc**	**acc**	**bac**	**bbc**	**bcc**	**cac**	**cbc**	**ccc**

Because we are interested in the number of permutations, the possibility of the first scoop being apple-flavored, then berry, and then cherry is considered to be a different outcome from the reverse order where the first scoop is cherry, then berry, and then apple-flavored. Thus, there are 27 possible ways in which Danny can select 3 scoops of ice cream where the order of the scoops is considered important. As you can see in this situation it doesn't matter if Danny chooses the same flavor three times in a row. Unlike the situation of the race in which once an individual crosses the finish line, that individual is no longer available to qualify again for another position; when apple flavor (**a**) is selected first, the same flavor **a** can be selected again (and even again). Therefore, to calculate the number of possible ways to arrange 3 different flavors of ice cream on a triple cone, we have $n = 3$ different flavors, selected a total of $k = 3$ different times or

$$\textit{Permutations with replacement} = n \bullet n \bullet \ldots (k\ times) = n^k$$

So, in our ice cream example we have

$$\textit{Permutations with replacement} = 3 \bullet 3 \bullet 3 = 3^3 = 27$$

Let's consider another example of permutations with replacement, also known as **permutations with repetition**. Kathy went to eat lunch at her favorite restaurant where she selected soup from ten different choices available each day. How many different permutations could Kathy make in a week? Here we have $n = 10$ soup selections and $k = 7$ repetitions (one bowl of soup each day for one week) or Permutations with replacement = $n^k \bullet 10^7 = 10{,}000{,}000$.

Summary of Permutations Without and With Replacement—Order is Important

Permutations	Notation	Formula
without replacement	$_nP_r$ or $P(n, r)$	$\frac{n!}{(n-r)!}$ where n is a positive integer and $0 \leq r \leq n$; when $n = r$, $_nP_r = n!$
with replacement	no specific notation	n^k

Exercise 12.1

1. $3! =$ _____
2. $6! =$ _____
3. $8! =$ _____
4. $8^3 =$ _____
5. $4^4 =$ _____
6. $3^3 =$ _____
7. $\frac{20!}{18!} =$ _____
8. $\frac{52!}{(52-3)!} =$ _____
9. ${}_6P_3 =$ _____
10. ${}_7P_2 =$ _____
11. ${}_8P_8 =$ _____
12. ${}_6P_2 =$ _____
13. ${}_9P_4 =$ _____
14. There are 10 entries in a horse race and each horse is equally likely to win. Only a total of three awards will be given: 1^{st}, 2^{nd}, and 3^{rd} place. What are the total number of possible outcomes for 1^{st}, 2^{nd}, and 3^{rd} place?
15. How many two-card hands can I draw from a deck when order matters (e.g., ace of spades followed by ten of clubs is different than ten of clubs followed by ace of spades)?
16. A baseball manager has decided who his 9 starting players are to be. How many different batting orders can be made?
17. How many different groups of 4-letters can be made using the letters “G”, “A”, “R”, and “Y” if each letter can be repeated?
18. How many different groupings of 4-letters can be made using the letters “G”, “A”, “R”, and “Y” if each letter can only be used once?
19. How many 3-digits numbers can be formed with the digits: 1, 2, 3, 4, 5, 6?
20. A byte is a number consisting of 8 digits, each digit can be either to 0 or to 1. How many different bytes are there?
21. A hexadecimal code consists of 4 alpha-numeric digits. Each digit can be either 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F. How many different permutations are there in a 4-digit hexadecimal number?
22. Seven letters, “a”, “b”, “c”, “d”, “e”, “f”, and “g” are placed into a hat. A letter is selected then written down and placed back into the hat again. This process continues until 3 letters have been selected. How many different permutations are there?
23. Perform the selection in problem #15, but once a letter is selected, it is not placed back in the hat.
24. A PIN code at your bank is made up of 4 digits, with replacement. (The same digit can be selected more than once). How many PIN codes are possible?
25. How many PIN codes are possible in problem #24, if the same digit cannot be used more than once?

12.2 Combinations

Unlike permutations in which the order of arrangements is important, **combinations** involve arrangements where the order is not important.

Combinations

Combinations refer to the possible ways that you can select a number of things from a larger group, where the order in which you select them does not matter. For example, let's consider we have four vases that are assigned the letters **a**, **b**, **c**, and **d.** Notice the difference between the number of different (without replacement) permutations and combinations of these four vases as shown below:

Permutations						Combinations
abcd	**adbc**	**bcad**	**cabd**	**cdab**	**dbac**	**abcd**
abdc	**adcb**	**bcda**	**cadb**	**cdba**	**dbca**	
acbd	**bacd**	**bdac**	**cbad**	**dabc**	**dcab**	
acdb	**badc**	**bdca**	**cbda**	**dacb**	**dcba**	

In this example we can see that there is only one combination for a much greater number of permutations. As a matter of fact for one combinations **abcd** there are 24 permutations. This is because in combinations the order doesn't matter as long as the same elements are chosen. Let's see how many permutations could be possible for a combination of only two elements **a** and **b**, and three elements **a**, **b** and **c**. Then we can investigate how we might calculate the number of possible combinations of chosen elements in an easier way by an adjustment to the permutations formula.

	Permutiations	Combinations
2 elements	**ab ba**	**ab**
3 elements	**abc bac cab acb bca cba**	**abc**

So with this information we can make a table and begin to see a pattern or relationship of how both permutations without replacement and combinations are related.

Combination of chosen elements	Permutations without replacement	Combinations
2	2 = 2!	1
3	6 = 3|	1
4	24 = 4|	1

Notice that the number of combinations is $\frac{1}{r!}$ times the number of permutations without replacement. Thus, we have the formula for the combinations of *n* elements taken *r* at a time:

$$Combinations = C(n,r) = P(n,r) \bullet \frac{1}{r!} = \frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!\,(n-r)!}$$

For example, if 10 students were trying to put together a team of 3 to work on a school project, the order of the three selected students is of no significance. The number of ways you could choose those 3 students would be called or referred to as a combination, and would be written as ${}_{10}C_3$, C(10,3), or $\binom{10}{3}$ and described as, "Ten taken three at a time." The number of possible different 3-student teams that can be selected given 10 students is

$$C(10,3) = \frac{P(10,3)}{3!} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{10\cdot9\cdot8\cdot7!}{(3\cdot2\cdot1)7!} = 10 \bullet 4 \bullet 3 = 120$$

Let's consider another example. In the regional math contest there can only be four students per school, however in your classroom there are 21 students, how many different groups of four students can be formed? In this problem we have $\boldsymbol{n} = 21$ students and we want to take $\boldsymbol{r} = 4$ at a time. Substituting the values into the combination formula, we have

$$C(21,4) = \frac{21!}{4!(21-4)!} = \frac{21!}{4!17!} = \frac{21\cdot20\cdot19\cdot18}{4\cdot3\cdot2\cdot1} = 5{,}985 \text{ possible groups of 4 students}$$

Have you ever wondered how many different poker hands there are in a standard deck of cards? A poker hand consists of 5 ($r = 5$) cards from a 52 card deck ($n = 52$). The 5 cards of the hand are all distinct and the order of cards in the hand does not matter. Therefore, if we compute the number of combinations C(52,5) we find there are 2,598,960 combinations of poker hands.

Recall in Section 12.1, we calculated the number of possible permutations of 6 runners achieving 1st and 2nd place was 30. If we consider how many different combinations of runners placed either 1st or 2nd place, without regard to order, we then have a combinations problems given by,

$${}_6C_2 = \frac{P(6,2)}{2!} = \frac{30!}{2!} = 15 \text{ different combinations that can obtain 1st or 2nd place.}$$

So, all we need to do is reduce the permutations formula by the number of different ways two runners could be in order.

If we consider how many different combinations of runners placed in the top 3 places, without regard to order, again we have a combinations problem given by

$${}_6C_3 = {}_6P_3/r! = 120/3! = 20 \text{ different combinations that can obtain 1st, 2nd, or 3rd place.}$$

Summary of Combinations without Replacements—Order Does Not Matter

Combinations	Notation	Formula
without replacement	${}_nC_r$ or $C(n, r)$ or $\binom{n}{r}$	$\frac{n!}{r!(n-r)!}$ or $\frac{P(n,r)}{r!}$ where n is a positive integer and $0 \leq r \leq n$

Exercise 12.2

1. $_{8}C_{5}$ = _____

2. $_{2}C_{2}$ = _____

3. $_{9}C_{5}$ = _____

4. $_{15}C_{9}$ = _____

5. $_{11}C_{7}$ = _____

6. $_{37}C_{2}$ = _____

7. $_{22}C_{3}$ = _____

8. $_{17}C_{7}$ = _____

9. $_{9}C_{8}$ = _____

10. $_{24}C_{7}$ = _____

11. Alice, Benjamin, Carl, and Daniel have been chosen by their teachers as candidates for a special project. However, only 3 students can participate in the project. How many combinations of 3 students are there?

12. How many different committees of 5 people can be formed from a group of 10 people?

13. How many five-card poker hands can I draw from a standard deck of 52 cards when order does <u>not</u> matter?

14. How many possible ways are there to obtain 3 heads in 5 coin tosses?

15. How many ways can you choose 5 starters from a team of 12 players?

16. A computer scientist has devised a password that consists of four letters of the alphabet (A through Z). None of the letters may be repeated and the order that the letters are entered does not matter. How many different possible passwords are possible?

17. Let us say there are five flavors of ice cream: banana, rocky road, lemon, strawberry and French vanilla. You can have three scoops, but can select only one of each flavor. How many different combinations are there?

18. How many ways can Bob, Mike, Joe, and Sue be arranged into a group of 3 if order does not matter?

19. Find the number of ways to take 20 pictures and arrange them into a group of 5, where order is not important.

20. There are 16 juniors and 8 seniors in the Service Club. The club is to send four representatives to the State Conference.

12.3 Probability

In Sections 12.1 and 12.2 you learned about permutations and combinations, both of which are examples of counting items that are useful in **probability** or determining the likelihood that an **event** (or **outcome**) will occur. The probability of an event that has no chance of occurring is 0 (or 0%) and the probability of an event that is certain to occur is 1 (or 100%). The higher the chances or likelihood of an event occurring, the closer that probability will be to 1.

Finding Probability

The probability ***P*** that a certain event will occur can be calculated as follows,

$$\boldsymbol{P} = \frac{\textit{number of ways an event can occur}}{\textit{number of possible outcomes}}$$

For example, if you wanted to compute the probability of rolling a 6 on a die (which has 6 sides numbered 1 through 6), we can write

$$\boldsymbol{P}(6) = \frac{\textit{number of ways of rolling a 6}}{\textit{number of possible outcomes}} = \frac{1}{6}$$

Interestingly, the probability of not rolling a 6 can be expressed as one minus the probability of rolling a 6, or

$$\boldsymbol{P}(\text{not } 6) = 1 - \boldsymbol{P}(6) = 1 - \frac{1}{6} = \frac{5}{6}$$

We obtain the same answer using the approach of computing the probability of rolling a 1, 2, 3, 4, or 5. Since there are 5 possible events out of 6 we have $\boldsymbol{P}(1, 2, 3, 4, \text{ or } 5) = \frac{5}{6}$. Notice also that

$$\boldsymbol{P}(6) + \boldsymbol{P}(\text{not } 6) = 1$$

In other words, the sum of the probability that an event will happen and the probability that the event will not happen is always 1.

What if you wanted to flip a coin and compute the probability that the result will be heads:

$$\boldsymbol{P}(\text{h}) = \frac{\textit{number of ways of obtaining heads}}{\textit{number of possible outcomes}} = \frac{1}{2}$$

What if you wanted to compute the probability of drawing a Jack out of a full deck of 52 cards? Since there are 4 different Jacks (jack of diamond, jack of hearts, jack of clubs, and jack of spades), we have

$$\boldsymbol{P}(\text{J}) = \frac{\textit{number of cards that are jacks}}{\textit{number of possible outcomes}} = \frac{4}{52} = \frac{1}{13}$$

Just remember to figure out the number of ways that the particular event CAN occur, and then divide by the total number of events possible.

We might also ask, what is the probability of rolling an odd number on the -sided die used in most dice games? Since there are three odd numbers (1, 3, and 5) out of a total of 6 possibilities, we can write the probability as follows

$$\boldsymbol{P}(\text{odd number on die}) = \frac{3}{6} = \frac{1}{2}$$

The probability of rolling an even number is the same, $\frac{1}{2}$, because again there are 3 even numbers (2, 4, and 6) on a die out of 6 total possible outcomes.

Let's consider still another example. Suppose we have a container that contains 6 red, 5 green, 3 pink, and 7 blue balls. If a single ball is chosen at random from the container, what is the probability of choosing a red ball? a green ball? a pink ball? a blue ball?

$$\boldsymbol{P}(\text{red}) = \frac{ways\ to\ choose\ red}{total\ nnumber\ of\ marbles} = \frac{6}{21} = \frac{2}{7}$$

$$\boldsymbol{P}(\text{green}) = \frac{ways\ to\ choose\ green}{total\ number\ of\ marbles} = \frac{5}{21} \qquad \boldsymbol{P}(\text{pink}) = \frac{ways\ to\ choose\ pink}{total\ number\ of\ marbles} = \frac{3}{21} = \frac{1}{7}$$

$$\boldsymbol{P}(\text{blue}) = \frac{ways\ to\ choose\ blue}{total\ number\ of\ marbles} = \frac{7}{21} = \frac{1}{3}$$

Let's use what we have previously learned to compute another probability. Returning to the example of the race with 6 runners, we might be interested in the probability that runner "a" will be first and runner "b" will be second. First, we determine the total number of possible outcomes of two runners out of 6, or compute the permutation

$${}_6P_2 = 30$$

Now, we are interested in one specific outcome "ab". So we have

$$\boldsymbol{P}(\text{ab}) = \frac{number\ of\ ways\ \text{"ab"}\ can\ occur}{number\ of\ permutations\ involving\ two\ runners} = \frac{1}{30}$$

If we were interested in two specific outcomes such as "a" first and "b" second, or "b" first and "a" second, we would have 2 events that could occur, so that the probability of these two outcomes is

$$\boldsymbol{P}(\text{ab or ba}) = \frac{number\ of\ ways\ \text{"ab"}\ or\ \text{"ba"}\ can\ occur}{number\ of\ possible\ permutatins\ involving\ two\ runners} = \frac{2}{30} = \frac{1}{15}$$

We can also use what we learned about combinations to compute probabilities. For example, a corporation had a lottery where the winner must select 4 numbers ranging from 1 to 99. What are the chances of having the winning combination of numbers? Since the order of the 4 numbers is not important in a lottery, we first compute the number of possible combinations, or

$${}_{99}C_4 = \frac{n!}{r!(n-r)!} = \frac{99!}{4!\bullet 95!} = \frac{99\bullet 98\bullet 97\bullet 96\bullet 95!}{4!\bullet 95!} = \frac{99\bullet 98\bullet 97\bullet 96}{4\bullet 3\bullet 2\bullet 1} = 3{,}764{,}376$$

Since only 1 winning combination is possible, we next compute the probability of winning:

$$\boldsymbol{P}(\text{winning lotto}) = \frac{\textit{number of possible winning lotto tickets}}{\textit{number of possible outcomes}} = \frac{1}{3{,}764{,}376} \approx 0.000000266 \text{ or}$$

about 0.0000266% chance of winning.

Probability of Independent Events

Suppose we are interested in the probability of obtaining 3 heads in a row. For the first toss, we have $\boldsymbol{P}(\text{h}) = \frac{1}{2}$. Now, just because we have tossed the coin once, does not mean that the probability of gaining heads is any different on the second or third toss. Since we are interested in the probability of the 1st toss being heads AND 2nd toss being heads AND 3rd toss being heads, the three independent probabilities multiply or

$$\boldsymbol{P}(\text{hhh}) = \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} = \frac{1}{8}$$

When two events do not affect the outcome of each other, we call these **independent events**. Suppose we are interested in the chances of rolling a 6 followed by a 5 on a die. The probability of rolling the 6 is $\frac{1}{6}$ AND the probability of rolling the 5 is $\frac{1}{6}$. The individual probabilities do not change just because we are rolling the die twice, so we multiply $\boldsymbol{P}(6)$ and $\boldsymbol{P}(5)$ to obtain,

$$\boldsymbol{P}(6 \text{ and } 5) = \frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}.$$

Now we might ask, what is the probability of obtaining three heads in a row followed by a 6 and 5 on the die. Again, we simply multiply these independent probabilities together as follows,

$$\boldsymbol{P}(\text{hhh AND } 6{,}5) = \boldsymbol{P}(\text{hhh}) \bullet \boldsymbol{P}(6 \text{ and } 5) = \frac{1}{8} \bullet \frac{1}{36} = \frac{1}{288}$$

Let's consider two more examples. What is the probability of selecting a king of hearts, and then a queen of hearts. The probability of selecting the king of hearts from a deck of 52 cards is $\boldsymbol{P}(\text{king of hearts}) = \frac{1}{52}$. If the king is placed back into the deck, the probability of selecting the queen of hearts is also $\boldsymbol{P}(\text{queen of hearts}) = \frac{1}{52}$. Thus, the probability of these two independent events is given by

$$\boldsymbol{P}(\text{king of hearts AND queen of hearts}) = \frac{1}{52} \bullet \frac{1}{52} = \frac{1}{2{,}704}.$$

Let's try another probability example. Suppose the probability of an earthquake occurring each year in Los Angeles is $\boldsymbol{P}(\text{earthquake occurs in Los Angeles in a year}) = 0.01$. We might ask, what is the probability of at least one earthquake occurring in the next 10 years. Now the way to think about this problem is this:

$$1 - \boldsymbol{P}(\text{earthquake does occur in a year}) = \boldsymbol{P}(\text{no earthquake in a year})$$

$$\text{or } 1 - 0.01 = 0.99 = \boldsymbol{P}(\text{no earthquake in a year})$$

So, first, we are really interested in the probability that no earthquake occurs in the next 10 years. Then, if we subtract this probability from 1, we will have the probability that at least one earthquake occurs in 10 years assuming the probability is independent, that is, the probability from year to year remains the same whether or not any earthquake occurs. This is likely a very poor assumption. Thus, we have

$$\boldsymbol{P}(\text{no earthquake in 10 years}) = \boldsymbol{P}(\text{no earthquake the } 1^{st} \text{ year}) \bullet \boldsymbol{P}(\text{no earthquake the } 2^{nd} \text{ year}) \bullet \ldots \bullet \boldsymbol{P}(\text{no earthquake the } 10^{th} \text{ year})$$

$$\text{or } \boldsymbol{P}(\text{no earthquake in 10 years}) = (0.99)^{10} \approx 0.904$$

Therefore, the probability of at least 1 earthquake in 10 years is

$$\boldsymbol{P}(\text{at least 1 earthquake in 10 years}) = 1 - \boldsymbol{P}(\text{no earthquake in 10 years}) = 1 - 0.904 = 0.096$$

or 9.6%

Let's consider one last example of the probability of independent events. 27% of students enrolled in Pearblossom Private School did school work regularly on the weekend and 90% of students participated in outside co-curricular activities. What is the probability that a student did school work regularly on the weekend and participated in outside co-curricular activities (assuming these events are independent)? Since these probabilities are independent, we can write

$$\boldsymbol{P}(\text{school work on weekend AND participates in outside co-curricular activities}) = \boldsymbol{P}(\text{school work on weekend}) \bullet \boldsymbol{P}(\text{participates in outside co-curricular activities})$$

Substituting our know probabilities, we have 0.27•0.90 = 0.243 or 24.3%

<u>Probability of Dependent Events</u>

Two events are **dependent** if the outcome or occurrence of the first affects the outcome or occurrence of the second so that the probability is changed. Now, let's say we do not place the king back into the deck. Again, the chance of selecting the king of hears from 52 cards is ***P***(king of hearts) = $\frac{1}{52}$; however, since now there are only 51 cards in the deck, ***P***(queen of hearts given a king of hearts was selected first) = $\frac{1}{51}$, so that without replacement, ***P***(king of hearts AND queen of hearts) = $\frac{1}{52} \bullet \frac{1}{51} = \frac{1}{2{,}652}$. Notice in this example that the **conditional probability** of selecting the queen was dependent upon selecting the king first. The notation for conditional probability of selecting a queen given selection of the king first is written ***P***(queen of hearts|king of hearts). In general, ***P***(B|A) is the conditional probability of an event B in relationship to an event A and is the probability that event B occurs given that event A has already occurred. When two events, A and B, are dependent, the probability of both occurring is: ***P***(A and B) = ***P***(A) • ***P***(B|A).

Let's consider a somewhat more challenging problem: the probability that 1 or more kings will be selected if three cards are picked (without replacement) from a standard deck of 52 cards. (There are 4 kings in a standard deck). Again, let's instead consider the probability that no king is selected in 3 picks. Then the probability that at least 1 king is selected will be given by

$$\boldsymbol{P}(\text{at least one king is selected in 3 picks}) = 1 - \boldsymbol{P}(\text{no kings are selected in 3 picks})$$

Since, initially, there are 4 kings, 48 cards are not kings in the deck, thus, the probability of not selecting a king on the first pick is $\frac{48}{52}$. Given that no king was selected, we now have 51 cards remaining, of which 47 cards are not kings. Thus, the probability of not selecting a king on the second pick is now $\frac{47}{51}$. Similarly, on the third pick, if we still have not selected a king, we now have 50 cards of which 46 are not kings, so the probability is $\frac{46}{50}$. These probabilities multiply together such that

$$\boldsymbol{P}(\text{no kings are selected in 3 picks}) =$$
$$\boldsymbol{P}(\text{no king selected on } 1^{\text{st}} \text{ pick})\bullet\boldsymbol{P}(\text{no king selected on } 2^{\text{nd}} \text{ pick})\bullet\boldsymbol{P}(\text{no king selected on } 3^{\text{rd}} \text{ pick})$$

Substituting our probabilities for each pick, we have

$$\boldsymbol{P}(\text{no kings are selected in 3 picks}) = \frac{48}{52}\bullet\frac{47}{51}\bullet\frac{46}{50} = \frac{564}{1275}$$

Thus, the $\boldsymbol{P}$(at least 1 king is selected in 3 picks) $= 1 - \frac{564}{1275} = \frac{711}{1275} \approx 0.558 = 55.8\%$

Exercise 12.3

1. What is the probability of flipping five heads in a row using a fair coin?

2. What is the probability of picking an Ace of Diamonds from a standard deck of 52 cards?

3. Alex, John, Henry, David, and Mark are running for three executive offices. What is the probability that Alex, Henry, and Mark will be selected?

4. Given the information in #3, what is the probability that Alex will be Pres., Henry will be vice-President, and Mark will be Secretary?

5. A poker hand consists of 5 cards randomly chosen from a deck of 52 cards. What is the probability you will be dealt 4 Aces and 1 King of Hearts (in any order)?

6. A poker hand consists of 5 cards randomly chosen from a deck of 52 cards. What is the probability you will be dealt 4 Aces out of the 5 cards you are dealt?

7. Rusty has a 7-volume set of Parelli® books on horses, numbered 1 through 7. What is the probability that she will read volume 3, then 1, and then 7, in that order?

8. Given the information in #7, What is the probability that Rusty will read Volumes 1, 3, and 7 in any order?

9. A box contains a nickel, a penny, and a dime. Find the probability of choosing first a dime and then, without replacing the dime, choosing a penny.

10. A coin is tossed and a die is rolled. What is the probability of getting heads on the coin and a 6 on the die?

11. One marble is randomly selected from a bag that contains 4 blue, 3 red, 5 yellow, and 4 green marbles. What is the probability of selecting either a blue marble or a red marble?

12. A bag that contains 5 green, 4 red, and 9 blue marbles. One marble is drawn randomly from the bag, then that marble is put back in the bag and another marble is drawn. What is the probability of drawing a green marble followed by a blue marble?

13. A lottery has you select 5 numbers out of the numbers 1 through 40. What is the probability of winning?

14. In a factory, there are 100 radios of which 6 are defective. You purchase 2 radios from this factory. What is the probability that both radios are defective?

15. Based on the information in question #10, what is the probability that at least one radio is defective?

16. In the United States 25% of children have asthma. What is the probability that two children chosen at random will have asthma?

17. A card is drawn from a deck of 52 standard cards and then replaced in the deck and the deck is shuffled. A second card is then drawn. What is the probability that a queen is drawn each time? (There are 4 queens in a standard deck of cards.)

18. A wine taster claims that she can distinguish four vintages or a particular wine labeled A, B, C, and D. To find the probability that she can do this by merely guessing (she is confronted with 4 unlabeled glasses) would require a calculation that uses

A. Permutations B. Combinations
C. Conditional Probability D. None of these

19. What is the probability of the wine taster identifying all 4 wine glasses correctly by merely guessing?

20. What is the probability that at least 1 of three people have the same month and day of birth (assuming 265 days in a year)?

12.4 Percents

Examples of percents or percentages are numbers with a percent sign such as 25%, 2%, 1½%, and 75.34%. Notice, these various percents can be expressed as whole numbers, mixed numbers (or fractions), and decimals. At first students might be intimidated working with percents, however this initial reaction is overcome by learning that "percent" simply means divide by 100. In other words, any percent can be changed to an "ordinary" number by removing the percent sign (%) and dividing the number by 100. Thus, given the examples of percents above, we could write their corresponding or equivalent numbers as 0.25 (or $\frac{1}{4}$), 0.02 (or $\frac{1}{50}$), 0.015 (or $\frac{3}{200}$), and 0.7534, where each percentage above was simply divided by 100 (or multiplied by 0.01) to obtain the decimal or fraction equivalent. So to change a percent written in general as *P*%, to a number, we simply use this formula,

$$P\% = \frac{P}{100}$$

Changing a Percent to a Fraction

Suppose we want to change $66\frac{2}{3}\%$ to a fraction (without the percent sign). Well, based on our above discussion, we simply need to take the number portion of the percent ($66\frac{2}{3}$) and divide by 100. Let's perform this operation step-by-step. So, to convert or change $66\frac{2}{3}\%$ to a fraction we first divide by 100, which produces this problem:

$$66\frac{2}{3}\% = \frac{66\frac{2}{3}}{100}$$

Next, to divide a mixed number, it is standard procedure to convert the mixed number to an improper fraction, so we are going to write $66\frac{2}{3}$ as $\frac{200}{3}$. You should have recalled that you take the denominator (the 3) times the whole number (66), then add the numerator (2), to obtain the numerator (or 3•66 + 2 = 200). So, now our problem becomes,

$$\frac{66\frac{2}{3}}{100} = \frac{\frac{200}{3}}{100}$$

We can rewrite this problem as

$$\frac{200}{3} \div 100$$

and use the rule that to divide fractions we can multiply by the inverse, or

$$\frac{200}{3} \bullet \frac{1}{100} = \frac{200}{300} = \frac{2}{3}$$

Some common percents and their fraction equivalents are given in the table below.

Table of Common Percents Expressed as Fractions

$1\% = \frac{1}{100}$	$10\% = \frac{1}{10}$	$20\% = \frac{1}{5}$	$25\% = \frac{1}{4}$	$30\% = \frac{3}{10}$	$33\frac{1}{3}\% = \frac{1}{3}$	$40\% = \frac{2}{5}$	$50\% = \frac{1}{2}$
$60\% = \frac{3}{5}$	$66\frac{2}{3}\% = \frac{2}{3}$	$70\% = \frac{7}{10}$	$75\% = \frac{3}{4}$	$80\% = \frac{4}{5}$	$90\% = \frac{9}{10}$	$100\% = 1$	$150\% = 1\frac{1}{2}$

Let's try another example of converting a percentage to a fraction. We will write 34% as a fraction by applying our rule of dividing by 100,

$$34\% = \frac{34}{100}$$

Next, if possible, we should always reduce the fraction to lowest terms, so

$$\frac{34}{100} = \frac{17}{50}$$

Now, let's consider changing a percentage that is expressed as a decimal number, such as 2.4%, to a fraction. Again we use the same approach as previously by dividing by 100, so

$$2.4\% = \frac{2.4}{100}$$

Next, we multiply by $\frac{10}{10}$ to remove the decimal digit (.4) in the numerator, and then reduce the fraction to simplest terms [since the numerator (24) and denominator (100) are both divisible by 8], Thus, we have

$$\frac{2.4 \bullet 10}{100 \bullet 10} = \frac{24}{1000} = \frac{3}{125}$$

Changing a Fraction to a Percent

To change a fraction to a percent, we simply multiply the fraction by 100. Let's do an example by converting $\frac{1}{8}$ to a percent. Applying our percentage rule in reverse, we simply multiply the fraction by 100, or

$$\frac{1}{8} \bullet 100 = \frac{100}{8}$$

Now since the resulting fraction is improper, let's simplify the fraction as follows,

$$\frac{100}{8} = \frac{25}{2} = 12\frac{1}{2}\% \text{ or } 12.5\%$$

Changing a Decimal to a Percent

To change a decimal to a percent, again, we simply multiply the decimal by 100. Again, let's do an example by converting 0.043 to a percent. Applying our percentage rule in reverse, we simply multiply the decimal by 100, which corresponds to moving the decimal point over two places to the left. Thus,

$$0.043 \cdot 100 = 4.3\% \text{ or } 4\tfrac{3}{10}\%$$

It is awkward to convert a repeating decimal to a percent, but it can be done! Consider the repeating decimal $0.1\overline{6}$. When we multiply by 100 to obtain a percent, we obtain $16.\overline{6}\%$. To convert the repeating decimal (the $.\overline{6}$) in this percentage to a fraction, we perform the following: Let $x = 0.\overline{6}$, then we compute $10x$ (or $10 \cdot 0.\overline{6} = 6.\overline{6}$) and perform the subtraction to obtain the result shown:

$$\begin{array}{rl} 10x &= 6.\overline{6} \\ - \quad x &= 0.\overline{6} \\ \hline 9x &= 6 \end{array}$$

Solving for x, we have $x = \frac{6}{9} = \frac{2}{3}$, so the percentage written as a repeating decimal $16.\overline{6}\% = 16\frac{2}{3}\%$ (which makes sense since $\frac{2}{3}$ is equivalent to the repeating decimal $0.\overline{6}$).

Let's consider another example of converting a repeating decimal to a percent; however, this time we will consider a decimal number that has two repeating digits: $0.\overline{62}$

As before, we multiply by 100 to convert the decimal number to a percentage and obtain $62.\overline{62}\%$. Next, to convert the repeating decimal (the $.\overline{62}$) to a fraction, let $x = 0.\overline{62}$ and then compute 100x (or $100 \cdot 0.\overline{62} = 62.\overline{62}$). Notice in this example that we multiply by 100 since there are two repeating decimals (and not by 10 as in our previous example with one digit repeating). Next we perform this subtraction to obtain the result shown:

$$\begin{array}{rl} 100x &= 62.\overline{62} \\ - \quad x &= 0.\overline{62} \\ \hline 99x &= 62 \end{array}$$

Solving for x, we have $x = \frac{62}{99}$. Thus, $62.\overline{62}\%$ is equivalent to $62\,\frac{62}{99}\%$.

Probability using Percents

Now at this point you are probably wondering how percentages (or percents) have anything to do with probability. So let's consider the following problem. A care package of food with a parachute is being dropped from a plane. The package will land somewhere in a rectangular field with dimension 400 m long and 200 m wide. Within this rectangular field is a cushioned area that is just 20 m square. What is the probability of the care package landing on the cushioned area? We first compute the total area of the larger field, 400 m x 200 m = 80,000 m^2. Then we compute the cushioned area 20 m x 20 m = 400 m^2. The probability of the care package landing in the cushioned area is the ratio of the cushioned area to the field, or

$$\boldsymbol{P}(\text{cushion landing}) = \frac{400\ m^2}{80{,}000\ m^2} = 0.005 \text{ or converting to a percent, } 0.5\% \text{ or } \tfrac{1}{2}\%$$

Since there is an 0.005 chance (or $\frac{5}{1000} = \frac{1}{200}$), this tells us that there is 1 chance in 200 drops that the care package will land in the cushioned area—assuming any location within the field is equally likely.

Let's try another application of percents to probability. Suppose there are a total 4 candidates running for office, let's call them simply **a**, **b**, **c**, and **d**. A random survey is conducted of 9850 individuals that plan to vote for 1 of these 4 candidates and 1850 votes are for candidate "**a**". What percentage of the total votes does this candidate expect to receive? We take the number of votes for this candidate and divide by the total number of voters in the survey to obtain

$$\frac{1{,}850}{9{,}850} = \frac{37}{197} \approx 0.1878 \quad \text{or} \quad 18.8\% \text{ (rounded to the nearest tenth percent)}$$

Assuming our sample was truly representative of the larger voting population, and there are 125,000 voters in the actual election, how many of these voters would we expect to vote for candidate "**a**"? We would expect 18.8% of the total 125,000 voters to vote for candidate "**a**". In mathematics, by converting the percentage to a decimal and since "of" means "times" (or the multiplication operator), we can write 18.8% of 125,000 as

$$0.188 \bullet 125{,}000 = 34{,}500 \text{ votes}$$

Taking a Percent of a Number and Other Variations

Suppose a coat that normally sells for $135.00 is on sale and marked down to 81% of the regular price (i.e., it is marked 19% off the original price). What is the sale price? This problem can be stated as 81% of $\$135 = s$, where s is the sale price. To take a percent of a number, we solve this equation by converting the percent to a decimal and multiply,

$$0.81 \bullet 135 = s$$

After performing this multiplication we find that the sales price, $s = 109.35$ (actually $109.35 when we properly label the units).

Let's consider two other variations on this same problem. Suppose we know the regular price and the sales price, but do not know the percentage. The problem then can be stated as: What percentage of the regular price ($135) is the sales price ($109.35)? Thus, we are interested in

$$P\% \text{ of } \$135 = 109.35$$

Let's remove the percent by dividing P by 100 and re-write the expression in terms of an algebraic equation:

$$\frac{P}{100} \bullet 135 = 109.35$$

Now we can solve for the percentage, P, by multiplying both sides by $\frac{100}{135}$ to obtain

$P = 109.35 \bullet \frac{100}{135} = 81$ which actually designates that the mark down price of $109.35 is 81% of the original price of $135—or, in other words, the mark down price is 19% off the original price.

Finally, let's say we know the mark down percentage (81%) and we know the sales price ($109.35), but desire to find the regular price. We know that 81% of the regular price is the sales price, or 81% of r = $109.35, if we let r ***be*** the unknown regular price. In algebraic terms, we have

$$0.81r = 109.35$$

We divide each side of the equation by 0.81 to obtain,

$$r = \frac{109.35}{0.81} = 135$$ or $135.00 when we apply the proper units.

Calculating the Percent of Increase or Decrease

Let's say that the mean temperature in Alaska increased from 13°C in January to 26°C in February. We might be interested in knowing the percent increase in temperature. The general formula for increase (or decrease) is given by

$$\text{Percent of change} = \frac{V_f - V_i}{|V_i|} \bullet 100$$

where V_f is the final value and V_i is the initial value. Notice that in the percent of change formula, we take the absolute value of the initial value, V_i. If V_f is greater than V_i (or $V_f > V_i$), then the percent of change is a percent of increase; otherwise, we have a percent of decrease. So, using our example values with $V_f = 26$ and $V_i = 13$, the percent of change is

$$\frac{26-13}{13} \bullet 100 = 100\% \text{ increase}$$

Notice that this formula works with negative values. Let's say the temperature in January increased from -13°C to 3°C in February. The percent of change

$$\frac{3-(-13)}{|-13|} \bullet 100 = \frac{16}{13} \approx 1.23 = 123\% \text{ increase}$$

Note: The denominator $|-13|$ denotes the absolute value of -13, which is positive 13. The denominator of the percent of change formula will always be a positive number.

Let's do one more example that demonstrates a percent of decrease. Let's say the January temperature decreased from -13°C to -19°C in February. The percent of change is

$$\frac{-19-(-13)}{|-13|} \bullet 100 = \frac{-6}{13} \approx -0.4615 = -46.15\% \text{ or a } 46.15\% \text{ decrease}$$

Notice in the above example, when we obtain a negative result (i.e., $V_f - V_i < 0$), we indicate that the change was -46.15% or a 46.15% decrease (and then do not show the leading negative sign—since we have used the word "decrease").

Exercise 12.4

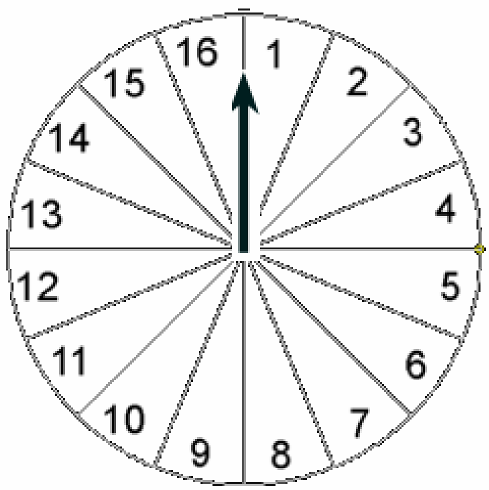
Convert each percent to a fraction in simplest terms.

1. 25% = _____
2. $\frac{2}{9}$% = _____
3. 8% = _____
4. $8\frac{2}{9}$% = _____
5. 2.5% = _____
6. 1.1% = _____
7. 25.64% = _____
8. 33 $\frac{1}{3}$% = _____
9. $87\frac{1}{2}$% = _____
10. 1.2% = _____

Convert the decimal or fraction to a percent in simplest terms.

11. $\frac{3}{5}$ = _____
12. $0.\overline{45}$ = _____
13. 0.06 = _____
14. $\frac{2}{7}$ = _____
15. 0.0013 = _____
16. 20% of 102 is what number?
17. 98% of 13 is what number?
18. 80 of what number is 90?
19. $33\frac{1}{3}$% of 99 is what number?
20. What percent of 26 is 4?
21. What percent of 85 is 40?
22. 9% of what number is 52?
23. What is the percent of increase from 20 to 25?
24. What is the percent of decrease from 25 to -5?
25. You purchase a computer that is on sale for 18% off the original price of $415.00. What is the sale price?
26. A computer originally priced at $350.00 was sold by the store manager for $305.00. What percent mark down did the manager apply?
27. A computer is purchased for $350.00 in Pearblossom, CA where the tax rate is $9\frac{1}{4}$%. What is the amount of tax on the purchase?

Questions 28-30 make reference to the spinner below on a circular board divided into 16 equal-sized segments.



28. What percentage of the spins will the spinner will land on an even number?
29. The spinner is spun 3 times. What is the percentage chance that the spinner will land first on 1, then on 2, and then on 3 in 3 spins? Round answer to nearest millionths.
30. What percentage of the spins will the spinner land on either 1 or 16?

12.5 Data Bias (Optional)

In statistics, it is important to learn and understand how to correctly collect data that is to be subsequently analyzed, especially if probability is involved. If the data is incorrectly collected, the figures and conclusions that are generated will be biased and incorrect--not properly reflecting the true population of interest. In this lesson, we consider some of the main types of errors in data collection.

Random Sampling

Say you were trying to put together the probability on the statistics of being a girl with blonde hair at Hollow Hills High School. Since there are over 2,000 students you conclude that it would be too difficult to count the number of girls vs. boys, and then the number of girls with blonde hair. You instead to decide to take a **random sampling**, that is a sample of the total population that is completely random in nature. However, which of the following would be examples of a true random sampling?

- You decide to sample all of the students on the cheerleading team.
- You decide to sample ALL of the Grade 10 girls.
- You stand at the school's entrance for 30 minutes and count all of the blonde girls that you see out of the total number of students that entered.

If you chose the last one, you would be correct. The first example is not a random sampling: (1) it is likely that students on the cheerleading team would be predominantly girls and (2) not all of the girls in Hollow Hills High would have an equally likely chance of being on the cheerleading team--which might have a stereotyped preference toward selecting blonde girls. Also, the number of blonde girls in Grade 10 may not be representative of the number of blondes in the other grades (e.g., Grades 9, 11, and 12), as in the second example. The last example, where you randomly count how many blonde girls enter the school, would be a stronger random sampling since all of the students have an equally likely chance of entering the school property.

It is important to randomly sample since if your sample is ***not*** random you will not have a probability that truly reflects the real statistics that characterize the larger population.

Sampling Bias

Sampling bias describes an error in statistics where either the interviewer or the respondent answers in a way that could compromise, or wrongly influence, the data and the subsequent probability, or the data is collected non-randomly. Sampling bias really is an example of when data is collected in a way that is not random.

For example, from the example previously, if a data collector were to only count the number of blonde girls on the cheerleading team at Hollow Hills High, this would be a perfect example of sampling bias, since the collection was not random. The subsequent calculated probability would not be representative of the school population.

Consider as another example a published study by an animal medical center that reported an amazingly high 90% survival rate of cats falling from apartment buildings seven floors or higher [Whitney WO, Mehlhaff CJ. High-rise syndrome in cats. J Am Vet Med Assoc 1988 Feb. 15;

192(4):542]. This percentage, upon closer inspection, is likely an over estimate of the true survival rate, since cats that had obviously died upon impact were never brought to the animal treatment center in the first place. If you found that your cat had fallen seven stories and was obviously dead upon impact on the concrete street below, would you bother taking this animal to a veterinarian for "treatment"?

Non-Response Bias

Non-response bias is yet another type of data collection error that stems from pulling a sample that is not random. Non-response bias occurs when some members of the population are less likely to respond than others, therefore throwing off the subsequent random sample.

For example, if an email was sent out at Hollow Hills High asking all blonde female students to respond, some may not answer (1) because they may not have a computer with Internet, (2) they may have thought it was unnecessary, (3) they may not have had time, or (4) just thought that the survey was not something they desired to participate in. This would then influence the results of the data collection and subsequent data analysis.

Therefore, it is important to make certain that the sample from which the probability is being calculated is selected randomly and in a way that would represent the actual population.

Reporting a Single mean of a "Bimodal" distribution and Failure to Stratify

Sometimes even trusted agencies resort to using improper and misleading statistics. The Centers for Disease Control and Prevention (CDC, Atlanta, GA) published a report stating the shingles rate among children aged less than 10 years old was approximately 80 cases per 100,000 person-years. This statistic was misleading because it represented an average or mean rate of two widely different **cohort**s (or groups) as follows: (1) vaccinated children demonstrated a shingles rate of 40 cases per 100,000 person-years and (2) unvaccinated children that had previously had natural chickenpox demonstrated a rate of 400 cases per 100,000 person years [1]. The single mean rate that was reported represented neither of the rates of the two cohorts very well. In statistics, reporting a single mean of a **bimodal distribution** is invalid. Similarly, sometimes statistics on shingles rates were reported among individuals aged less than 20 years old. Again, it is often preferable to **stratify** or separate the age groups into smaller age categories such as children aged 0 to 10 years old and adolescents aged 11 to 19 years old**—**especially when the shingles rate in these two groups substantially differs. Failure to stratify data can lead to missing important trends that would otherwise become manifest through a proper statistical analysis. The mean or average effectiveness or efficacy of the varicella (chickenpox) vaccine during 1997 to 2001 was initially reported by the CDC as 78.9% in the Antelope Valley region; however, using a yearly (or annual) trend, the efficacy was declining and was lower than 60% by 2002 [2,3].

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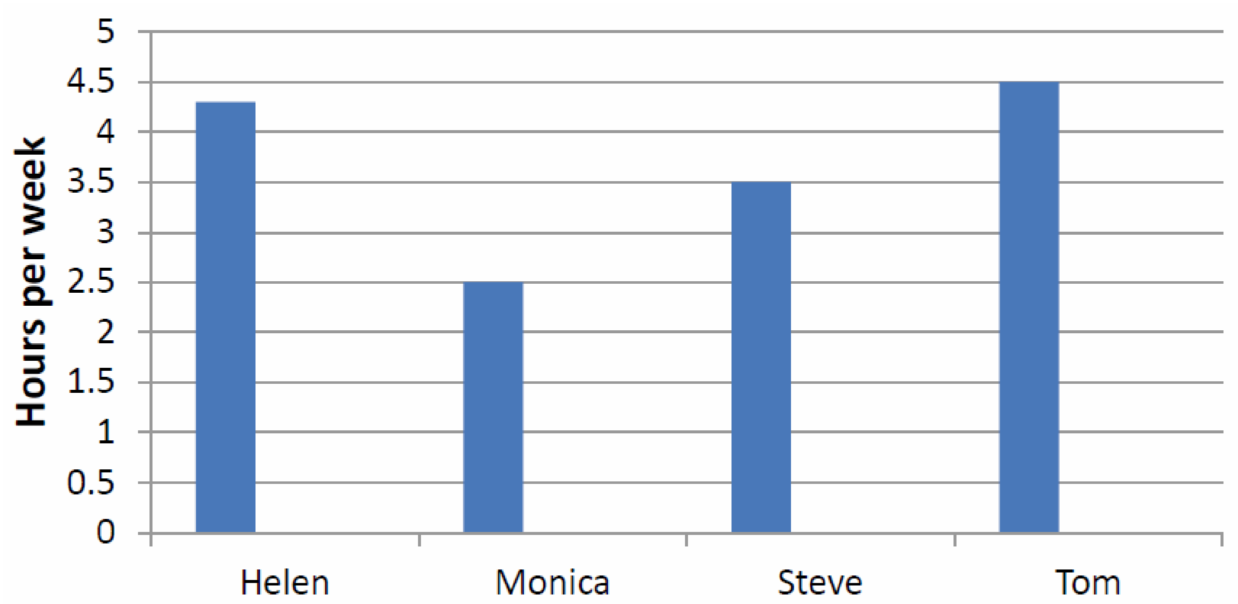
1. Goldman GS. Response to letter to editor by Jumaan: Goldman's role in the Varicella Active Surveillance Project. Vaccine 2004;22(25–26):3232–6.
2. Goldman GS. Universal varicella vaccination: efficacy trends and effect on herpes zoster. Int J Toxicol 2005;24(4):203–13.
3. Goldman GS, King PG. Review of the United States universal varicella vaccination program: Herpes zoster incidence rates, cost-effectiveness, and vaccine efficacy based primarily on the Antelope Valley Varicella Active Surveillance Project data. Vaccine 2012 Mar; 31(13):1680-94 (http://dx.doi.org/10.1016/j.vaccine.2012.05.050).

12.6 Analyzing Data - Graphs

In the previous lesson you learned how to analyze data with plots. In this lesson, you will be learning how to analyze and interpret data with graphs.

Interpreting Graphs

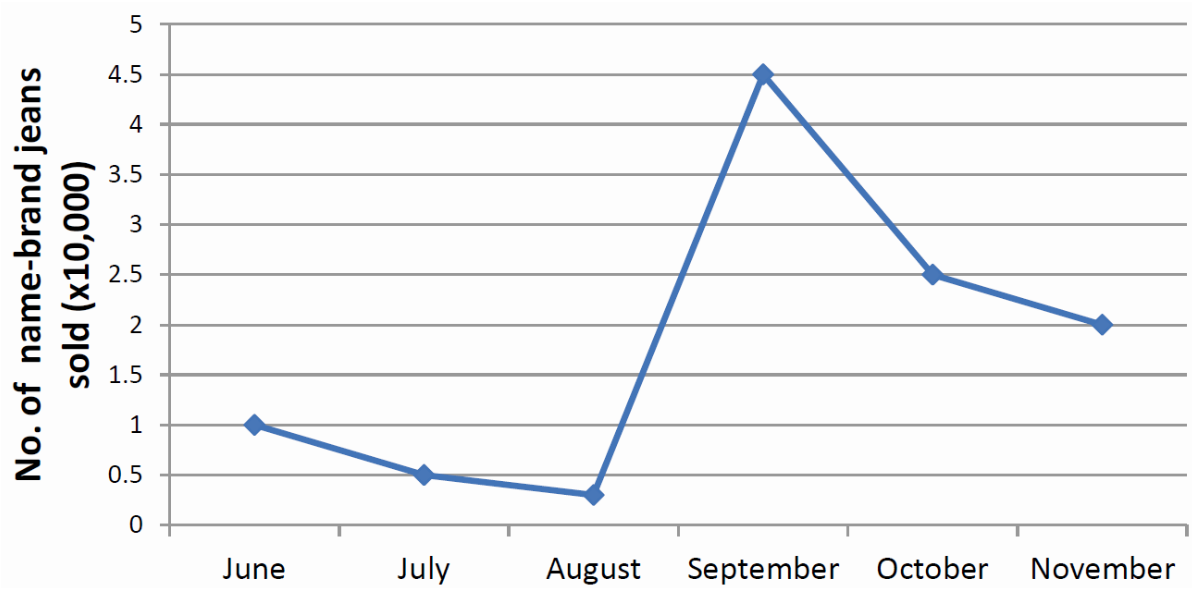
Graphs are helpful ways to interpret data, since they allow us to easily represent the data in a way that can be clearly seen and understood. Graphs can come in a number of different forms including frequency tables, bar graphs, line graphs, circle graphs or picture graphs. In order to understand a graph, you first must understand what is being measured in the graph, and how it is being measured (or the units of measurement). For example, consider a graph that showed how many hours of chores your friends do each week. It might look something like this bar graph:



Notice, that along the bottom (horizontal axis) of the graph is the name of the individual that is doing the chores, and along the left side (vertical axis) is the time (in hours) spent on chores. From this graph, we can see that Tom spends the most time doing chores per week (4.5 hours) and Monica spends the least (only 2.5 hours).

Interpreting a Line Graph

A graph can also be in the form of a line, which allows us to see trends and patterns. For example, if we were to graph the number of name-brand jeans sold by month to students, it might look something like this:

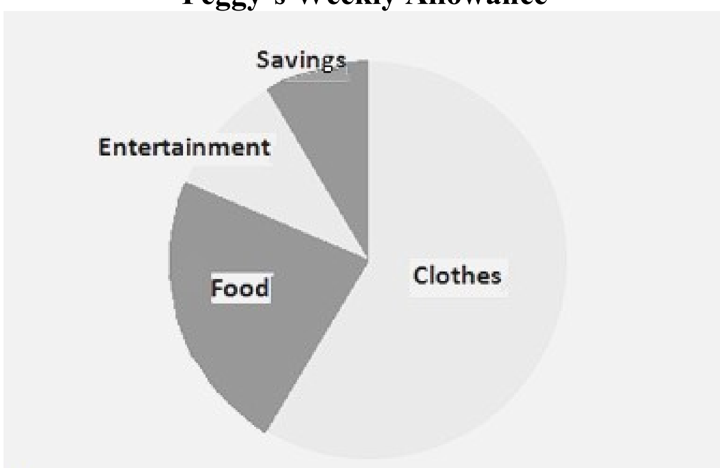


When do students purchase the most name-brand jeans? The peak is in September (when they go back to school) when there is 45,000 pairs of jeans sold. When do students buy the least quantity of name-brand jeans? The minimum occurs in August, when the weather is hot and students prefer other clothing.

Interpreting a Circle Graph

A graph can also be shown in the form of a circle graph, or pie graph. For example, if we were to look at how Peggy spent her allowance each week, it might look something like this:

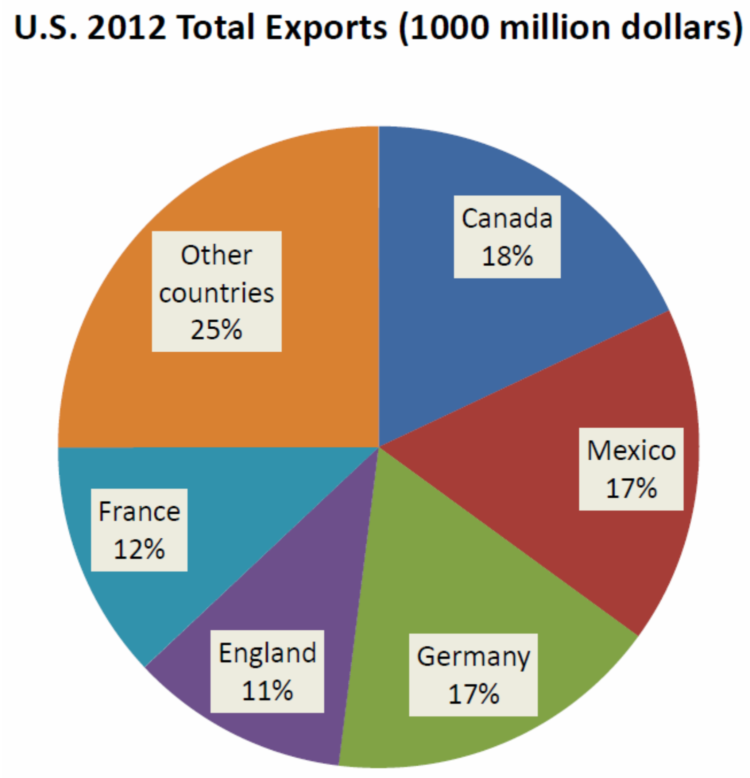
Peggy's Weekly Allowance



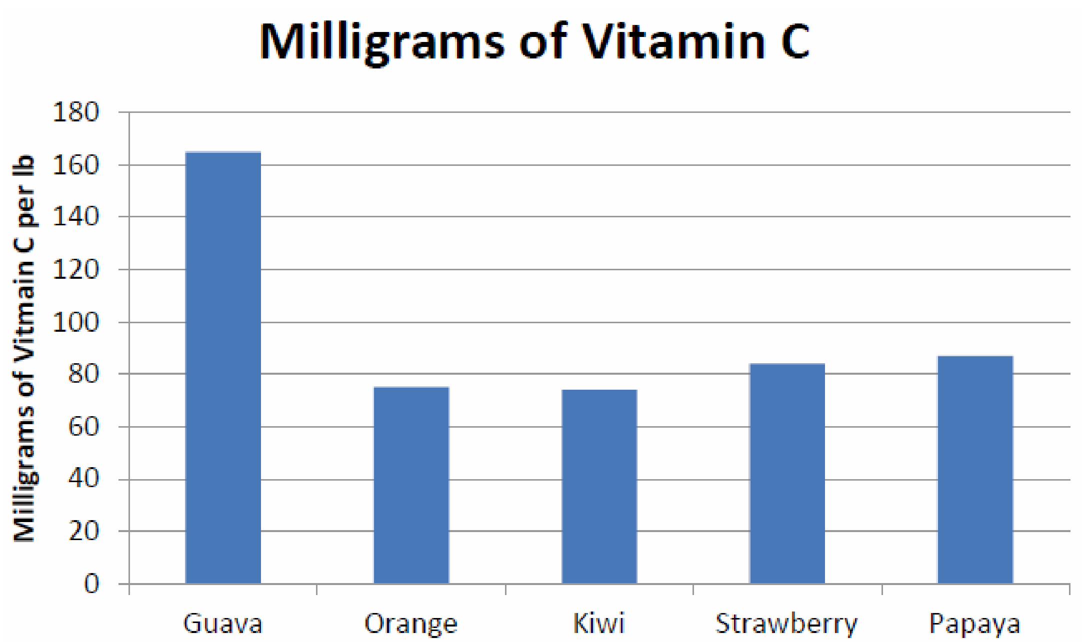
From this pie chart, we observe that Peggy spends most of her money on clothes, and saves only a small fraction of her money. She spends more money on food than on entertainment (movies, etc.). Note that unless we were told how much money Peggy makes per week and the percentage of the circle that each category occupies, we would not be able to determine the amount spent in each of the different categories.

Exercise 12.6

Answer the questions pertaining to each of the following graphs.



1. To which country does the U.S. export the least amount of money in products?
2. Which named country or countries rank second among the U.S. exports?
3. Which country buys approximately 180 million dollars in products from the U.S.?
4. How much money in products did the U.S. sell to France in 2012?
5. Based on the graph, can you tell how many countries the U.S. sold products to?



6. Charles needs to increase his intake of Vitamin C. Which fruit should he eat in order to increase it as fast as possible?

7. Which fruit has the second highest concentration of vitamin C?

8. The daily vitamin C requirement for men is 90 mg/day. Which fruit should a man eat if he is to achieve the requirement by ingesting the fewest pounds of fruit?

9. The daily vitamin C requirement for a child 9 to 13 years old is 45 mg, which fruits should they eat?

10. Could you tell by the graph which fruit you would prefer to eat?

12.7 Misleading Statistics

It is therefore appropriate to conclude a chapter on statistics by briefly considering the danger of misleading statistics—statistics that have no sound basis or are misconstrued.

Correlation Does Not Equal Causation

The first example of a statistic that could be misleading is that correlation does not equal causation. In other words, just because the probability of two events appear to be related, does not necessarily mean that they are. For example, as ice cream sales increase, the rate of deaths due to drowning increase. On the surface, one might conclude that the increase in drowning is caused by the increase in ice cream consumption. Therefore, the "obvious" conclusion is that ice cream consumption causes drowning.

This conclusion fails to recognize the importance of time and temperature in relationship to ice cream sales and drownings. Ice cream is sold during the hot summer months at a much greater rate than during colder times, and it is during these hot summer months that people are more likely to engage in activities involving water, such as swimming. The increased drowning deaths are simply caused by more exposure to water-based activities, not the consumption of ice cream. Thus, the "obvious" conclusion is false. Therefore, you should not assume that just because two events appear to be related, that they are.

Self-Selected Population

At times, certain companies may advertise their product with statements like, "99% of customers were happy with this product!" "97% of doctor's interviewed preferred product X over product Y." Be careful. Companies such as the above often do not describe or explain where they took their sample from, and oftentimes the sample itself is quite biased and compromised, making the stated statistics misleading and biased.

Confounders

At times, statistics may be accurately calculated from a random sample, but have **confounders** or other **variables** representing conditions that compromise the statistical analysis. For example, if a poll was taken for accidents in different age groups, the poll would most likely show that 16-year olds are the safest drivers who have the least amount of accidents. Is this true? Hardly! 16-year olds tend to be erratic, new, inexperienced and are more likely to take risks while driving. However, since 16-year olds on average drive far less than say, a thirty-year old, it would appear when polled, and the variable of actual time spent driving is omitted, that 16-year olds are the safest drivers. Such confounders can compromise the statistics.

Use Caution

There are many more examples, but the lesson is: be careful. As this chapter hopefully taught you, the chances of a certain event happening can be calculated and you can take this new-found knowledge and start applying it right in your own life.

Often vaccine manufacturers promote "Vaccines are safe!" Healthcare provides will often admit that they may cause only mild adverse effects, such as swelling or redness at the injection site. In recent times, however, vaccine safety has been compromised by failure of the vaccine researchers and sponsors to (1) measure and compare adverse vaccine reactions with a true placebo, (2) perform long-term testing, (3) test the combination and interaction effects of one vaccine administered concurrently with other vaccines and other accumulated doses, and (4) provide data that is openly accessible to independent experts for independent re-evaluation.

I often hear parents say, "But my child cannot attend school unless he (or she) is vaccinated." Although this is what school and healthcare officials have promoted, there is a provision for a parent or guardian to sign a religious or philosophical exemption (in most states) and these exemptions satisfy legal requirements.

From 1990 to 2000 there was an 800% increase in autism spectrum disorders in children. The Thimerosal, a preservative that is 50% mercury by weight, used in many vaccines during that period was at a relatively "low" level in each of the childhood vaccines that contained it. However, healthcare agencies, including the FDA, neglected to compute the accumulated dose delivered to children from the administration of multiple vaccines--a calculation that someone in middle school could do! The CDC investigated the effect of Thimerosal and initially described a relationship that demonstrated autism spectrum disorders increasing with dosage. However, when the CDC published their paper after several revisions, they reported no association between autism and vaccines. This led to a congressional investigation. However, when the initial study data were requested, the CDC claimed "all the original datasets had been lost."

Adverse reactions to vaccines and other interventions invariably start as a small number of poorly described reports which are anecdotal and easily attributed to chance. As the numbers increase, those in authority in public health discern something might be wrong and closer scrutiny is needed. When the numbers reach the hundreds, decision makers have to persuade themselves that every adverse reaction is a false alarm—not a single one is a true association. When that happens, the numbers support a causal relationship except for the skeptics and those with conflicts of interest—who accept nothing but "scientific proof."

Public health officials and their respective medical establishments in the United States and United Kingdom often ignore important evidence, especially with regard to vaccines, stating "the weight of currently available scientific evidence does not support the hypothesis..." U.S. professor Donald W. Miller, Jr., MD and British lawyer Clifford G. Miller, Esq, explain, "Editors can subvert peer review by selecting only reviewers who will reject papers that run counter to—or praise papers that support—the interests of journal's advertisers or its owners. Lines of independent research contradicting conventional wisdom can systematically remain unpublished." They continue, "Such hard-to-publish research may prove that what the scientific community generally accepts as correct is, in fact, wrong. Research follows the funding, resulting in a wealth of publications favoring the funding interests. This can have a disproportionate effect on the 'weight' of evidence, especially for epidemiologic evidence in court." [On Evidence, Medical and Legal, Journal of American Physicians and Surgeons, Fall 2005;10(3):70–75].

Recent medical journal papers have highlighted dangers associated with Hepatitis B, rotavirus, HPV vaccines, DTaP, MMR, and others. The general public is often left in the dark due to publication bias and search engine bias on the Internet ["Web of Trust"] that (a) promotes websites sponsored by the pharmaceutical industry and (b) suppresses information that challenges vaccine safety. Despite study limitations, unvaccinated children were healthier than their vaccinated peers in terms of the chronic conditions studied (e.g., asthma, eczema, ear infections, tonsillitis, etc.). [Kemp T, Pearce N, Fitzharris P, Crane J, Fegusson D, St. George I, Wickens K, Beasley R. Is infant immunization a risk factor for childhood asthma or allergy? *Epidemiology* 1997, Nov.; 8(6):678-80.]

Exercise 12.7

State whether the following are solid or misleading examples of statistics and explain why:

1. The money housewives spend on groceries when they purchase our garbage bags decreased according to the statistics. Therefore, our product decreases the household general expenses.

2. 99% of the scientists in our R&D department recommended our product.

3. According to statistics people that wash their hands before preparing their foods have shown less cases of food poisoning. Therefore, they are less likely to contaminate the food they are preparing.

4. 95% of the population interviewed in the area of mental health services mention that they are unhappy. Therefore, we can conclude that depression in the population has dramatically increased.

5. 80% of the employees with a car in a corporation feel that they need a raise. Therefore the employees need to sell their cars.

6. The sales in a clothing store increases when the weather gets warmer. Therefore, the owners have to set the thermostat to a warmer temperature to ensure sales.

7. Less than 95% of the car accidents that occurred in the winter season in a small town were attributed to weather conditions; therefore driving should be banned depending on the weather conditions.

8. 95% of the subjects that increased their intake of vitamin C showed less seasonal respiratory diseases. Therefore, to reduce the risk of catching a seasonal cold you should take more vitamin C.

Answers to Chapter 12 Exercises

Exercise 12.1
1. 6; **2.** 720; **3.** 40320; **4.** 512; **5.** 256; **6.** 27; **7.** 380; **8.** 132600; **9.** 120;; **10.** 42; **11.** 40320; **12.** 30; **13.** 524; **14.** ${}_{10}P_3$ = 720; **15.** ${}_{52}P_2$=2652; **16.** ${}_{9}P_9$=362880; **17.** 4^4=256; **18.** ${}_{4}P_4$=4!=24; **19.** 6^3=216; **20.** 2^8=256; **21.** 16^4=65536; **22.** 7^3=343; **23.** ${}_{7}P_3$=210; **24.** 10^4=10000; **25.** ${}_{10}P_4$=5040

Exercise 12.2
1. 56; **2.** 1; **3.** 126; **4.** 5005; **5.** 330; **6.** 666; **7.** 1540; **8.** 19,448; **9.** 9; **10.** 346104; **11.** ${}_{4}C_3$=4; **12.** ${}_{10}C_5$=252; **13.** ${}_{52}C_5$=201376; **14.** ${}_{5}C_3$=10; **15.** ${}_{12}C_5$=792; **16.** ${}_{26}C_4$=14950; **17.** ${}_{5}C_3$=10; **18.** ${}_{4}C_3$=4; **19.** ${}_{20}C_5$= 15504; **20.** ${}_{24}C_4$=10626

Exercise 12.3
1. $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$; **2.** $\frac{1}{52}$; **3.** $\frac{1}{C(5,3)} = \frac{1}{10}$; **4.** $\frac{1}{P(5,3)} = \frac{1}{60}$; **5.** $\frac{1}{C(52,5)} = \frac{1}{2{,}598{,}960}$;

6. $\frac{48}{C(52,5)} = \frac{48}{2{,}598{,}960} = \frac{1}{54145}$; **7.** $\frac{1}{P(7,3)} = \frac{1}{210}$; **8.** $\frac{1}{C(7,3)} = \frac{1}{35}$; **9.** $\frac{1}{3}\bullet\frac{1}{2} = \frac{1}{6}$; **10.** $\frac{1}{2}\bullet\frac{1}{6} = \frac{1}{12}$;

11. $\frac{4\ blue+3\ red}{16} = \frac{7}{16}$; **12.** $\frac{5}{18}\bullet\frac{9}{18} = \frac{5}{36}$; **13.** $\frac{1}{C(40,5)} = \frac{1}{658{,}008}$; **14.** $\frac{6}{100}\bullet\frac{5}{99} = \frac{1}{330}$;

15. 1– ***P***(none are defective) = $1 - \frac{94}{100}\bullet\frac{93}{99} = 1 - \frac{1457}{1650} = \frac{193}{1650}$; **16.** 0.25•0.2 = 0.0625 or 6.25%;

17. $\frac{4}{52}\bullet\frac{4}{52} = \frac{1}{169}$; **18.** A; **19.** $\frac{\mathbf{1}}{\boldsymbol{P}(\mathbf{4,4})} = \frac{1}{24}$; **20.** ***P*(**at least 1 has the same birthday) =

1 – ***P***(none have the same birthday) = $1 - \frac{365}{365}\bullet\frac{364}{365}\bullet\frac{363}{365} = 1 - \frac{132132}{133225} \approx 1 - 0.9918 = 0.0082$ or 0.82%

Exercise 12.4
1. $\frac{1}{4}$; **2.** $\frac{1}{450}$; **3.** $\frac{2}{25}$; **4.** $\frac{37}{450}$; **5.** $\frac{1}{40}$; **6.** $\frac{11}{1000}$; **7.** $\frac{641}{2500}$; **8.** $\frac{1}{3}$; **9.** $\frac{7}{8}$; **10.** $\frac{3}{250}$; **11.** 60%; **12.** $45\frac{5}{11}$%;

13. 6%; **14.** $28\frac{4}{7}$%; **15.** 0.13%; **16.** 20.4 or $20\frac{2}{5}$; **17.** 12.74; **18.** $112\frac{1}{2}$; **19.** 33; **20.** $11\frac{1}{9}$%;

21. $47\frac{1}{17}$%; **22.** $577\frac{7}{9}$; **23.** 25%; **24.** 120%; **25.** $340.30; **26.** $12\frac{6}{7}$%; **27.** $32.38; **28.** $\frac{1}{2}$ = 50%;

29. $\left(\frac{1}{16}\right)^3 = \frac{1}{4096} \approx 0.000244 = 0.0244\%$; **30.** $\frac{1}{8}$ = 0.125 = 12.5%

Exercise 12.5
no problems

Exercise 12.6
1. not possible to answer; **2.** Mexico, Germany; **3.** Canada; **4.** 120 million; **5.** no; **6.** guava; **7.** papaya; **8.** guava; **9.** all meet the requirements; **10.** no

Exercise 12.7

1. misleading statistics, correlation does not mean causation; the fact that the sample group spent less money on groceries may be a coincidence and not necessarily the consequence of using a product;

2. misleading statistics; self-selected population; we do not know which criteria were used by the R&D department of the company that is advertising their own product;

3. solid statistic; details about the population sampled, attempt to explain in a sound way the correlation in a cause-effect way; even though the conclusion could be questioned from different angles the argument is in general solid.

4. misleading statistics; self-selected population; we do not know exactly how many people in the general population have depression, but it would be logical to find an increased number of people with clinical depression in the mental health services department; a flawed study design leads to a misleading statistic.

5. misleading conclusion; although the statistic may be correct, the conclusion reached is not based on any sound correlation (the desire for a raise and employee has a car) or background information;

6. misleading statistic; correlation does not equal causation; while weather or temperature could be a variable that influences the sales of summer clothes and other items, this does not mean that owners need to keep high temperatures inside the store; perhaps a more sound approach would be to change the clothes and items according to the season;

7. misleading conclusion; although the statistic might be correct, the conclusion reached is not based on any sound correlation or background information; weather condition is not the only safety factor to take into consideration while driving. an in-depth analysis of additional factors would be necessary to assess if it is safe or not to drive based on certain weather conditions;

8. solid statistic; details are provided about the population sampled, there is an attempt to explain in a sound way a cause-effect correlation; although the conclusion could be questioned from different angles, the argument is generally solid.

Appendix I: Trigonometric Values Corresponding to Angles 0 to 90 Degrees

Degree	Sine	Cosine	Tangent
0	0.00000	1.00000	0.00000
1	0.01745	0.99985	0.01746
2	0.03490	0.99939	0.03492
3	0.05234	0.99863	0.05241
4	0.06976	0.99756	0.06993
5	0.08716	0.99619	0.08749
6	0.10453	0.99452	0.10510
7	0.12187	0.99255	0.12278
8	0.13917	0.99027	0.14054
9	0.15643	0.98769	0.15838
10	0.17365	0.98481	0.17633
11	0.19081	0.98163	0.19438
12	0.20791	0.97815	0.21256
13	0.22495	0.97437	0.23087
14	0.24192	0.97030	0.24933
15	0.25882	0.96593	0.26795
16	0.27564	0.96126	0.28675
17	0.29237	0.95630	0.30573
18	0.30902	0.95106	0.32492
19	0.32557	0.94552	0.34433
20	0.34202	0.93969	0.36397
21	0.35837	0.93358	0.38386
22	0.37461	0.92718	0.40403
23	0.39073	0.92050	0.42447
24	0.40674	0.91355	0.44523
25	0.42262	0.90631	0.46631
26	0.43837	0.89879	0.48773
27	0.45399	0.89101	0.50953
28	0.46947	0.88295	0.53171
29	0.48481	0.87462	0.55431
30	0.50000	0.86603	0.57735
31	0.51504	0.85717	0.60086
32	0.52992	0.84805	0.62487
33	0.54464	0.83867	0.64941
34	0.55919	0.82904	0.67451
35	0.57358	0.81915	0.70021
36	0.58779	0.80902	0.72654
37	0.60182	0.79864	0.75355
38	0.61566	0.78801	0.78129
39	0.62932	0.77715	0.80978
40	0.64279	0.76604	0.83910
41	0.65606	0.75471	0.86929
42	0.66913	0.74314	0.90040
43	0.68200	0.73135	0.93252
44	0.69466	0.71934	0.96569
45	0.70711	0.70711	1.00000
46	0.71934	0.69466	1.03553
47	0.73135	0.68200	1.07237
48	0.74314	0.66913	1.11061
49	0.75471	0.65606	1.15037
50	0.76604	0.64279	1.19175
51	0.77715	0.62932	1.23490
52	0.78801	0.61566	1.27994
53	0.79864	0.60182	1.32704
54	0.80902	0.58779	1.37638
55	0.81915	0.57358	1.42815
56	0.82904	0.55919	1.48256
57	0.83867	0.54464	1.53986
58	0.84805	0.52992	1.60033
59	0.85717	0.51504	1.66428
60	0.86603	0.50000	1.73205
61	0.87462	0.48481	1.80405
62	0.88295	0.46947	1.88073
63	0.89101	0.45399	1.96261
64	0.89879	0.43837	2.05030
65	0.90631	0.42262	2.14451
66	0.91355	0.40674	2.24604
67	0.92050	0.39073	2.35585
68	0.92718	0.37461	2.47509
69	0.93358	0.35837	2.60509
70	0.93969	0.34202	2.74748
71	0.94552	0.32557	2.90421
72	0.95106	0.30902	3.07768
73	0.95630	0.29237	3.27085
74	0.96126	0.27564	3.48741
75	0.96593	0.25882	3.73205
76	0.97030	0.24192	4.01078
77	0.97437	0.22495	4.33148
78	0.97815	0.20791	4.70463
79	0.98163	0.19081	5.14455
80	0.98481	0.17365	5.67128
81	0.98769	0.15643	6.31375
82	0.99027	0.13917	7.11537
83	0.99255	0.12187	8.14435
84	0.99452	0.10453	9.51436
85	0.99619	0.08716	11.43005
86	0.99756	0.06976	14.30067
87	0.99863	0.05234	19.08114
88	0.99939	0.03490	28.63625
89	0.99985	0.01745	57.28996
90	1.00000	0.00000	undefined

Appendix II: Square Roots of Numbers 1 through 200

n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$
1	1	26	5.09902	51	7.14143	76	8.71780	101	10.04988	126	11.22497	151	12.28821	176	13.26650
2	1.41421	27	5.19615	52	7.21110	77	8.77496	102	10.0995	127	11.26943	152	12.32883	177	13.30413
3	1.73205	28	5.29150	53	7.28011	78	8.83176	103	10.14889	128	11.31371	153	12.36932	178	13.34166
4	2	29	5.38516	54	7.34847	79	8.88819	104	10.19804	129	11.35782	154	12.40967	179	13.37909
5	2.23607	30	5.47723	55	7.41620	80	8.94427	105	10.24695	130	11.40175	155	12.44990	180	13.41641
6	2.44949	31	5.56776	56	7.48331	81	9	106	10.29563	131	11.44552	156	12.49000	181	13.45362
7	2.64575	32	5.65685	57	7.54983	82	9.05539	107	10.34408	132	11.48913	157	12.52996	182	13.49074
8	2.82843	33	5.74456	58	7.61577	83	9.11043	108	10.39230	133	11.53256	158	12.56981	183	13.52775
9	3	34	5.83095	59	7.68115	84	9.16515	109	10.44031	134	11.57584	159	12.60952	184	13.56466
10	3.16228	35	5.91608	60	7.74597	85	9.21954	110	10.48809	135	11.61895	160	12.64911	185	13.60147
11	3.31662	36	6	61	7.81025	86	9.27362	111	10.53565	136	11.66190	161	12.68858	186	13.63818
12	3.4641	37	6.08276	62	7.87401	87	9.32738	112	10.58301	137	11.70470	162	12.72792	187	13.67479
13	3.60555	38	6.16441	63	7.93725	88	9.38083	113	10.63015	138	11.74734	163	12.76715	188	13.71131
14	3.74166	39	6.24500	64	8	89	9.43398	114	10.67708	139	11.78983	164	12.80625	189	13.74773
15	3.87298	40	6.32456	65	8.06226	90	9.48683	115	10.72381	140	11.83216	165	12.84523	190	13.78405
16	4	41	6.40312	66	8.12404	91	9.53939	116	10.77033	141	11.87434	166	12.88410	191	13.82027
17	4.12311	42	6.48074	67	8.18535	92	9.59166	117	10.81665	142	11.91638	167	12.92285	192	13.85641
18	4.24264	43	6.55744	68	8.24621	93	9.64365	118	10.86278	143	11.95826	168	12.96148	193	13.89244
19	4.35890	44	6.63325	69	8.30662	94	9.69536	119	10.90871	144	12	169	13	194	13.92839
20	4.47214	45	6.70820	70	8.36660	95	9.74679	120	10.95445	145	12.04159	170	13.03840	195	13.96424
21	4.58258	46	6.78233	71	8.42615	96	9.79796	121	11	146	12.08305	171	13.07670	196	14
22	4.69042	47	6.85565	72	8.48528	97	9.84886	122	11.04536	147	12.12436	172	13.11488	197	14.03567
23	4.79583	48	6.92820	73	8.54400	98	9.89949	123	11.09054	148	12.16553	173	13.15295	198	14.07125
24	4.89898	49	7	74	8.60233	99	9.94987	124	11.13553	149	12.20656	174	13.19091	199	14.10674
25	5	50	7.07107	75	8.66025	100	10	125	11.18034	150	12.24745	175	13.22876	200	14.14214

Appendix III. List of Prime Numbers from 2 to 9733

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151
1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	1231	1237	1249	1259	1277	1279	1283	1289	1291
1297	1301	1303	1307	1319	1321	1327	1361	1367	1373
1381	1399	1409	1423	1427	1429	1433	1439	1447	1451
1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583
1597	1601	1607	1609	1613	1619	1621	1627	1637	1657
1663	1667	1669	1693	1697	1699	1709	1721	1723	1733
1741	1747	1753	1759	1777	1783	1787	1789	1801	1811
1823	1831	1847	1861	1867	1871	1873	1877	1879	1889
1901	1907	1913	1931	1933	1949	1951	1973	1979	1987
1993	1997	1999	2003	2011	2017	2027	2029	2039	2053
2063	2069	2081	2083	2087	2089	2099	2111	2113	2129
2131	2137	2141	2143	2153	2161	2179	2203	2207	2213
2221	2237	2239	2243	2251	2267	2269	2273	2281	2287
2293	2297	2309	2311	2333	2339	2341	2347	2351	2357
2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531
2539	2543	2549	2551	2557	2579	2591	2593	2609	2617
2621	2633	2647	2657	2659	2663	2671	2677	2683	2687
2689	2693	2699	2707	2711	2713	2719	2729	2731	2741
2749	2753	2767	2777	2789	2791	2797	2801	2803	2819
2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999
3001	3011	3019	3023	3037	3041	3049	3061	3067	3079
3083	3089	3109	3119	3121	3137	3163	3167	3169	3181
3187	3191	3203	3209	3217	3221	3229	3251	3253	3257
3259	3271	3299	3301	3307	3313	3319	3323	3329	3331
3343	3347	3359	3361	3371	3373	3389	3391	3407	3413
3433	3449	3457	3461	3463	3467	3469	3491	3499	3511
3517	3527	3529	3533	3539	3541	3547	3557	3559	3571
3581	3583	3593	3607	3613	3617	3623	3631	3637	3643
3659	3671	3673	3677	3691	3697	3701	3709	3719	3727
3733	3739	3761	3767	3769	3779	3793	3797	3803	3821
3823	3833	3847	3851	3853	3863	3877	3881	3889	3907
3911	3917	3919	3923	3929	3931	3943	3947	3967	3989
4001	4003	4007	4013	4019	4021	4027	4049	4051	4057
4073	4079	4091	4093	4099	4111	4127	4129	4133	4139
4153	4157	4159	4177	4201	4211	4217	4219	4229	4231
4241	4243	4253	4259	4261	4271	4273	4283	4289	4297
4327	4337	4339	4349	4357	4363	4373	4391	4397	4409

4421	4423	4441	4447	4451	4457	4463	4481	4483	4493
4507	4513	4517	4519	4523	4547	4549	4561	4567	4583
4591	4597	4603	4621	4637	4639	4643	4649	4651	4657
4663	4673	4679	4691	4703	4721	4723	4729	4733	4751
4759	4783	4787	4789	4793	4799	4801	4813	4817	4831
4861	4871	4877	4889	4903	4909	4919	4931	4933	4937
4943	4951	4957	4967	4969	4973	4987	4993	4999	5003
5009	5011	5021	5023	5039	5051	5059	5077	5081	5087
5099	5101	5107	5113	5119	5147	5153	5167	5171	5179
5189	5197	5209	5227	5231	5233	5237	5261	5273	5279
5281	5297	5303	5309	5323	5333	5347	5351	5381	5387
5393	5399	5407	5413	5417	5419	5431	5437	5441	5443
5449	5471	5477	5479	5483	5501	5503	5507	5519	5521
5527	5531	5557	5563	5569	5573	5581	5591	5623	5639
5641	5647	5651	5653	5657	5659	5669	5683	5689	5693
5701	5711	5717	5737	5741	5743	5749	5779	5783	5791
5801	5807	5813	5821	5827	5839	5843	5849	5851	5857
5861	5867	5869	5879	5881	5897	5903	5923	5927	5939
5953	5981	5987	6007	6011	6029	6037	6043	6047	6053
6067	6073	6079	6089	6091	6101	6113	6121	6131	6133
6143	6151	6163	6173	6197	6199	6203	6211	6217	6221
6229	6247	6257	6263	6269	6271	6277	6287	6299	6301
6311	6317	6323	6329	6337	6343	6353	6359	6361	6367
6373	6379	6389	6397	6421	6427	6449	6451	6469	6473
6481	6491	6521	6529	6547	6551	6553	6563	6569	6571
6577	6581	6599	6607	6619	6637	6653	6659	6661	6673
6679	6689	6691	6701	6703	6709	6719	6733	6737	6761
6763	6779	6781	6791	6793	6803	6823	6827	6829	6833
6841	6857	6863	6869	6871	6883	6899	6907	6911	6917
6947	6949	6959	6961	6967	6971	6977	6983	6991	6997
7001	7013	7019	7027	7039	7043	7057	7069	7079	7103
7109	7121	7127	7129	7151	7159	7177	7187	7193	7207
7211	7213	7219	7229	7237	7243	7247	7253	7283	7297
7307	7309	7321	7331	7333	7349	7351	7369	7393	7411
7417	7433	7451	7457	7459	7477	7481	7487	7489	7499
7507	7517	7523	7529	7537	7541	7547	7549	7559	7561
7573	7577	7583	7589	7591	7603	7607	7621	7639	7643
7649	7669	7673	7681	7687	7691	7699	7703	7717	7723
7727	7741	7753	7757	7759	7789	7793	7817	7823	7829
7841	7853	7867	7873	7877	7879	7883	7901	7907	7919
7927	7933	7937	7949	7951	7963	7993	8009	8011	8017
8039	8053	8059	8069	8081	8087	8089	8093	8101	8111
8117	8123	8147	8161	8167	8171	8179	8191	8209	8219
8221	8231	8233	8237	8243	8263	8269	8273	8287	8291
8293	8297	8311	8317	8329	8353	8363	8369	8377	8387
8389	8419	8423	8429	8431	8443	8447	8461	8467	8501
8513	8521	8527	8537	8539	8543	8563	8573	8581	8597
8599	8609	8623	8627	8629	8641	8647	8663	8669	8677
8681	8689	8693	8699	8707	8713	8719	8731	8737	8741
8747	8753	8761	8779	8783	8803	8807	8819	8821	8831
8837	8839	8849	8861	8863	8867	8887	8893	8923	8929
8933	8941	8951	8963	8969	8971	8999	9001	9007	9011
9013	9029	9041	9043	9049	9059	9067	9091	9103	9109
9127	9133	9137	9151	9157	9161	9173	9181	9187	9199
9203	9209	9221	9227	9239	9241	9257	9277	9281	9283
9293	9311	9319	9323	9337	9341	9343	9349	9371	9377
9391	9397	9403	9413	9419	9421	9431	9433	9437	9439
9461	9463	9467	9473	9479	9491	9497	9511	9521	9533
9539	9547	9551	9587	9601	9613	9619	9623	9629	9631
9643	9649	9661	9677	9679	9689	9697	9719	9721	9733

Appendix IV. Trigonometric Values for Some Common Angles

A	sin A	cos A	tan A	cot A	sec A	cosec A
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
90°	1	0	∞	0	∞	1

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More About the Author

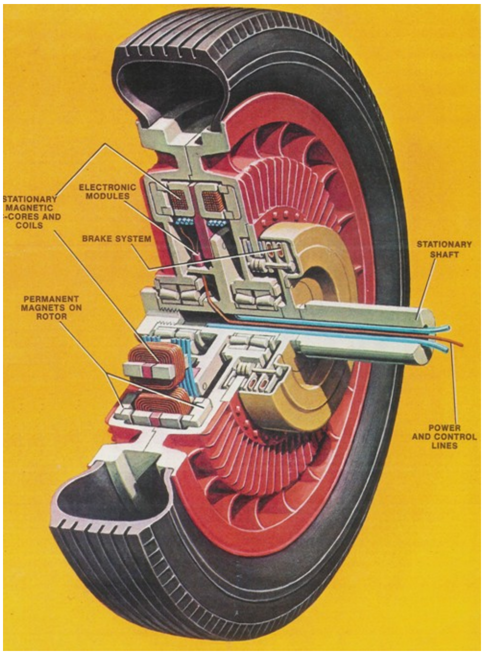
Gary S. Goldman holds a Ph.D. in Computer Science from Pacific Western University in Los Angeles and graduated with honors in 1977 from California State University, Fullerton (CSUF) with a double major: B.S. Engineering (Electronic emphasis) and B.S. Computer Science. He was elected member of the Phi-Kappa-Phi honor society and in 1976 received the Outstanding University Engineering Student Award, presented by the Orange County Engineering Education Council (OCEC). At graduation he received a special Merit Award in recognition of scholarly commitment and outstanding Academic Achievement in Computer Science, presented by CSUF.

At CSUF, Goldman was employed as a computer consultant assisting faculty and staff. He later served as a part-time assistant professor for the Engineering and Quantitative Methods departments instructing both graduate and undergraduate courses in statistics, programming, digital simulation, and digital logic design and switching theory.

Dr. Goldman served for eight years (from January, 1995 until his resignation in October of 2002) as Research Analyst for the Varicella Active Surveillance Project (VASP) in Antelope Valley, in a cooperative project with the Centers for Disease Control and Prevention (CDC, Atlanta, GA). He developed a model that quantified the seasonal variation in chickenpox based on school enrollment (clustering) and high ambient air temperature. Additionally, he created a database and data entry programs for several hundred demographic and clinical variables pertaining to chickenpox and shingles. This included logic to detect case duplicates (important for application of capture-recapture methods) and link cases originating from the same household. He also supplied the initiative and background material for the proposal to add shingles to the active surveillance program. Additionally, he wrote statistical analyses to (a) investigate 2nd varicella infections (later published in peer-reviewed medical journal), (b) study varicella susceptibility (presented at a symposium and published later outside VASP), (c) quantify transmission of varicella in households, (d) determine chickenpox vaccine efficacy by year (published later outside VASP), (e) track outbreaks of chickenpox in schools, (f) perform a cost-benefit analysis of universal varicella vaccination taking into account the closely related herpes zoster epidemiology (published later outside VASP), and (g) perform capture-recapture to measure reporting completeness of chickenpox cases to the surveillance project (published in *The Journal of the American Medical Association--JAMA*). Finally, Goldman computed both (1) true shingles incidence rate among children with previous histories of chickenpox and (2) true shingles incidence rate among vaccinated children so as to investigate trends in shingles incidence in a community under moderate to widespread varicella vaccination (published later outside VASP).

Presently, Dr. Goldman serves as a consulting computer scientist and is on the board of directors of Pearblossom Private School, Inc. which provides distance education to over 1,500 independent study students each year in grades K through 12 throughout the United States (see www.PearblossomSchool.com). The Western Association of Schools and Colleges (WASC) has granted the school the longest six-year term of accreditation that expires on June 30, 2017. Dale J. Mitchell, the Commission Chair, writes, “This action was taken after a careful study of the Visiting Committee Report [prepared by David E. Brown, PhD, Executive Director of WASC and Visiting Committee Chairman] which noted many laudable aspects of the school.... Please accept our congratulations on the quality of instruction being offered in your school.”

From 2004 through 2010 he served as Editor-in-Chief of the peer-reviewed journal, *Medical Veritas*: The Journal of Medical Truth (see www.MedicalVeritas.com). Through this endeavor Dr. Goldman has had the pleasure of interfacing with world-renowned scientists, researchers, and physicians. Dr. Goldman has served on the Editorial Board and as a reviewer for the *Journal of the American Medical Association* (*JAMA*), *Human and Experimental Toxicology* (*HET*), Vaccine*, Research and Reviews in BioSciences* The American Journal of Managed Care (AJMC), *Expert Review of Vaccines* (*ERV*), *Expert Review of Dermatology* (*ERD*), *Journal of the European Academy of Dermatology and Venereology* (*JEADV*), *Epidemiology and Infection*, *The Open Allergy Journal*, *British Medical Journal* (*BMJ*), *Annals of Virology and Research, and BioMed Research International (Biomed Res Int)*.



1980 U.S. Patent for
Power Wheel

Vice-president, Systems
Development, Cascade Graphics

Gary S. Goldman with first microcomputer-based
(Apple II w/68000 mother board) graphics system in 1980

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