## Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- Eliminating square roots.
- Integrals of tangents and secants.
- Products of sines and cosines.

## Product of sines and cosines

Remark: There is a procedure to compute integrals of the form

$$I = \int \sin^m(x) \cos^n(x)\, dx.$$

(a) If $m = 2k + 1$, (odd), then $\sin^{(2k+1)}(x) = \big(\sin^2(x)\big)^k \sin(x)$;

$$I = \int \big(1 - \cos^2(x)\big)^k \cos^n(x) \sin(x)\, dx.$$

Substitute $u = \cos(x)$, so $du = -\sin(x)\, dx$, hence

$$I = -\int (1 - u^2)^k\, u^n\, du.$$

We now need to integrate a polynomial.

## Product of sines and cosines

Remark: There is a procedure to compute integrals of the form

$$I = \int \sin^m(x)\, \cos^n(x)\, dx.$$

(b) If $n = 2k + 1$, (odd), then $\cos^{(2k+1)}(x) = \big(\cos^2(x)\big)^k \cos(x)$;

$$I = \int \sin^m(x)\, \big(1 - \sin^2(x)\big)^k \cos(x)\, dx.$$

Substitute $u = \sin(x)$, so $du = \cos(x)\, dx$, hence

$$I = \int u^m\, (1 - u^2)^k\, du.$$

Again, we now need to integrate a polynomial.

## Product of sines and cosines

Remark: There is a procedure to compute integrals of the form

$$I = \int \sin^m(x)\, \cos^n(x)\, dx.$$

(c) If both $m$ and $n$ are even, say $m = 2k$ and $n = 2\ell$, then

$$I = \int \sin^{2k}(x)\, \cos^{2\ell}(x)\, dx = \int \big(\sin^2(x)\big)^k \big(\cos^2(x)\big)^\ell\, dx.$$

Now use the identities

$$\sin^2(x) = \frac{1}{2}\big(1 - \cos(2x)\big), \qquad \cos^2(x) = \frac{1}{2}\big(1 + \cos(2x)\big).$$

Depending whether $k$ or $\ell$ are odd, repeat (a), (b) or (c).

## Product of sines and cosines

Example

Evaluate $I = \int \sin^5(x)\,dx$.

Solution: Since $m = 5$ is odd, we write it as $m = 4 + 1$,

$$I = \int \sin^{4+1}(x)\,dx = \int \big(\sin^2(x)\big)^2 \sin(x)\,dx$$

$$I = \int \big(1 - \cos^2(x)\big)^2 \sin(x)\,dx.$$

Introduce the substitution $u = \cos(x)$, then $du = -\sin(x)\,dx$,

$$I = -\int (1-u^2)^2\,du = -\int (1 - 2u^2 + u^4)\,du.$$

$$I = -u + 2\,\frac{u^3}{3} - \frac{u^5}{5} + c.$$

We conclude $I = -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + c$. $\triangleleft$

## Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\,dx$.

Solution: Since $m = 6$ is even, we write it as $m = 2(3)$,

$$I = \int \big(\sin^2(x)\big)^3\,dx = \int \Big(\frac{1}{2}[1 - \cos(2x)]\Big)^3\,dx$$

$$I = \frac{1}{8}\int \big(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\big)\,dx.$$

The first two terms are: $\int (1 - 3\cos(2x))\,dx = x - \frac{3}{2}\sin(2x)$.

The third term can be integrated as follows,

$$\int 3\cos^2(2x)\,dx = 3\int \frac{1}{2}(1 + \cos(4x))\,dx = \frac{3}{2}\Big(x + \frac{1}{4}\sin(4x)\Big).$$

## Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\,dx$.

Solution: So far we have found that

$$I = \frac{1}{8}\Big[x - \frac{3}{2}\sin(2x) + \frac{3}{2}\Big(x + \frac{1}{4}\sin(4x)\Big)\Big] - \frac{1}{8}\int \cos^3(2x)\,dx.$$

The last term $J = \int \cos^3(2x)\,dx$ can we computed as follows,

$$J = \int \cos^2(2x)\,\cos(2x)\,dx = \int \big(1 - \sin^2(2x)\big)\,\cos(2x)\,dx.$$

Introduce the substitution $u = \sin(2x)$, then $du = 2\cos(2x)\,dx$.

$$J = \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\Big(u - \frac{u^3}{3}\Big) = \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x).$$

$$I = \frac{1}{8}\Big[x - \frac{3}{2}\sin(2x) + \frac{3}{2}x + \frac{3}{8}\sin(4x) - \frac{1}{2}\sin(2x) + \frac{1}{6}\sin^3(2x)\Big] + c.$$

## Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- **Eliminating square roots.**
- Integrals of tangents and secants.
- Products of sines and cosines.

## Eliminating square roots

Remarks:

- Recall the double angle identities:

$$\sin^2(\theta) = \frac{1}{2}\big[1 - \cos(2\theta)\big], \qquad \cos^2(\theta) = \frac{1}{2}\big[1 + \cos(2\theta)\big].$$

  These identities can be used to simplify certain square roots.

- The same holds for Pythagoras Theorem,

$$\sin^2(\theta) = 1 - \cos^2(\theta), \qquad \cos^2(\theta) = 1 - \sin^2(\theta).$$

Example

Evaluate $I = \int_0^{\pi/8} \sqrt{1 + \cos(8x)}\, dx$.

Solution: Use that : $1 + \cos(8x) = 2\cos^2(4x)$. Hence,

$$I = \sqrt{2} \int_0^{\pi/8} \cos(4x)\, dx = \frac{\sqrt{2}}{4} \sin(4x)\Big|_0^{\pi/8} \;\Rightarrow\; I = \frac{\sqrt{2}}{4}.$$

## Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- Eliminating square roots.
- **Integrals of tangents and secants.**
- Products of sines and cosines.

## Integrals of tangents and secants

Remark: Recall the identities:

$$\tan'(x) = \sec^2(x) = \tan^2(x) + 1.$$

First equation comes from quotient rule, the second from Pythagoras Theorem. These identities can be used to compute

$$I = \int \tan^{2k}(x)\,dx, \qquad k \in \mathbb{N}.$$

Example

Evaluate $I = \int \tan^2(x)\,dx$.

Solution: The identity on the far left above implies

$$I = \int \big(\tan'(x) - 1\big)\,dx \quad \Rightarrow \quad I = \tan(x) - x + c. \qquad \triangleleft$$

## Integrals of tangents and secants

Example

Find a recurrence formula to compute $I = \int \tan^{2k}(x)\,dx$, $k \in \mathbb{N}$.

Solution: Recall: $\tan'(x) = \sec^2(x) = \tan^2(x) + 1$.

$$I = \int \tan^{(2k-2)}(x)\,\tan^2(x)\,dx = \int \tan^{(2k-2)}(x)\big(\tan'(x) - 1\big)\,dx$$

$$I = \int \tan^{(2k-2)}(x)\,\tan'(x)\,dx - \int \tan^{(2k-2)}(x)\,dx.$$

In the first term on the right, $u = \tan(x)$, then $du = \tan'(x)\,dx$,

$$\int \tan^{(2k-2)}(x)\,\tan'(x)\,dx = \int u^{(2k-2)}\,du = \frac{u^{(2k-1)}}{(2k-1)}.$$

$$I = \frac{1}{(2k-1)}\,\tan^{(2k-1)}(x) - \int \tan^{2(k-1)}(x)\,dx. \qquad \triangleleft$$

## Integrals of tangents and secants

Example

Evaluate $I = \int \sec^3(x)\,dx$.

Solution: Recall: $\tan'(x) = \sec^2(x) = \tan^2(x) + 1$.

Rewrite the integral as follows,

$$I = \int \sec(x)\,\sec^2(x)\,dx = \int \sec(x)\,\tan'(x)\,dx.$$

Where we used that $\sec^2(x) = \tan'(x)$. Integrate by parts,

$$u = \sec(x), \quad dv = \tan'(x)\,dx \;\Rightarrow\; du = \sec'(x)\,dx, \quad v = \tan(x).$$

$$I = \sec(x)\tan(x) - \int \tan(x)\,\sec'(x)\,dx.$$

Recall: $\sec'(x) = \dfrac{\sin(x)}{\cos^2(x)} = \sec(x)\,\tan(x)$.



Example

Evaluate $I = \int \sec^3(x)\,dx$.

Solution: $I = \sec(x)\tan(x) - \int \tan(x)\,\sec'(x)\,dx$, and we also know $\sec'(x) = \sec(x)\tan(x)$.

$$I = \sec(x)\tan(x) - \int \sec(x)\,\tan^2(x)\,dx$$

$$I = \sec(x)\tan(x) - \int \sec(x)\,(\sec^2(x) - 1)\,dx$$

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx - \int \sec^3(x)\,dx.$$

$$\int \sec^3(x)\,dx = \frac{1}{2}\,\sec(x)\tan(x) + \frac{1}{2}\,\ln\bigl(\sec(x) + \tan(x)\bigr) + c.$$

## Integrals of tangents and secants

Recall: $\int \sec(x)\,dx = \ln\big(\sec(x) + \tan(x)\big) + c.$

Proof:

$$I = \int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx$$

$$I = \int \frac{1}{\cos(x)}\,\frac{[1+\sin(x)]}{\cos(x)}\,\frac{\cos(x)}{[1+\sin(x)]}\,dx$$

$$I = \int \frac{[1+\sin(x)]}{\cos^2(x)}\,\frac{1}{\left(\frac{[1+\sin(x)]}{\cos(x)}\right)}\,dx$$

$$I = \int \left(\frac{[1+\sin(x)]}{\cos(x)}\right)'\,\frac{1}{\left(\frac{[1+\sin(x)]}{\cos(x)}\right)}\,dx$$

## Integrals of tangents and secants

Recall: $\int \sec(x)\,dx = \ln\big(\sec(x) + \tan(x)\big) + c.$

$$I = \int \left(\frac{[1+\sin(x)]}{\cos(x)}\right)'\,\frac{1}{\left(\frac{[1+\sin(x)]}{\cos(x)}\right)}\,dx$$

$$I = \int \big(\sec(x) + \tan(x)\big)'\,\frac{1}{\big(\sec(x) + \tan(x)\big)}\,dx$$

Substitute $u = \sec(x) + \tan(x)$, then

$$I = \int \frac{du}{u} = \ln(u) + c.$$

So we obtain the formula,

$$\int \sec(x)\,dx = \ln\big(\sec(x) + \tan(x)\big) + c.$$

## Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- Eliminating square roots.
- Integrals of tangents and secants.
- **Products of sines and cosines.**

## Products of sines and cosines

Remark: The identities

$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big]$$

$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta-\phi) + \sin(\theta+\phi)\big]$$

$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) + \cos(\theta+\phi)\big].$$

can be used to compute integrals of the form

$$\int \sin(mx)\,\sin(nx)\,dx, \quad \int \sin(mx)\,\cos(nx)\,dx,$$

$$\int \cos(mx)\,\cos(nx)\,dx.$$

## Products of sines and cosines

Example

Evaluate: $I = \int \sin(3x)\,\cos(4x)\,dx.$

Solution: Recall: $\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta-\phi)+\sin(\theta+\phi)\big].$

The formula above implies,

$$I = \frac{1}{2}\int \big[\sin((3-4)x)+\sin((3+4)x)\big]\,dx,$$

that is,

$$I = \frac{1}{2}\int \big[-\sin(x)+\sin(7x)\big]\,dx.$$

This integral is simple to do,

$$I = \frac{1}{2}\Big[\cos(x) - \frac{1}{7}\,\cos(7x)\Big] + c. \qquad \triangleleft$$