DIFFERENTIAL EQUATIONS PRACTICE PROBLEMS: ANSWERS

1. Find the solution of $y' + 2xy = x$, with $y(0) = -2$.

   This is a linear equation. The integrating factor is $e^{\int 2x\ dx} = e^{x^2}$. Multiplying through by this, we get

$$\begin{aligned} y'e^{x^2} + 2xe^{x^2}y &= xe^{x^2} \\ (e^{x^2}y)' &= xe^{x^2} \\ e^{x^2}y &= \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + C \\ y &= \frac{1}{2} + Ce^{-x^2}. \end{aligned}$$

   Putting in the initial condition gives $C = -5/2$, so $y = \dfrac{1}{2} - \dfrac{5}{2}e^{=x^2}$.

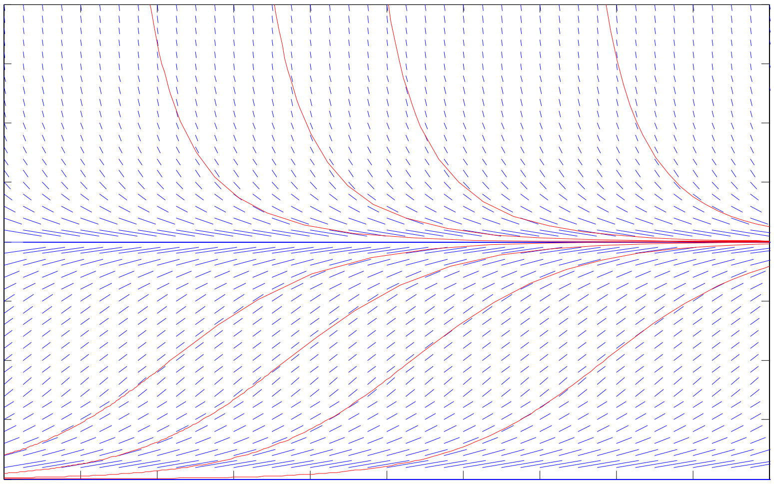
2. Find the general solution of $xy' = y - (y^2/x)$.

   A number of substitutions will work here. The simplest is $y = ux$, so $y' = u + u'x$. Rewriting the equation with u and x eventually gives a separable equation:

$$\begin{aligned} x(u + u'x) &= ux - \frac{u^2x^2}{x} = ux - u^2x \\ \frac{du}{dx}x^2 &= -u^2x \\ \int -u^{-2}du &= \int \frac{1}{x}\ dx \\ \frac{1}{u} &= \ln x + C \\ u &= \frac{1}{\ln x + C} \\ y &= \frac{x}{\ln x + C}. \end{aligned}$$

3. Suppose that the frog population $P(t)$ of a small lake satisfies the differential equation $\frac{dP}{dt} = kP(200 - P)$.

   (a) Find the equilibrium solutions. Sketch them and using the equation, sketch several solution curves, choosing some with initial points above and between the equilibrium solutions.



   The equilibrium solutions are $P = 0$ (unstable) and $P = 200$ (stable).

(b) In the year 2000, its population was 100 and growing at the rate of 5 per year. Predict the lake's frog population in 2008. $\left(\text{Note : } \frac{1}{P(200-P)} = \frac{1/200}{P} + \frac{1/200}{(200-P)}.\right)$

This is a separable equation:

$$\begin{aligned}
\int \frac{1}{P(200-P)} dP &= \int k\ dt \\
\int \frac{1}{200}\left(\frac{1}{P} + \frac{1}{200-P}\right) &= \int k\ dt \\
\frac{1}{200}\left(\ln(P) - \ln(200-P)\right) &= \int k\ dt \\
\ln(200-P) - \ln(P) &= \ln\left(\frac{200-P}{P}\right) = -200kt + C \\
\frac{200}{P} - 1 &= Ke^{-200kt}.
\end{aligned}$$

Taking 2000 as the base year, the initial condition $P(0) = 100$ gives $K = 1$. Putting $P = 100$ and $\left.\frac{dP}{dt}\right|_{t=0} = 5$ gives $k = \frac{5}{10,000}$. Thus:

$$\begin{aligned}
\frac{200}{P} - 1 &= e^{-200\frac{5}{10,000}(8)} = e^{-4/5} \\
P &= \frac{200}{1+e^{-4/5}} \approx 138.
\end{aligned}$$

4. Find the general solution of the differential equation $y'' - y' = e^x - 9x^2$.

   Using the differential operator D, the homogeneous equation $y'' - y' = 0$ becomes $D^2 - D = 0$ which has solutions $D = 1$ and $D = 0$, corresponding to $Dy = y$ ($y = e^x$) and $Dy = 0$ ($y =$ constant). Thus, the general solution to the homogeneous equation is $y_h = c_1 + c_2e^x$. We now find a particular solution to the original equation using undetermined coefficients. Our guess might be $y_p = Ae^x + Bx^2 + Cx + D$, But e^x duplicates part of the homogeneous solution as does the derivative of Cx (the constant c_1). So we multiply by a high enough power of x to avoid this; x will do:

$$\begin{aligned}
y_p &= Axe^x + Bx^3 + Cx^2 + Dx \\
y_p' &= Axe^x + Ae^x + 3Bx^2 + 2Cx + D \\
y_p'' &= Axe^x + 2Ae^x + 6Bx + 2C \\
y_p'' - y_p' &= Ae^x - 3Bx^2 + (6B - 2C)x + (2C - D).
\end{aligned}$$

   We set this equal to $e^x - 9x^2$, which gives: $A = 1$, $B = 3$, $C = 9$ and $D = 18$. Since the general solution to a linear DE is the general solution to the associated homogeneous equation + a particular solution to the original, the general solution is $y = c_1 + c_2e^x + xe^x + 3x^3 + 9x^2 + 18x$.

5. A mass of 2 kg is attached to a spring with constant $k = 8$ Newtons/meter.

   (a) Find the natural frequency of this system.

   The system equation (no driving force) is $2x'' + 8x = 0$ or $x'' + 4x = 0$. This gives $D^2 + 4 = 0$ so $D = \pm 2i$. Thus, the solution is $x(t) = c_1 \cos 2t + c_2 \sin 2t$, and the frequency is $\omega_0 = 2$ (radians per second or $1/\pi$ hertz).

   (b) If the motion is also subject to a damping force with $c = 4$ Newtons/(meter/sec), and the mass is initially pulled 1 meter beyond its equilibrium point and released (without initial velocity), find the motion, $x(t)$. (You may leave your answer in any form.)

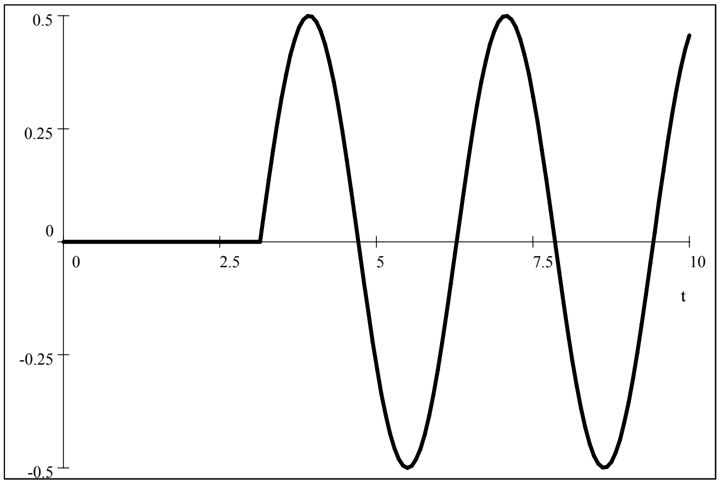
We could use Laplace methods here, but we'll use the D operator again. The equation $2x''+4x'+8x = 0$, which becomes $D^2+2D+4=0$, having roots $D = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$. This gives:

$$\begin{aligned} x(t) &= e^{-t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right)\right) \\ x'(t) &= -e^{-t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right)\right) + e^{-t}\left(-c_1\sqrt{3}\sin\left(\sqrt{3}t\right) + c_2\sqrt{3}\cos\left(\sqrt{3}t\right)\right). \end{aligned}$$

The initial conditions $x(0)=1$, $x'(0)=0$ now give $c_1 = 1$, $c_2 = 1/\sqrt{3}$, so $x(t) = e^{-t}\left(\cos\left(\sqrt{3}t\right) + (1/\sqrt{3})\sin\left(\sqrt{3}t\right)\right)$

6. Find *and sketch* the solution to the initial value problem $y''+4y = \delta(t-\pi)$, $y(0)=y'(0)=0$.

   Taking the Laplace transform gives $s^2Y+4Y = e^{-\pi s}$, so $Y(s) = e^{-\pi s}\left(\frac{1}{s^2+4}\right)$. Now $\frac{1}{s^2+4} = \frac{1}{2}\left(\frac{2}{s^2+4}\right)$ untransforms into $\frac{1}{2}\sin 2t$, so $Y(s)$ untransforms into: $y(t) = u(t-\pi)\frac{1}{2}\sin 2t$.



7. Find the inverse Laplace transform of $F(s) = \frac{s^2+4}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$, so $s^2+4 = A\left(s^2+1\right) + (Bs+C)\,s$. Letting $s=0$ gives $A=4$. Letting $s=1$ and $s=-1$ gives the equations $B+C=-3$ and $B-C=-3$, so $B=-3$ and $C=0$. Thus,

$$\begin{aligned} F(s) &= \frac{4}{s} - \frac{3s}{s^2+1} \\ f(t) &= 4 - 3\cos t. \end{aligned}$$

8. Let $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}$

   (a) Find A^{-1}, the inverse of A.

   We put the identity next to A and row reduce the augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 4 & 3 \\ 1 & 5 & 5 & 1 & 5 & 5 \\ 2 & 5 & 1 & 2 & 5 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \overbrace{\begin{array}{ccc} -20 & 11 & 5 \\ 9 & -5 & -2 \\ -5 & 3 & 1 \end{array}}^{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

(b) Use your answer above to solve $A\vec{x} = \vec{b}$ where $\vec{b} = (1, 0, -1)$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20 & 11 & 5 \\ 9 & -5 & -2 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -25 \\ 11 \\ -6 \end{bmatrix}.$$

9. The matrix

$$B = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

has eigenvalues $\lambda = 4, -2$. Find a basis of eigenvectors.

We row reduce $B - \lambda I$ for $\lambda = 4$ and $\lambda = -2$:

$$\begin{aligned} B - 4I &= \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} z = s \\ y = z = s \\ x = z = s \end{matrix} \\ B + 2I &= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} z = s \\ y = t \\ x = -s - t \end{matrix} \end{aligned}$$

For $\lambda = 4$, letting $s = 1$ gives the eigenvector $(1, 1, 1)$ (as a column); for $\lambda = -2$, letting $s = 0$, $t = -1$ gives $(1, -1, 0)$ and letting $s = -1, t = 0$ gives $(1, 0, -1)$. Eigenvectors for distinct eigenvalues are always independent, and the two vectors for the eigenvalue $\lambda = -2$ are clearly independent (neither is a multiple of the other). Thus, these three vectors are a basis for the eigenspace of B; this eigenspace is all of $\mathbb{R}^3$.

10. The reduced row echelon form for the matrix A below has been computed by Matlab:

$$A = \begin{pmatrix} 2 & -4 & -1 & 2 \\ -3 & 6 & 1 & -5 \\ 5 & -10 & -4 & -1 \end{pmatrix} \qquad \text{rref}(A) = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Use this to find all solutions of

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 2 \\ -3x_1 + 6x_2 + x_3 &= -5 \\ 5x_1 - 10x_2 - 4x_3 &= -1 \end{aligned}$$

and express your answer in vector form.

Thinking of the row-reduced matrix as an augmented matrix we see that there is no restriction on x_2, so let $x_2 = s$. The second row says $x_3 = 4$ and the first row says $x_1 - 2x_2 = 3$ or $x_1 = 3 + 2s$. In vector form we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}.$$

11. Let $\mathbf{v}_1 = (2, 1, 3)$, $\mathbf{v}_2 = (1, 5, 9)$, and $\mathbf{w} =(1, -1, -1)$. Is $\mathbf{w}$ in span$(\mathbf{v}_1, \mathbf{v}_2)$? Find a basis for span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$. What is the *dimension* of span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$?

We make these vectors into the column of a matrix $\mathbf{A}$. A linear dependence among the vectors is then a solution to the equation $\mathbf{AX} = \mathbf{0}$. So we row reduce to see if there is a non-trivial ($\mathbf{X} \neq \mathbf{0}$) one:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 5 & -1 \\ 3 & 9 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} z = s \\ y = (1/3)s \\ x = -(2/3)s \end{matrix}$$

Thus, $-(2/3)\mathbf{v}_1 + (1/3)\mathbf{v}_2 + \mathbf{w} = \mathbf{0}$ is a non-trivial dependency, allow us to solve for $\mathbf{w}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. So Span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$ = Span$(\mathbf{v}_1, \mathbf{v}_2)$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are clearly not multiples of one another, they are independent, hence form a basis for Span$(\mathbf{v}_1, \mathbf{v}_2)$. Thus, Span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$ has dimension 2.

12. Consider the following system of first-order differential equations:

$$\begin{aligned} x_1' &= 9x_1 + 5x_2 \qquad x_1(0) = 1 \\ x_2' &= -6x_1 - 2x_2 \qquad x_2(0) = 0 \end{aligned}$$

Use eigenvalues and eigenvectors to find the solution.
In matrix form these equations become

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \overbrace{\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

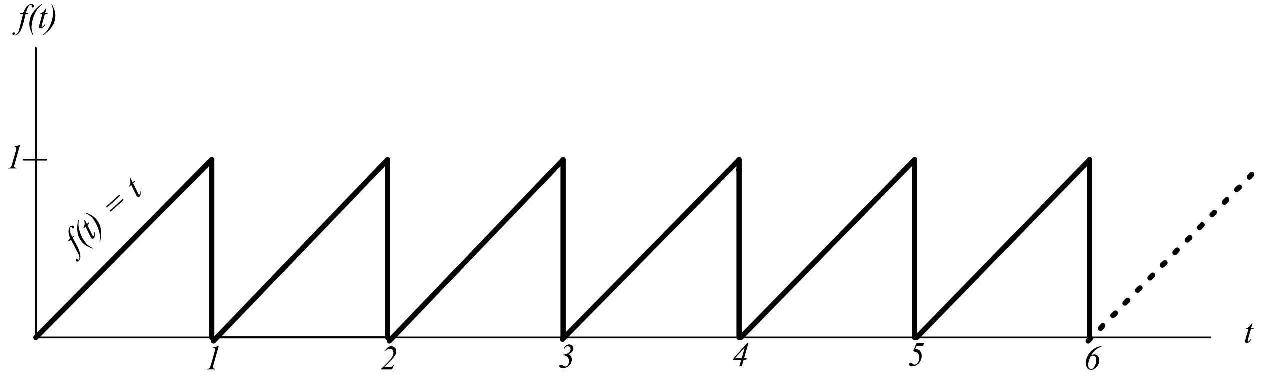
The characteristic polynomial for $\mathbf{A}$ is $\det(\mathbf{A} - \mathbf{xI}) = \begin{vmatrix} 9-x & 5 \\ -6 & -2-x \end{vmatrix} = x^2 - 7x + 12 = (x-3)(x-4)$, so the eigenvalues are $\lambda = 3, 4$.

$$\begin{aligned} \mathbf{A} - \mathbf{3I} &= \begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5/6 \\ 0 & 0 \end{bmatrix} \text{ which gives } \mathbf{v}_3 = \begin{bmatrix} 5 \\ -6 \end{bmatrix} \\ \mathbf{A} - \mathbf{4I} &= \begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ which gives } \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

So the general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 6 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$. Letting $t = 0$, the initial conditions give $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5c_1 \\ -6c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ -c_2 \end{bmatrix}$ or the equations $6c_1 + c_2 = 0$ and $5c_1 + c_2 = 1$, with solutions $c_1 = -1$ and $c_2 = 6$. This gives the solution

$$\begin{aligned} x_1 &= -5e^{3t} + 6e^{4t} \\ x_2 &= 6e^{3t} - 6e^{4t}. \end{aligned}$$

13. Here is a "sawtooth" function $f(t)$:



The first "tooth" is the function $f_1(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$.

(a) From the definition $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$, show that $F_1(s) = \dfrac{1-e^{-s}}{s^2} - \dfrac{e^{-s}}{s}$. (Use integration by parts; you only have to integrate from 0 to 1.)

$F_1(s) = \int_0^\infty f_1(t)e^{-st}\ dt = \int_0^1 te^{-st}\ dt$. We use the parts $u = t$, and $dv = e^{-st}\ dt$: $\int u\ dv = uv - \int v\ du = -\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st}\ dt = -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$. Thus

$$\begin{aligned}
\int\limits_0^1 te^{-st}\ dt &= \left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Bigg|_0^1 \\
&= \left(-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}\right) - \left(0 - \frac{1}{s^2}\right) \\
&= \frac{1}{s^2}\left(1 - e^{-s}\right) - \frac{1}{s}e^{-s}.
\end{aligned}$$

(b) For a periodic function f of period p, $F(s) = \frac{1}{1 - e^{-ps}}\int_0^p f(t)e^{-st}\ dt$. Use this and part (a) to show that, for the sawtooth:

$$F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})}.$$

This is simple algebra.