Lecture 6

Black-Scholes PDE

Lecture Notes
by Andrzej Palczewski

Pricing function

Let the dynamics of underlining S_t be given in the risk-neutral measure $\mathbb{Q}$ by

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

If the contingent claim X equals

$$X = h(S_T)$$

for some function h, then the price of X at time t is given by

$$V_t = V(S_t, t),$$

where $V(s,t)$ is given by the formula

$$V(s,t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big(h(S_T)|S_t = s\big).$$

Replication strategy

We call (ψ, ϕ) the replication strategy for contingent claim X, if the value process is given by

$$V_t = \psi_t B_t + \phi_t S_t,$$

where B_t is the bank account.

Goal

Find the function $V(s,t) : (0,\infty) \times [0,T] \rightarrow [0,\infty)$.

With the function $V(s,t)$ we can:

- compute the price of the contingent claim: at t it equals

$$V(S_t, t).$$

- find the replicating strategy

$$\phi_t = \frac{\partial V}{\partial s}(S_t, t),$$
$$\psi_t = e^{-rt}\big(V(S_t, t) - \phi_t S_t\big).$$

The Black-Scholes PDE

Theorem. If $X = h(S_T)$ then there exists a function $V : (0, \infty) \times [0, T] \to \mathbb{R}$ such that

$$V_t = V(S_t, t).$$

This function is a solution to the Black-Scholes partial differential equation

$$\frac{\partial V(s,t)}{\partial t} + rs\frac{\partial V(s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} - rV(s,t) = 0$$

with the terminal condition

$$V(s,T) = h(s)$$

for any $s > 0$ and $t \in [0, T]$.

Applications of B-S PDE

We can compute V analytically for:

- vanilla call and put options,
- binary call and put options,
- call and put options on the foreign exchange market.

We cannot compute analytically but can solve B-S PDE numerically for:

- options with non-standard payoffs,
- American call and put options,
- Asian options,
- some other complicate contingent claims.

Derivation of the Black-Scholes PDE

$$V_t = V(S_t, t).$$

Once we are at t, the value V_t is no longer random as it is $\mathcal{F}_t$ measurable.

Applying Itô's formula gives

$$dV_t = d\big(V(S_t,t)\big) = \left(\sigma S_t \frac{\partial V(S_t,t)}{\partial s}\right) d\tilde{W}_t + \left(r S_t \frac{\partial V(S_t,t)}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S_t,t)}{\partial s^2} + \frac{\partial V(S_t,t)}{\partial t}\right) dt.$$

The discounted value process $\hat{V}_t = e^{-rt} V_t$ is a **martingale** under the risk-neutral measure $\mathbb{Q}$.

Compute $d\hat{V}_t$

$$d\hat{V}_t = d\big(e^{-rt}V_t\big) = \left(e^{-rt}\sigma S_t \frac{\partial V(S_t,t)}{\partial s}\right) d\tilde{W}_t$$
$$+ e^{-rt}\left(rS_t \frac{\partial V(S_t,t)}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S_t,t)}{\partial s^2} + \frac{\partial V(S_t,t)}{\partial t} - rV(S_t,t)\right) dt.$$

Observation. $\hat{V}_t$ is a martingale with respect to $\mathbb{Q}$, so the term by dt must be zero.

Hence the function $V(s,t)$ satisfies

$$rs\frac{\partial V(s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + \frac{\partial V(s,t)}{\partial t} - rV(s,t) = 0,$$

which is the Black-Scholes PDE.

Boundary conditions

B-S PDE $+$ Conditions $=$ Unique solution

Boundary conditions are required to establish uniqueness of the solution to the Black-Scholes PDE. Their role is to impose some **economically justified constraints** on the solution of the PDE.

We have to be able to find conditions without knowing the formula for the function V.

- Terminal condition:

$$V(s, T) = h(s), \quad s > 0.$$

- Left-boundary: what happens to $V(s, t)$ when s approaches 0.
- Right-boundary: what happens to $V(s, t)$ when s approaches ∞.

Call option

For a **call option** with payoff $(S_T - K)^+$ good boundary conditions are:

$$V(s,t) \approx 0, \quad \text{for } s \textbf{ very small},$$
$$V(s,t) \approx s, \quad \text{for } s \textbf{ very large}.$$

$$\begin{cases} \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} = 0, \\ V(s,T) = (s-K)^+, \quad s > 0, \\ \lim_{s\to 0} V(s,t) = 0, \quad t \in [0,T], \\ \lim_{s\to\infty} \frac{V(s,t)}{s} = 1, \quad t \in [0,T]. \end{cases}$$

Put option

For a **put option** with payoff $(K - S_T)^+$ good boundary conditions are:

$$V(s,t) \approx K e^{-r(T-t)}, \quad \text{for } s \text{ very small},$$

$$V(s,t) \approx 0, \quad \text{for } s \text{ very large}.$$

$$\begin{cases} \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} = 0, \\ \\ V(s,T) = (K-s)^+, \quad s > 0, \\ \\ \lim_{s\to 0} V(s,t) = K e^{-r(T-t)}, \quad t \in [0,T], \\ \lim_{s\to\infty} V(s,t) = 0, \quad t \in [0,T]. \end{cases}$$

General form of the Black-Scholes PDE

$V(s,t)$ is a unique solution to the **Black-Scholes PDE**

$$\begin{cases} \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} = 0, \\ V(s,T) = h(s), \quad s > 0, \\ \lim_{s\to 0} V(s,t) = r_1(t), \quad t \in [0,T], \\ \lim_{s\to\infty} \frac{V(s,t)}{r_2(s,t)} = 1 \quad \text{or} \quad \lim_{s\to\infty} V(s,t) = r_2(t), \quad t \in [0,T], \end{cases}$$

where r_1 and r_2 are appropriately chosen to match h.

Change of variables

Consider the following **time dependent** change of variables:

$$x := \log s + (r - \frac{1}{2}\sigma^2)(T - t),$$

$$\tau := \frac{\sigma^2}{2}(T - t),$$

$$y(x, \tau) := e^{-r(T - \frac{2\tau}{\sigma^2})} V\left(e^{x - (\frac{2r}{\sigma^2} - 1)\tau},\ T - \frac{2\tau}{\sigma^2}\right).$$

The Black-Scholes PDE is transformed into the **heat equation**

$$\frac{\partial y}{\partial \tau}(x, \tau) - \frac{\partial^2 y}{\partial x^2}(x, \tau) = 0.$$

Properties

- time $t = 0$ corresponds to $\tau = \frac{\sigma^2}{2} T$,
- time $t = T$ corresponds to $\tau = 0$,
- terminal condition $V(s, T) = h(s)$ changes to the **initial condition** $y(x, 0) = e^{-rT} h(e^x)$,
- unlike s, the variable x is between $-\infty$ and ∞,
- new PDE is much **simpler** to solve numerically than the Black-Scholes PDE.

Transformation completed

The function y from the previous slide is a unique solution to the following PDE

$$
\begin{cases}
\dfrac{\partial y}{\partial \tau}(x,\tau) - \dfrac{\partial^2 y}{\partial x^2}(x,\tau) = 0, \\[2ex]
y(x,0) = e^{-rT} h(e^x), \quad x \in \mathbb{R}, \\[2ex]
\lim\limits_{x\to -\infty} y(x,\tau) = e^{-r(T-\frac{2\tau}{\sigma^2})} r_1\Big(T - \dfrac{2\tau}{\sigma^2}\Big), \quad \tau \in \Big[0, \dfrac{\sigma^2}{2}T\Big], \\[2ex]
\lim\limits_{x\to\infty} \dfrac{y(x,\tau)}{e^{-r(T-\frac{2\tau}{\sigma^2})} r_2\big(e^{x-(\frac{2r}{\sigma^2}-1)\tau}, T-\frac{2\tau}{\sigma^2}\big)} = 1, \\[2ex]
\qquad \text{or} \quad \lim\limits_{x\to\infty} y(x,\tau) = e^{-r(T-\frac{2\tau}{\sigma^2})} r_2\Big(T - \dfrac{2\tau}{\sigma^2}\Big), \quad \tau \in \Big[0, \dfrac{\sigma^2}{2}T\Big].
\end{cases}
$$

Dividend-paying assets

When assets pay dividend with a constant dividend yield d then $V(s,t)$ is a unique solution to the following **Black-Scholes PDE**

$$
\begin{cases}
\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + (r-d)s\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} = 0, \\
V(s,T) = h(s), \quad s > 0, \\
\lim_{s \to 0} V(s,t) = r_1(t), \quad t \in [0,T], \\
\lim_{s \to \infty} \frac{V(s,t)}{r_2(s,t)} = 1 \quad \text{or} \quad \lim_{s \to \infty} V(s,t) = r_2(t), \quad t \in [0,T],
\end{cases}
$$

where r_1 and r_2 are appropriately chosen to match h.

Change of variables

The **time dependent** change of variables for dividend-paying assets:

$$x := \log s + (r - d - \frac{1}{2}\sigma^2)(T - t),$$

$$\tau := \frac{\sigma^2}{2}(T - t),$$

$$y(x,\tau) := e^{-r(T - \frac{2\tau}{\sigma^2})} V\left(e^{x - (\frac{2(r-d)}{\sigma^2} - 1)\tau},\ T - \frac{2\tau}{\sigma^2}\right).$$

Then the Black-Scholes PDE is transformed into the **heat equation**

$$\frac{\partial y}{\partial \tau}(x,\tau) - \frac{\partial^2 y}{\partial x^2}(x,\tau) = 0.$$

Transformed equation

The function y from the previous slide is a unique solution to the following PDE

$$
\begin{cases}
\dfrac{\partial y}{\partial \tau}(x,\tau) - \dfrac{\partial^2 y}{\partial x^2}(x,\tau) = 0, \\[2ex]
y(x,0) = e^{-rT} h(e^x), \quad x \in \mathbb{R}, \\[2ex]
\lim\limits_{x \to -\infty} y(x,\tau) = e^{-r(T - \frac{2\tau}{\sigma^2})} r_1\Big(T - \dfrac{2\tau}{\sigma^2}\Big), \quad \tau \in \Big[0, \dfrac{\sigma^2}{2}T\Big], \\[2ex]
\lim\limits_{x \to \infty} \dfrac{y(x,\tau)}{e^{-r(T-\frac{2\tau}{\sigma^2})} r_2\big(e^{x - (\frac{2(r-d)}{\sigma^2} - 1)\tau}, T - \frac{2\tau}{\sigma^2}\big)} = 1, \\[2ex]
\qquad \text{or} \quad \lim\limits_{x \to \infty} y(x,\tau) = e^{-r(T-\frac{2\tau}{\sigma^2})} r_2\Big(T - \dfrac{2\tau}{\sigma^2}\Big), \quad \tau \in \Big[0, \dfrac{\sigma^2}{2}T\Big].
\end{cases}
$$

Another change of variables

For some problems the proposed changes of variables are not very convenient because straight lines $S = c$ became time dependent in the new variables. Then the following **time independent** change of variables can be useful:

$$x := \log s, \quad \tau := \frac{\sigma^2}{2}(T - t),$$

$$y(x,\tau) := \exp\left(\frac{1}{2}(q_d - 1)x + \left(\frac{1}{4}(q_d-1)^2 + q\right)\tau\right) V\left(e^x,\ T - \frac{2\tau}{\sigma^2}\right),$$

$$\text{where} \quad q := \frac{2r}{\sigma^2}, \quad q_d := \frac{2(r-d)}{\sigma^2}.$$

In this case the Black-Scholes PDE is transformed also into the heat equation

$$\frac{\partial y}{\partial \tau}(x,\tau) - \frac{\partial^2 y}{\partial x^2}(x,\tau) = 0.$$

Boundary conditions

For the change of variables from the previous slide the boundary conditions are as follows

$$y(x,0) = e^{\frac{1}{2}(q_d-1)x} h(e^x), \quad x \in \mathbb{R},$$

$$\lim_{x\to-\infty} y(x,\tau) = \exp\Big(\big(\frac{1}{4}(q_d-1)^2 + q\big)\tau\Big) r_1\Big(T - \frac{2\tau}{\sigma^2}\Big), \quad \tau \in \Big[0, \frac{\sigma^2}{2}T\Big],$$

$$\lim_{x\to\infty} \frac{y(x,\tau)}{\exp\Big(\frac{1}{2}(q_d-1)x + \big(\frac{1}{4}(q_d-1)^2+q\big)\tau\Big) r_2\big(e^x, T - \frac{2\tau}{\sigma^2}\big)} = 1, \ \ \tau \in \Big[0, \frac{\sigma^2}{2}T\Big].$$