Fun With Stupid Integral Tricks

1. Compute
$$\int \frac{x^4 + 2x^3 + 3x^2 + 2x + 1}{x^2 + 1}\ dx$$
If we carry out the long division, we will get a polynomial plus a term of the form $Ax/(x^2+1)$ and a term of the form $B/(x^2+1)$. Since we can deal with all of these terms (using substitution for the first type and $\tan^{-1}$ for the second type), this strategy will succeed.
$$\begin{aligned}\int \frac{x^4 + 2x^3 + 3x^2 + 2x + 1}{x^2 + 1}\ dx &= \int x^2 + 2x + 2 - \frac{1}{x^2+1}\ dx \\ &= \frac{x^3}{3} + x^2 + 2x - \tan^{-1} x + C\end{aligned}$$

2. Compute
$$\int e^u \sin u\ du$$
Since the derivative of e^u is e^u and derivatives of sin eventually get back to sin, it is reasonable to try out integration by parts:
$$\begin{aligned}\int e^u \sin u\ du &= -e^u \cos u + \int e^u \cos u\ du \\ &= -e^u \cos u + e^u \sin u - \int e^u \sin u\ du\end{aligned}$$
Solving for the integral we want then gives
$$\int e^u \sin u\ du = \frac{e}{2}(\sin u - \cos u) + C$$

3. Compute

$$\int \cos^4\varphi - \sin^4\varphi \; d\varphi$$

Since we don't know any nice formulas involving fourth powers of trig functions, we can try to use the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ to reduce the powers:

$$\cos^4 x = \cos^2 x \cdot \cos^2 x = \cos^2 x - \cos^2 x \sin^2 x$$

$$\sin^4 x = \sin^2 x \cdot \sin^2 x = \sin^2 x - \sin^2 x \cos^2 x$$

So we discover the surprising identity

$$\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x = \cos(2x)$$

Our integral is then just

$$\int \cos(2\varphi)\; d\varphi = \frac{1}{2}\sin(2\varphi) + C$$

4. Compute

$$\int \frac{\tan\theta}{\sqrt{\cos^4\theta - 1}}\; d\theta$$

As a first attempt, we might try to split up $\tan\theta$ to see what we get:

$$\int \frac{\tan\theta}{\sqrt{\cos^4\theta - 1}}\; d\theta = \int \frac{\sin\theta}{\cos\theta\sqrt{\cos^4\theta - 1}}\; d\theta$$

This doesn't seem much better. On the other hand, we know that things like $dx/\sqrt{x^2-1}$ are OK to integrate, so we can try to put our integrand in that form. To do this, we would define $x = \cos^2\theta$, so that $dx = -2\cos\theta\sin\theta\; d\theta$, giving

$$\begin{aligned}
\int \frac{\sin\theta}{\cos\theta\sqrt{\cos^4\theta - 1}} \cdot \frac{dx}{-2\cos\theta\sin\theta} &= -\frac{1}{2}\int \frac{dx}{\cos^2\theta\sqrt{\cos^4\theta - 1}} \\
&= -\frac{1}{2}\int \frac{dx}{x\sqrt{x^2-1}} \\
&= -\frac{1}{2}\sec^{-1} x + C \\
&= \frac{-1}{2}\sec^{-1}(\cos^2\theta) + C
\end{aligned}$$

5. Compute

$$\int \frac{x-1}{x^2-2x+5}\, dx$$

By completing the square in the denominator, we will end up with terms of the form $Ax/(Bx^2+C)$ or $A/(Bx^2+C)$, both of which are OK to integrate. $x^2-2x+5=(x-1)^2+4$, so we get (over-detailed calculation which can be simplified if you remember the dx/a^2+x^2 integral)

$$\begin{aligned}\int \frac{x-1}{x^2-2x+5}\, dx &= \int \frac{x-1}{(x-1)^2+4}\, dx \\ &= \frac{1}{4}\int \frac{x-1}{(\frac{x-1}{2})^2+1}\, dx \\ &= \frac{1}{2}\int \frac{\frac{x-1}{2}}{(\frac{x-1}{2})^2+1}\, dx\end{aligned}$$

Setting $u=\left(\frac{x-1}{2}\right)^2$, $du=\frac{x-1}{2}dx$, so we have

$$\frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}\left(\frac{(x-1)^2}{4}\right)+C$$

6. Compute

$$\int \frac{\sqrt{1-\beta^2}}{\beta^2}\, d\beta$$

The $\sqrt{1-\beta^2}$ makes me think of trig functions. In particular, if $\beta=\sin\theta$, then $\sqrt{1-\beta^2}=\cos\theta$. This is a sort of "backwards" substitution, but perfectly valid. Since $\beta=\sin\theta$, $d\beta=\cos\theta\ d\theta$ and the integral is just

$$\begin{aligned}\int \frac{\sqrt{1-\beta^2}}{\beta^2}\, d\beta &= \int \frac{\cos\theta}{\sin^2\theta}\cdot\cos\theta\ d\theta \\ &= \int \frac{1-\sin^2\theta}{\sin^2\theta}\, d\theta \\ &= \int \sec^2\theta - 1\ d\theta \\ &= \tan\theta - \theta + C\end{aligned}$$

To get back to β, just use the fact that since $\beta = \sin\theta$, $\theta = \sin^{-1}\beta$, so the last line is equal to $\tan(\sin^{-1}\beta) - \sin^{-1}\beta + C$.

7. Compute

$$\int \sinh y \sin y \; dy$$

This looks like a typical place for integration by parts, since both functions in the integrand have derivatives which eventually cycle around to the original function. Starting with $f = \sinh y$, $dg = \sin y \; dy$ gives

$$\begin{aligned}\int \sinh y \sin y \; dy &= -\sinh y \cos y + \int \cosh y \cos y \; dy \\ &= -\sinh y \cos y + \cosh y \sin y - \int \sinh y \sin y \; dy\end{aligned}$$

Isolating our integral and solving gives

$$\int \sinh y \sin y \; dy = \frac{1}{2}(\cosh y \sin y - \sinh y \cos y)$$

8. Find the antiderivatives of tan and tanh.

To compute the antiderivative of tan, let us start by splitting it up into functions we understand better. Since the numerator is pretty much the derivative of the denominator, a substitution jumps right out at us:

$$\begin{aligned}\int \tan x \; dx &= \int \frac{\sin x}{\cos x} \; dx \\ &= \int \frac{-du}{u} \\ &= -\log|u| + C = -\log|\cos x| + C\end{aligned}$$

Similarly for the hyperbolic tangent:

$$\begin{aligned}\int \tanh x \; dx &= \int \frac{\sinh x}{\cosh x} \; dx \\ &= \int \frac{du}{u} \\ &= \log|u| + C = \log|\cosh x| + C\end{aligned}$$

9. Show that if f is any continuous function and n any (positive) number,

$$\int_{n^{-1}}^{n} f(x + \frac{1}{x}) \frac{\log x}{x}\ dx = 0$$

It might be hard to see how to proceed since f and n could be anything at all. As an experiment, we might notice that setting $y = 1/x$ doesn't change the form of f, since

$$x + 1/x = 1/y + y$$

Also helpful is that when $x = n$, $y = n^{-1}$ and when $x = n^{-1}$, $y = n$ so the substitution $x = 1/y$ will also reverse the bounds of integration. $x = 1/y$ means $dx = -1/y^2$, so the integral becomes

$$\int_{n^{-1}}^{n} f(x + \frac{1}{x}) \frac{\log x}{x}\ dx = \int_{n}^{n^{-1}} f(\frac{1}{y} + y) \frac{\log(1/y)}{1/y}\ \frac{-dy}{y^2}$$

Using the fact that $\log(1/y) = -\log y$ and canceling some of the ys floating around gives

$$\begin{aligned}\int_{n}^{n^{-1}} f(\frac{1}{y} + y) \frac{\log(1/y)}{1/y}\ \frac{-dy}{y^2} &= \int_{n}^{n^{-1}} f(\frac{1}{y} + y) \frac{\log y}{y}\ dy \\ &= -\int_{n^{-1}}^{n} f(\frac{1}{y} + y) \frac{\log y}{y}\ dy\end{aligned}$$

Since we went from the original integral to one that looks the exact same but with a minus in front, the whole thing must actually equal zero!

10. Compute

$$\int x^n \log x\ dx$$

for $n \in \{2, 1, 0\}$ Since the derivative of log is nice and simple, we might try integration by parts with $f = \log x$ and $dg = x^n\ dx$. Starting with $n = 2$, we get

$$\begin{aligned}\int x^2 \log x\ dx &= \frac{x^3}{3} \log x - \int \frac{x^3}{3} \cdot \frac{1}{x}\ dx \\ &= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C\end{aligned}$$

With $n = 1$, we take the same approach:

$$\begin{aligned}\int x \log x \, dx &= \frac{x^2}{2} \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx \\ &= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C\end{aligned}$$

Finally, with $n = 0$ we can take the same strategy, with the strange-seeming choice $f = \log x$, $dg = dx$ to get

$$\begin{aligned}\int \log x \, dx &= x \log x - \int x \cdot \frac{1}{x} \, dx \\ &= x \log x - x + C\end{aligned}$$

11. Compute the definite integral

$$\int_0^1 e^x e^{\sqrt{1-x}} \, dx$$

This problem doesn't work!

12. Find the antiderivatives of sec and $\sec^3$.

To integrate $\sec x$, we can use the sneaky trick from the book, or we can try a more direct approach:

$$\int \sec x \, dx = \int \frac{dx}{\cos x}$$

To get the $\cos x$ taken care of, let us set $u = \cos x$ so that $du = -\sin x \, dx$. So we have

$$\begin{aligned}\int \frac{dx}{\cos x} &= \int \frac{-du}{\cos x \sin x} \\ &= \int \frac{-du}{u\sqrt{1-u^2}} \\ &= \operatorname{sech}^{-1} u + C \\ &= \operatorname{sech}^{-1} \cos x + C\end{aligned}$$

For $\sec^3 x$, there are several things we could try (integration by parts, substitution, identities, etc). Let us use the fact that $\sec^2 x$ is the derivative of $\tan x$ to lead into an integration by parts:

$$\begin{aligned}\int \sec^3 x\ dx &= \int \sec x\ d\tan x \\ &= \sec x \tan x - \int \tan x\ d\sec x \\ &= \sec x \tan x - \int \tan^2 x \sec x\ dx\end{aligned}$$

Using the identity $1 + \tan^2 x = \sec^2 x$, we get

$$\sec x \tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

Solving for $\int \sec^3$ and using the integral of sec that we just derived, we get

$$\int \sec^3 x\ dx = \frac{1}{2}(\sec x \tan x + \operatorname{sech}^{-1} \cos x) + C$$