Moment-Distribution Method

Structural Analysis
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Theory of Structures-II
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Moment-Distribution Method

- Classical method.
- Used for Beams and Frames.
- Developed by Hardy Cross in 1924.
- Used by Engineers for analysis of small structures.
- It does not involve the solution of many simultaneous equations.

Moment-Distribution Method

- For beams and frames without sidesway, it does not involve the solution of simultaneous equations.

- For frames with sidesway, number of simultaneous equations usually equals the number of independent joint translations.

- In this method, Moment Equilibrium Equations of joints are solved iteratively by considering the moment equilibrium at one joint at a time, while the remaining joints are considered to be restrained.

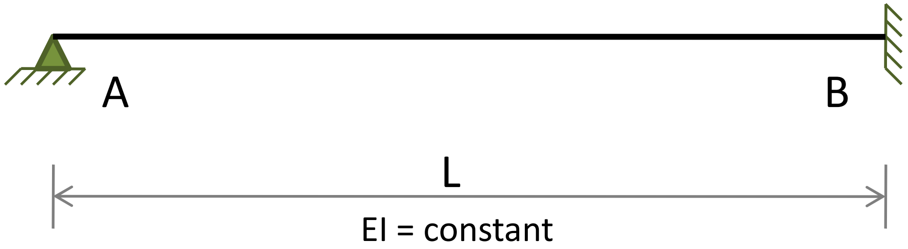
Definitions and Terminology

Sign Convention

- Counterclockwise member end moments are considered positive.
- Clockwise moments on joints are considered positive.

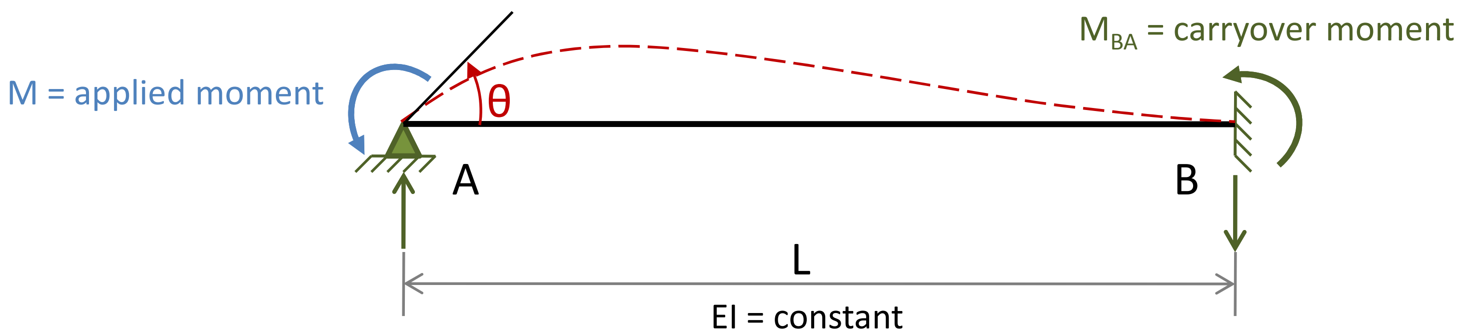
Member Stiffness

- Consider a prismatic beam AB, which is hinged at end A and fixed at end B.



Member Stiffness

If we apply a moment M at the end A, the beam rotates by an angle θ at the hinged end A and develops a moment M_{BA} at the fixed end B, as shown.



The relationship between the applied moment M and the rotation θ can be established using the slope-deflection equation.

Member Stiffness

By substituting $M_{nf} = M$, $\theta_n = \theta$, and $\theta_f = \Psi = FEM_{nf} = 0$ into the slope-deflection equation, we obtain

$$M = \left(\frac{4EI}{L}\right)\theta \tag{1}$$

"The bending stiffness, $\bar{K}$, *of a member is defined as the moment that must be applied at an end of the member to cause a unit rotation of that end."*

By setting $\theta = 1$ rad in Eq. 1, we obtain the expression for the bending stiffness of the beam of figure to be

$$\bar{K} = \frac{4EI}{L} \tag{2}$$

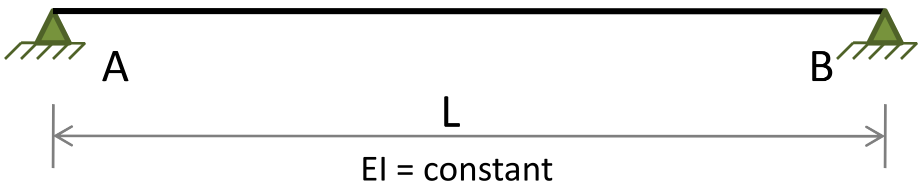
Member Stiffness

when the modulus of elasticity for all the members of a structure is the same (constant), it is usually convenient to work with the relative bending stiffness of members in the analysis.

"The relative bending stiffness, K, of a member is obtained by dividing its bending stiffness, $\overline{K}$ *, by 4E."*

$$K = \frac{\overline{K}}{4E} = \frac{I}{L} \qquad (3)$$

- Now suppose that the far end B of the beam is hinged as shown.

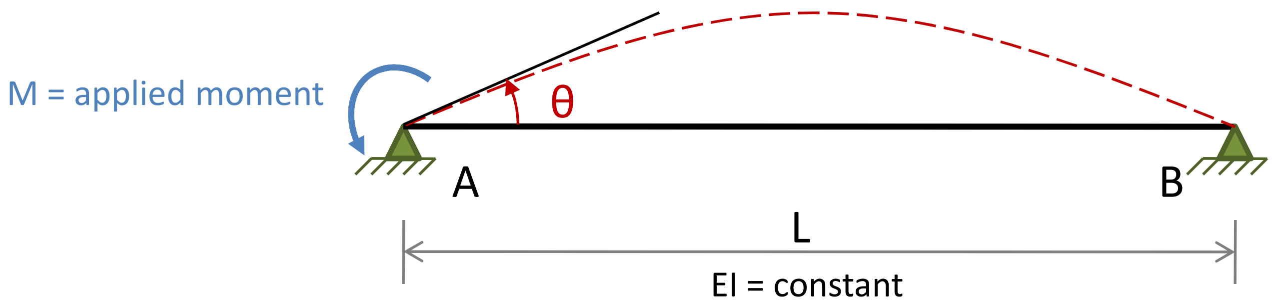


Member Stiffness

The relationship between the applied moment M and the rotation θ of the end A of the beam can now be determined by using the modified slope-deflection equation.

By substituting $M_{rh} = M$, $\theta_r = \theta$, and $\Psi = FEM_{rh} = FEM_{hr} = 0$ into MSDE, we obtain

$$M = \left(\frac{3EI}{L}\right)\theta \tag{4}$$



Member Stiffness

By setting θ = 1 rad, we obtain the expression for the bending stiffness of the beam of figure to be

$$\bar{K} = \frac{3EI}{L} \tag{5}$$

A comparison of Eq. 2 & Eq. 5 indicates that the stiffness of the beam is reduced by 25% when the fixed support at B is replaced by a hinged support.

The relative bending stiffness of the beam can now be obtained by dividing its bending stiffness by 4E.

$$K = \frac{\bar{K}}{4E} = \frac{3}{4}\left(\frac{I}{L}\right) \tag{6}$$

Member Stiffness

Relationship b/w applied end moment M and the rotation θ

$$M = \begin{cases} \left(\dfrac{4EI}{L}\right)\theta & \text{if far end of member is fixed} \\ \left(\dfrac{3EI}{L}\right)\theta & \text{if far end of member is hinged} \end{cases} \qquad (7)$$

Bending stiffness of a member

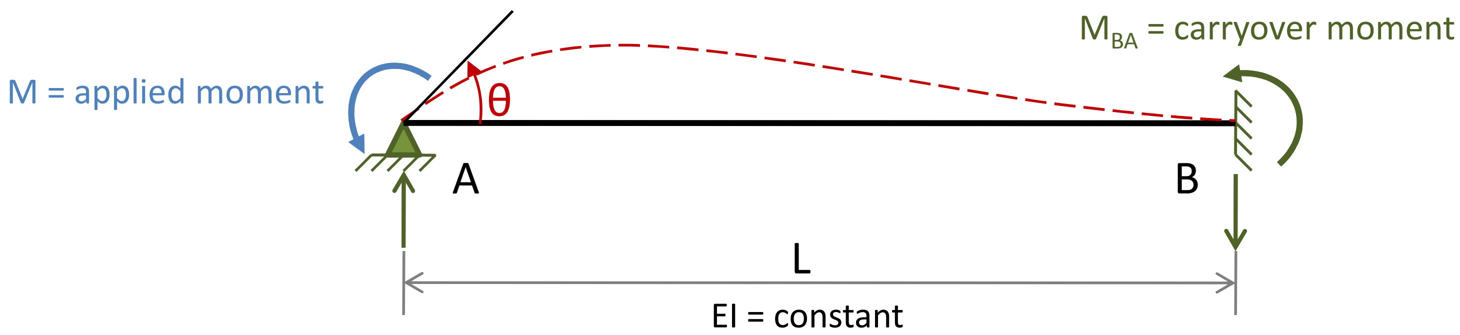
$$\overline{K} = \begin{cases} \dfrac{4EI}{L} & \text{if far end of member is fixed} \\ \dfrac{3EI}{L} & \text{if far end of member is hinged} \end{cases} \qquad (8)$$

Relative bending stiffness of a member

$$K = \begin{cases} \dfrac{I}{L} & \text{if far end of member is fixed} \\ \dfrac{3}{4}\dfrac{I}{L} & \text{if far end of member is hinged} \end{cases} \qquad (9)$$

Carryover Moment

Let us consider again the hinged-fixed beam of Figure.



When a moment M is applied at the hinged end A of the beam, a moment M_{BA} develops at the fixed end B.

The moment M_{BA} is termed the *carryover moment*.

Carryover Moment

To establish the relationship b/w the applied moment M and the carryover moment M_{BA}, we write the slope deflection equation for M_{BA} by substituting $M_{nf} = M_{BA}$, $\theta_f = \theta$, and $\theta_n = \Psi = FEM_{nf} = 0$ into SDE

$$M_{BA} = \left(\frac{2EI}{L}\right)\theta \tag{10}$$

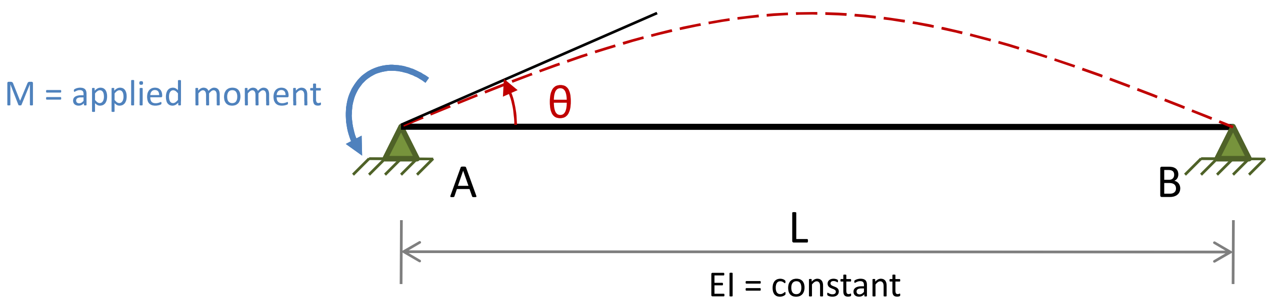
By substituting $\theta = ML/(4EI)$ from Eq. 1 into Eq. 10, we obtain

$$M_{BA} = \frac{M}{2} \tag{11}$$

Eq. 11 indicates, when a moment of magnitude M is applied at the hinged end of the beam, one-half of the applied moment is carried over to the far end, provided that the far end is fixed. The direction of M_{BA} and M is same.

Carryover Moment

When the far end of the beam is hinged as shown, the carryover moment M_{BA} is zero.



$$M_{BA} = \begin{cases} \dfrac{M}{2} & \text{if far end of member is fixed} \\ 0 & \text{if far end of member is hinged} \end{cases} \qquad (12)$$

Carryover Factor (COF)

"The ratio of the carryover moment to the applied moment (M_{BA}/M) is called the carryover factor of the member."

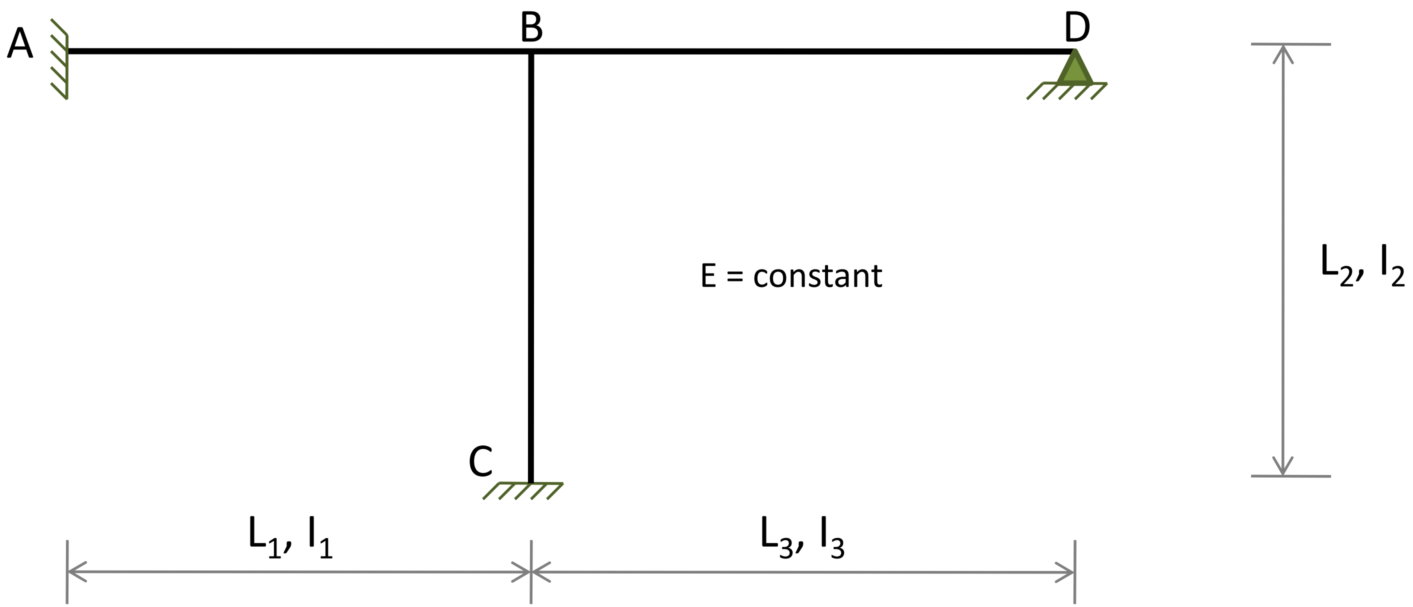
It represents the fraction of the applied moment M that is carried over to the far end of the member. By dividing Eq. 12 by M, we can express the carryover factor (COF) as

$$COF = \begin{cases} \dfrac{1}{2} & \text{if far end of member is fixed} \\ 0 & \text{if far end of member is hinged} \end{cases} \qquad (13)$$

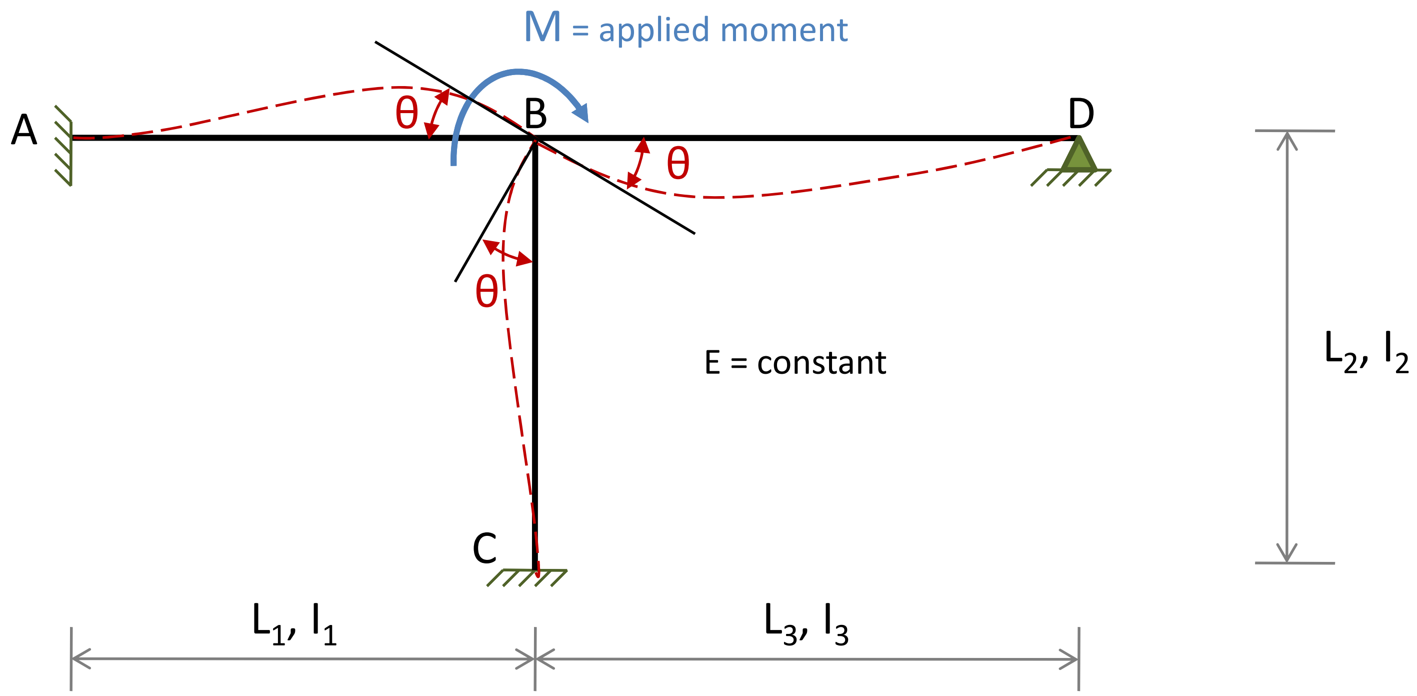
Distribution Factors

When analyzing a structure by the moment-distribution method, an important question that arises is how to distribute a moment applied at a joint among the various members connected to that joint.

Consider the three-member frame shown in figure below.



Suppose that a moment M is applied to the joint B, causing it to rotate by an angle θ as shown in figure below.

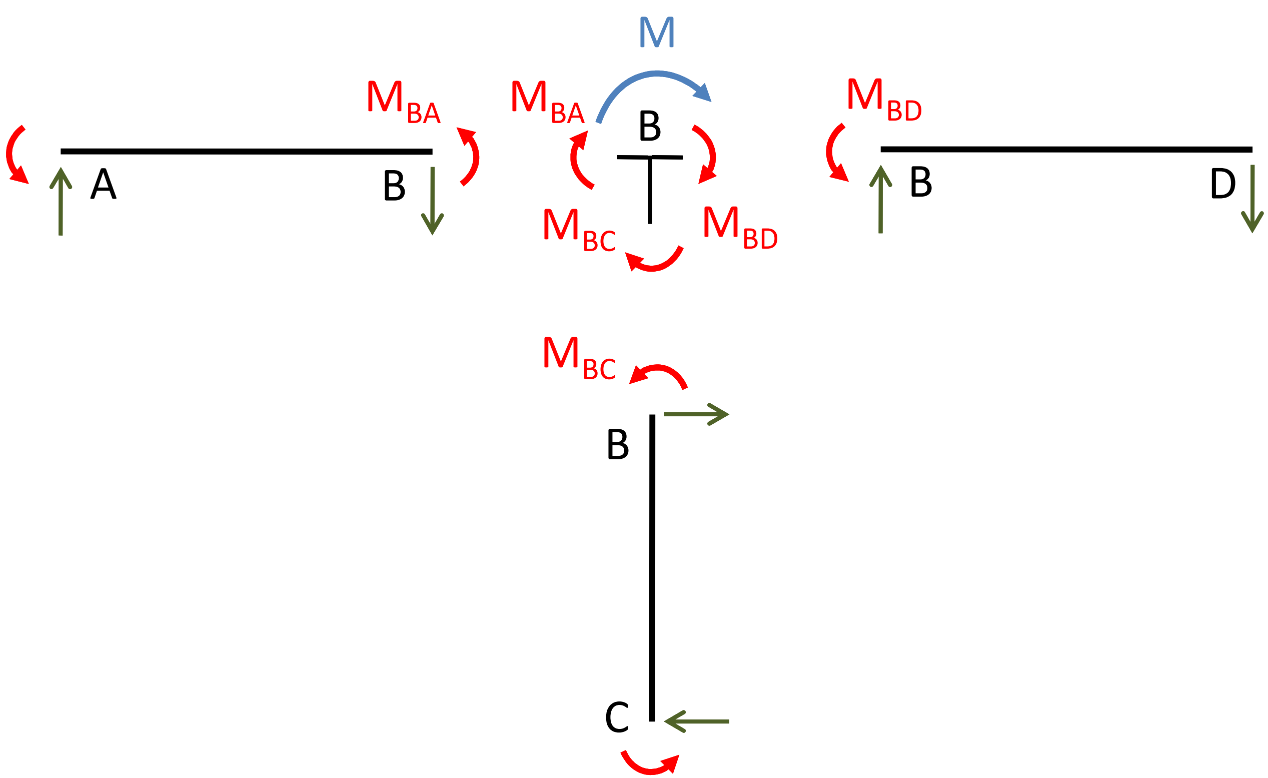


To determine what fraction of applied moment is resisted by each of the three members AB, BC, and BD, we draw free-body diagrams of joint B and of the three members AB, BC, and BD.

By considering the moment equilibrium of the free body of joint B ($\sum M_B = 0$), we write

$$M + M_{BA} + M_{BC} + M_{BD} = 0$$

$$M = -\left(M_{BA} + M_{BC} + M_{BD}\right) \tag{14}$$



Since members AB, BC, and BD are rigidly connected to joint B, the rotations of the ends B of these members are the same as that of the joint.

The moments at the ends B of the members can be expressed in terms of the joint rotation θ by applying Eq. 7.

Noting that the far ends A and C, respectively, of members AB and BC are fixed, whereas the far end D of member BD is hinged, we apply Eq. 7 through Eq. 9 to each member to obtain

$$M_{BA} = \left(\frac{4EI_1}{L_1}\right)\theta = \overline{K}_{BA}\theta = 4EK_{BA}\theta \tag{15}$$

$$M_{BC} = \left(\frac{4EI_2}{L_2}\right)\theta = \overline{K}_{BC}\theta = 4EK_{BC}\theta \tag{16}$$

$$M_{BD} = \left(\frac{3EI_3}{L_3}\right)\theta = \overline{K}_{BD}\theta = 4EK_{BD}\theta \tag{17}$$

Substitution of Eq. 15 through Eq. 17 into the equilibrium equation Eq. 14 yields

$$M = -\left(\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2} + \frac{3EI_3}{L_3} \right)\theta$$

$$= -\left(\overline{K}_{BA} + \overline{K}_{BC} + \overline{K}_{BD} \right)\theta = -\left(\sum \overline{K}_B \right)\theta \qquad (18)$$

in which $\sum \overline{K}_B$ represents the sum of the bending stiffnesses of all the members connected to joint B.

"The rotational stiffness of a joint is defined as the moment required to cause a unit rotation of the joint."

From Eq. 18, we can see that the rotational stiffness of a joint is equal to the sum of the bending stiffnesses of all the members rigidly connected to the joint.

The negative sign in Eq. 18 appears because of the sign convention.

To express member end moments in terms of the applied moment M, we first rewrite Eq. 18 in terms of the relative bending stiffnesses of members as

$$M = -4E\left(K_{BA} + K_{BC} + K_{BD}\right)\theta = -4E\left(\sum K_B\right)\theta$$

$$\theta = -\frac{M}{4E\sum K_B} \qquad (19)$$

By substituting Eq. 19 into Eqs. 15 through 17, we obtain

$$M_{BA} = -\left(\frac{K_{BA}}{\sum K_B}\right)M \qquad (20)$$

$$M_{BC} = -\left(\frac{K_{BC}}{\sum K_B}\right)M \qquad (21)$$

$$M_{BD} = -\left(\frac{K_{BD}}{\sum K_B} \right) M \tag{22}$$

From Eqs. 20 through 22, we can see that the applied moment M is distributed to the three members in proportion to their relative bending stiffnesses.

"The ratio $K/\sum K_B$ for a member is termed the distribution factor of that member for end B, and it represents the fraction of the applied moment M that is distributed to end B of the member."

Thus Eqs. 20 through 22 can be expressed as

$$M_{BA} = -DF_{BA}M \tag{23}$$

$$M_{BC} = -DF_{BC}M \tag{24}$$

$$M_{BD} = -DF_{BD}M \tag{25}$$

in which $DF_{BA} = K_{BA}/\sum K_B$, $DF_{BC} = K_{BC}/\sum K_B$, and $DF_{BD} = K_{BD}/\sum K_B$, are the distribution factors for ends B of members AB, BC, and BD, respectively.

For example, if joint B of the frame is subjected to a clockwise moment of 150 k-ft ($M = 150$ k-ft) and if $L_1 = L_2 = 20$ ft, $L_3 = 30$ ft, and $I_1 = I_2 = I_3 = I$, so that

$$K_{BA} = K_{BC} = \frac{I}{20} = 0.05I$$

$$K_{BD} = \frac{3}{4}\left(\frac{I}{30}\right) = 0.025I$$

then the distribution factors for the ends B of members AB, BC, and BD are given by

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.05I}{(0.05 + 0.05 + 0.025)I} = 0.4$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.05I}{0.125I} = 0.4$$

$$DF_{BD} = \frac{K_{BD}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.025I}{0.125I} = 0.2$$

These distribution factors indicate that 40% of the 150 k-ft moment applied to joint B is exerted at end B of member AB, 40% at end B of member BC, and the remaining 20% at end B of member BD.

The moments at ends B of the three members are

$$M_{BA} = -DF_{BA}M = -0.4(150) = -60\ \text{k-ft} \quad \text{or} \quad 60\ \text{k-ft}$$

$$M_{BC} = -DF_{BC}M = -0.4(150) = -60\ \text{k-ft} \quad \text{or} \quad 60\ \text{k-ft}$$

$$M_{BD} = -DF_{BD}M = -0.2(150) = -30\ \text{k-ft} \quad \text{or} \quad 30\ \text{k-ft}$$

Based on the foregoing discussion, we can state that, in general, *"the distribution factor (DF) for an end of a member that is rigidly connected to the adjacent joint equals the ratio of the relative bending stiffness of the member to the sum of the relative bending stiffnesses of all the members framing into the joint"*; that is

$$DF = \frac{K}{\sum K} \tag{26}$$

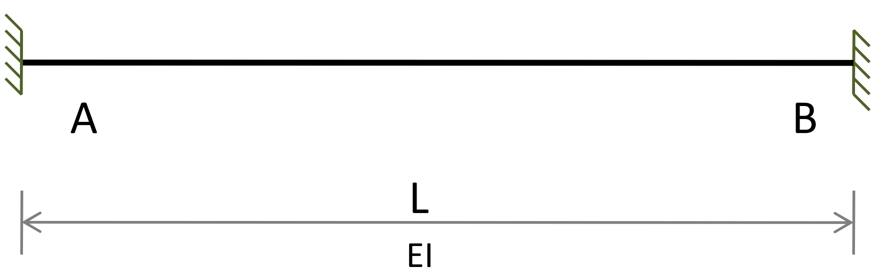
"The moment distributed to (or resisted by) a rigidly connected end of a member equals the distribution factor for that end times the negative of the moment applied to the adjacent joint."

Fixed-End Moments

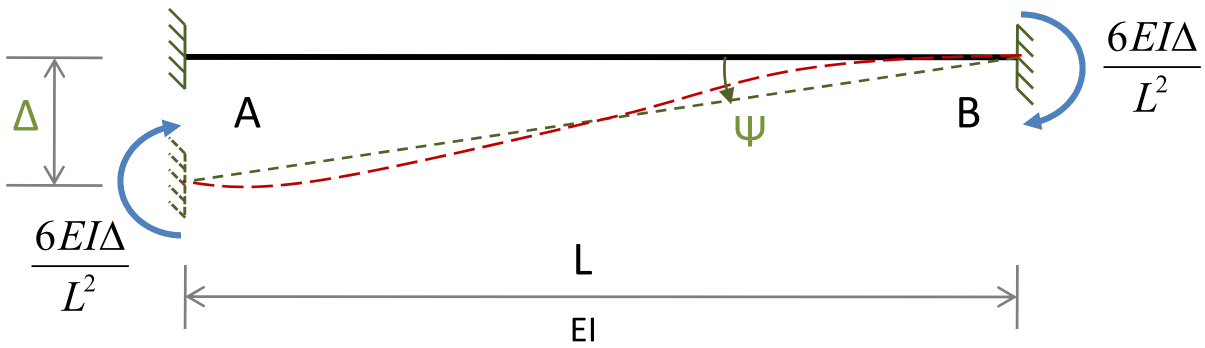
The fixed end moment expressions for some common types of loading conditions as well as for relative displacements of member ends are given inside the back cover of book.

In the MDM, the effects of joint translations due to support settlements and sidesway are also taken into account by means of fixed-end moments.

Consider the fixed beam of Figure.



A small settlement Δ of the left end A of the beam with respect to the right end B causes the beam's chord to rotate counterclockwise by an angle $\Psi = \Delta/L$.

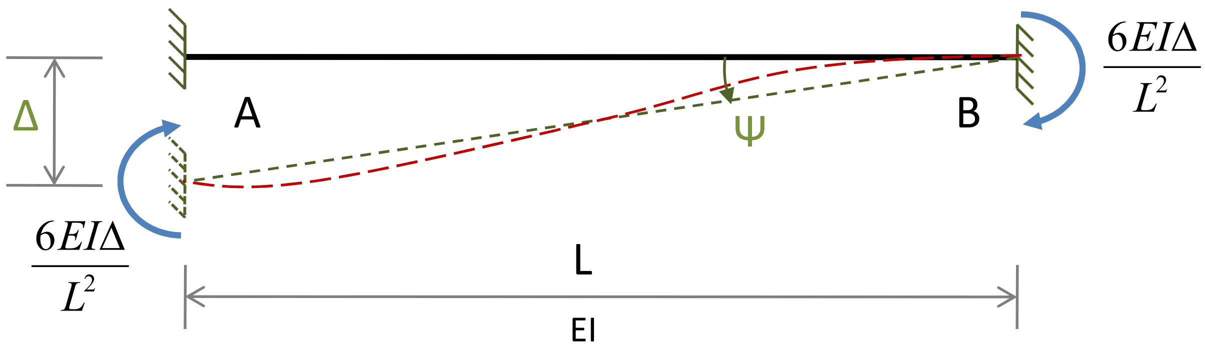


By writing the SDE for the two end moments with $\Psi = \Delta/L$ and by setting θ_A, θ_B, and FEM_{AB} and FEM_{BA} due to external loading, equal to zero, we obtain

$$FEM_{AB} = FEM_{BA} = -\frac{6EI\Delta}{L^2}$$

in which FEM_{AB} and FEM_{BA} denote the FEM due to the relative translation Δ between the two ends of the beam.

Note that the magnitudes as well as the directions of the two FEM are the same.



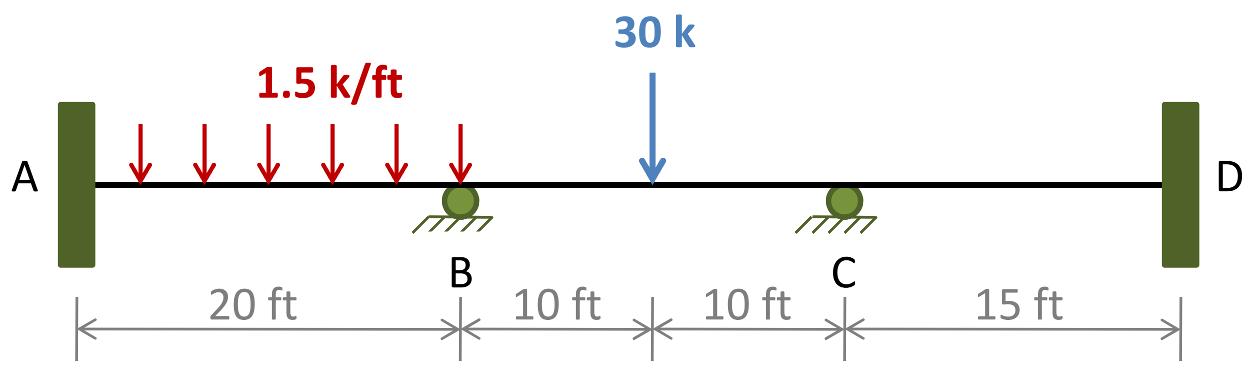
It can be seen from the figure that when a relative displacement causes a chord rotation in the CCW direction, then the two FEMs act in the CW (-ve) direction to maintain zero slopes at the two ends of the beam.

Conversely, if the chord rotation due to a relative displacement is CW, then both FEM act in CCW (+ve) direction.

Moment-Distribution Method

• MDM	Moment Distribution Method
• MD Table	Moment Distribution Table
• COM	Carryover Moment
• COF	Carryover Factor
• DM	Distributed Moment
• UM	Unbalanced Moment

Basic Concept of the Moment Distribution Method



EI = constant
E = 29,000 ksi
I = 500 in^4

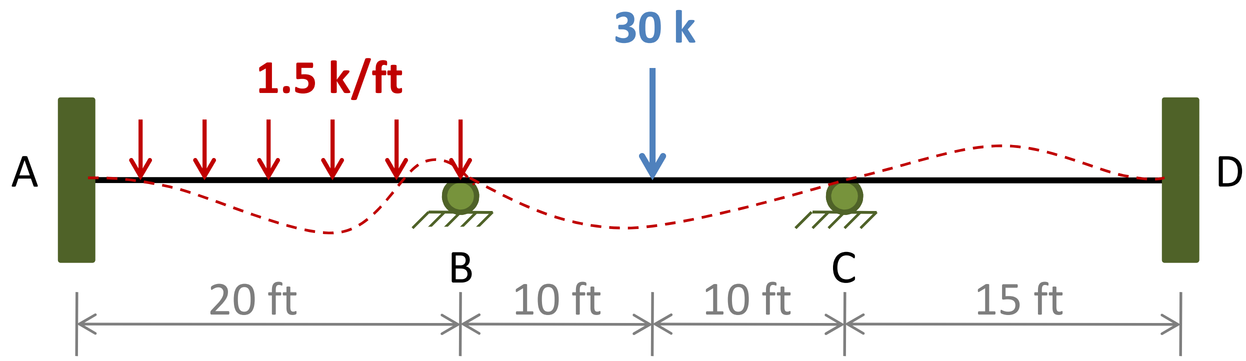
Distribution Factors

The first step in the analysis is to calculate the distribution factors at those joints of the structure that are free to rotate.

The distribution factor for an end of a member is equal to the relative bending stiffness of the member divided by the sum of relative bending stiffnesses of all the members connected to the joint.

$$DF = \frac{K}{\sum K} \tag{26}$$

Basic Concept of the Moment Distribution Method



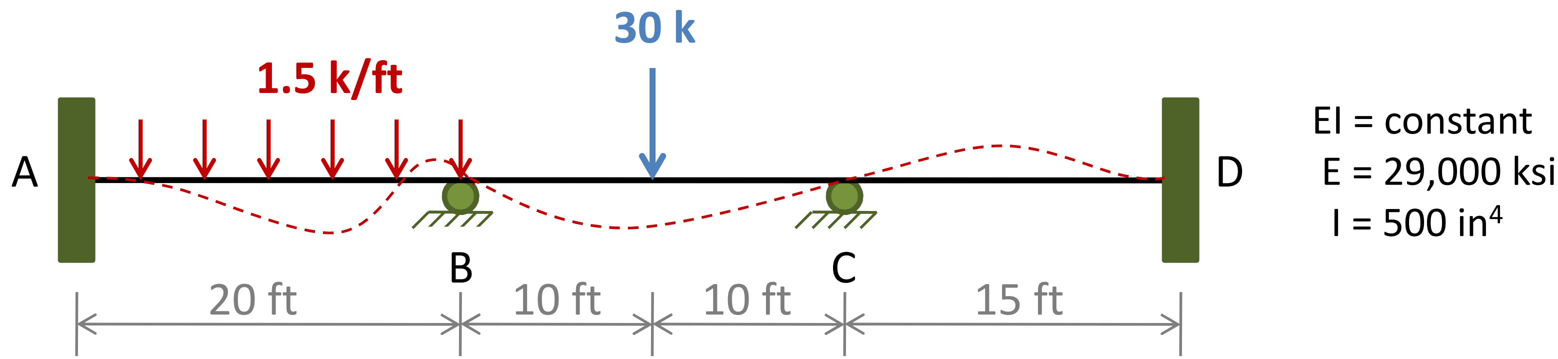
EI = constant
E = 29,000 ksi
$I = 500 \text{ in}^4$

We can see that only joint B and C of the continuous beam are free to rotate. The distribution factors at joint B are

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{I/20}{2I/20} = 0.5$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{I/20}{2I/20} = 0.5$$

Basic Concept of the Moment Distribution Method



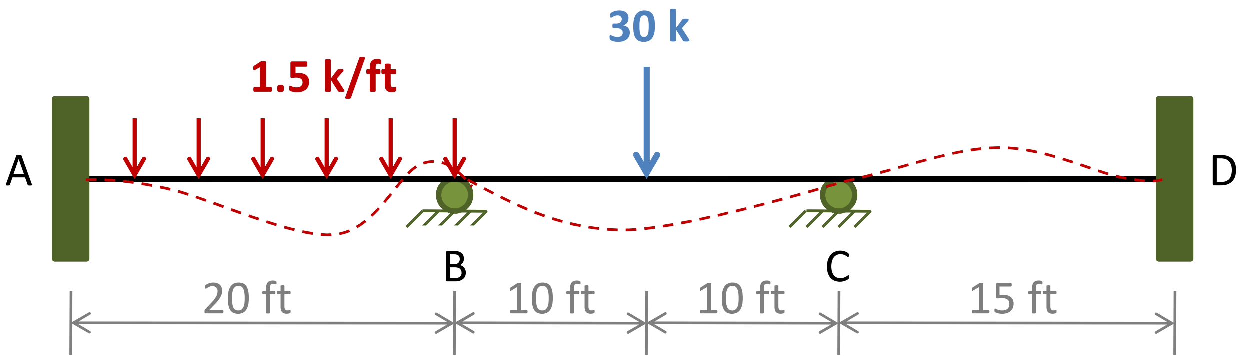
Similarly at joint C

$$DF_{CB} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{I/20}{(I/20) + (I/15)} = 0.429$$

$$DF_{CD} = \frac{K_{CD}}{K_{CB} + K_{CD}} = \frac{I/15}{(I/20) + (I/15)} = 0.571$$

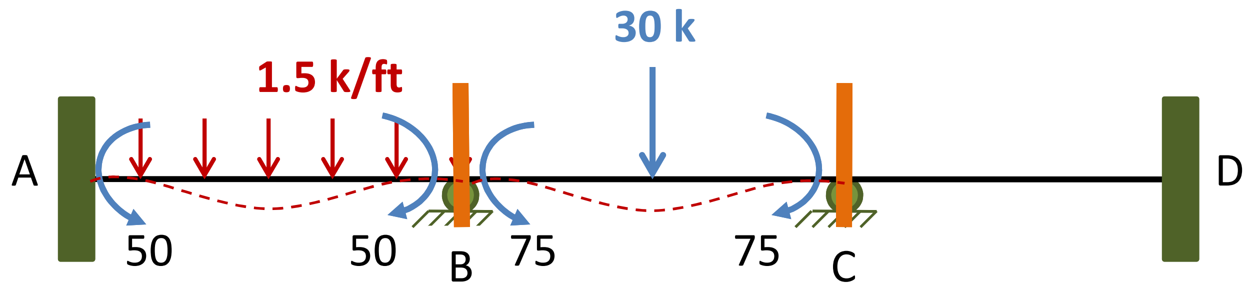
Note that the sum of distribution factors at each joint must always equal 1. The DF are recorded in boxes directly beneath the corresponding member ends on top of the MD Table.

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors

0.5	0.5	0.429	0.571



Fixed End Moments

Next, by assuming that joints B and C are restrained against rotation by imaginary clamps applied to them, we calculate the FEM that develop at the ends of each member. (1. line MD Table)

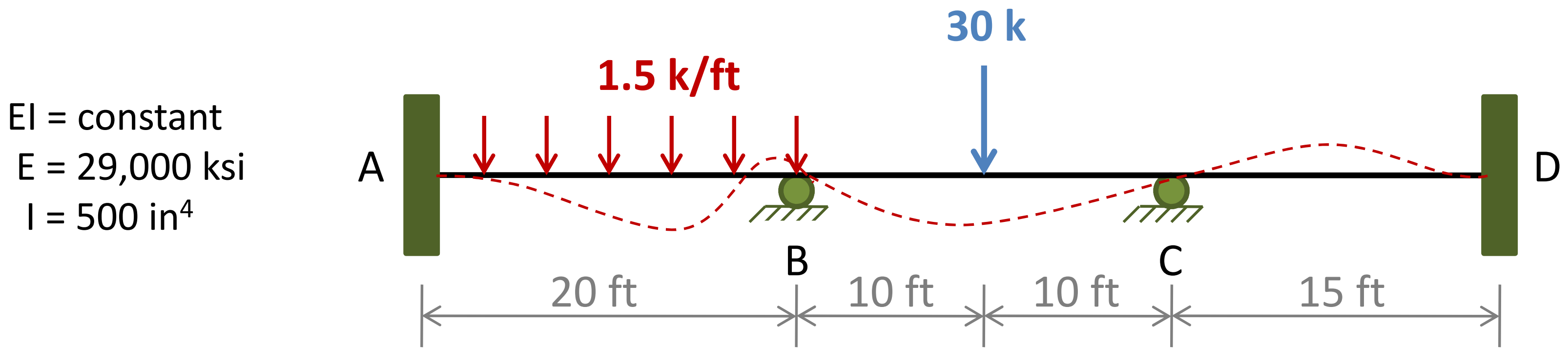
$$FEM_{AB} = \frac{1.5(20)^2}{12} = 50\,\mathrm{k-ft} \qquad \text{or} \qquad +50\,\mathrm{k\text{-}ft}$$

$$FEM_{BA} = \frac{1.5(20)^2}{12} = 50\,\mathrm{k-ft} \qquad \text{or} \qquad -50\,\mathrm{k\text{-}ft}$$

$$FEM_{BC} = \frac{30(20)}{8} = 75\,\mathrm{k-ft} \qquad \text{or} \qquad +75\,\mathrm{k\text{-}ft}$$

$$FEM_{CB} = 75\,\mathrm{k-ft} \qquad \text{or} \qquad -75\,\mathrm{k\text{-}ft}$$

$$FEM_{CD} = FEM_{DC} = 0$$

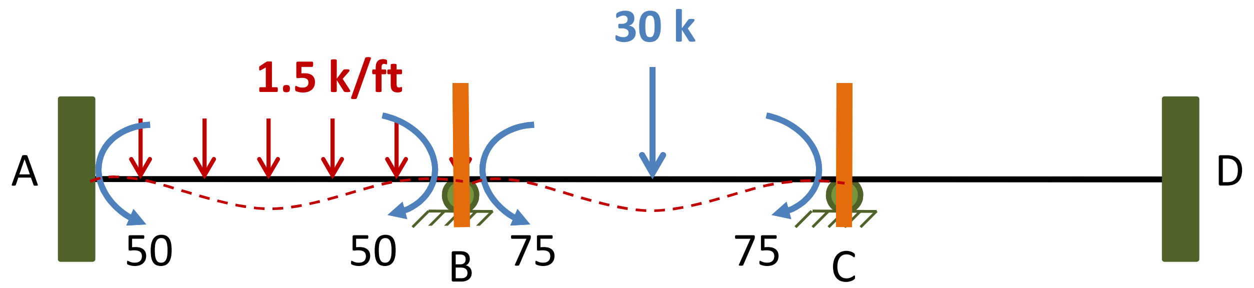


Distribution Factors

	0.5	0.5		0.429	0.571

1.Fixed-end Moments

+50	-50	+75		-75	

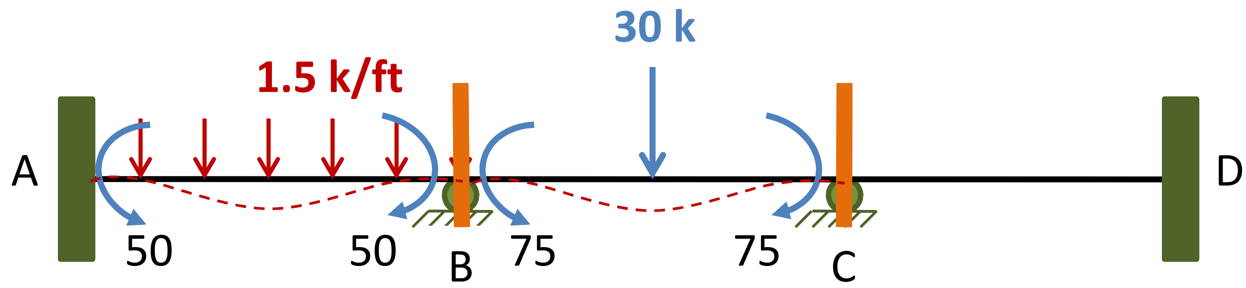


Balancing Joint *C*

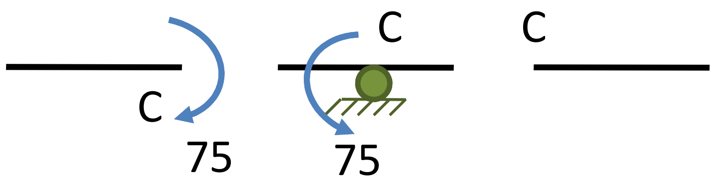
Since joints B and C are actually not clamped, we release them, one at a time. Let us begin at joint C.

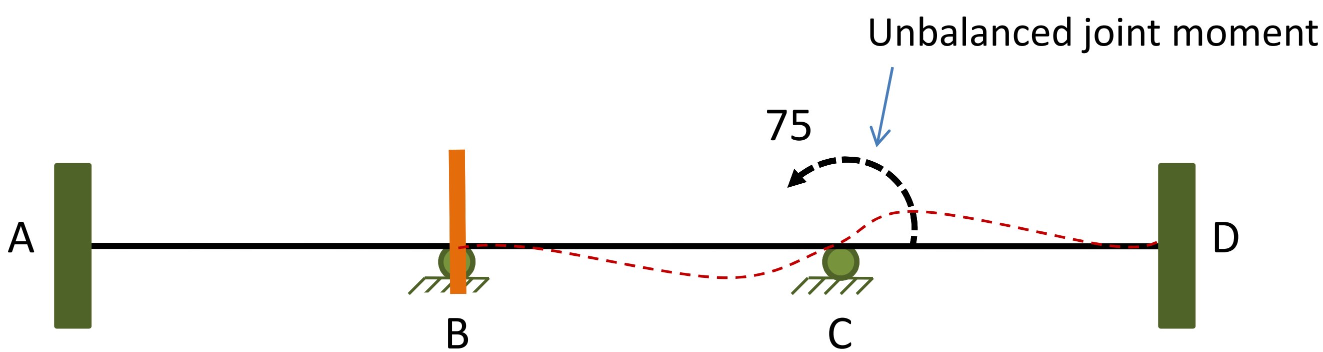
From fig. we can see that there is a -75 k-ft (clockwise) FEM at end C of member BC, whereas no moment exists at end C of member CD.

As long as joint C is restrained against rotation by the clamp, the -75 k-ft unbalanced moment is absorbed by the clamp.



When the imaginary clamp is removed to release the joint, the -75 k-ft unbalanced moment acts at the joint, causing it to rotate in the CCW direction until it is in equilibrium.

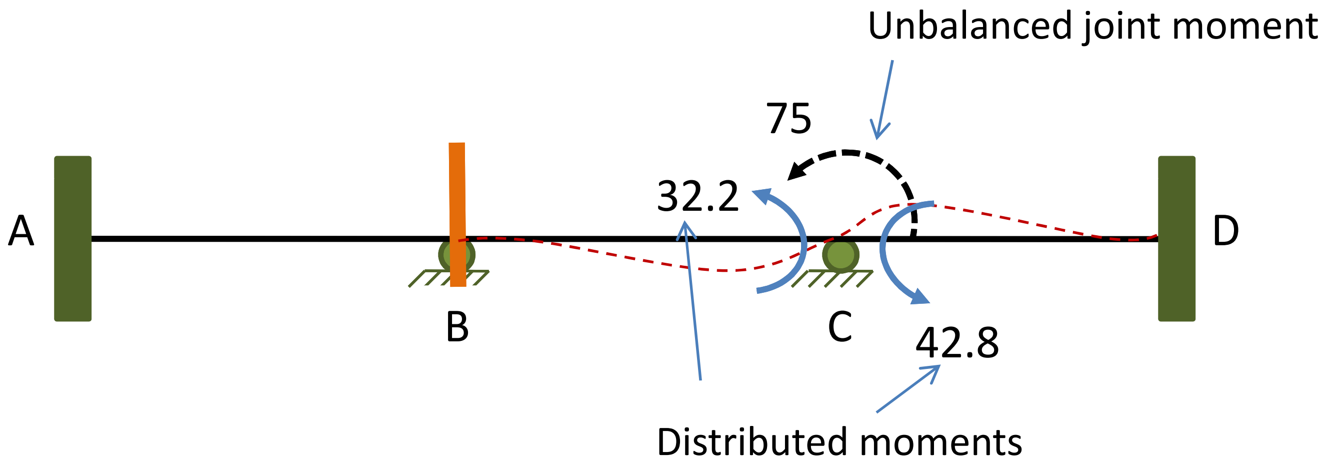




The rotation of joint C causes the distributed moments, DM_{CB} and DM_{CD}, to develop at ends C of members BC and CD, which can be evaluated by multiplying the negative of the unbalanced moment (+75 k-ft) by distribution factors DF_{CB} and DF_{CD}, respectively.

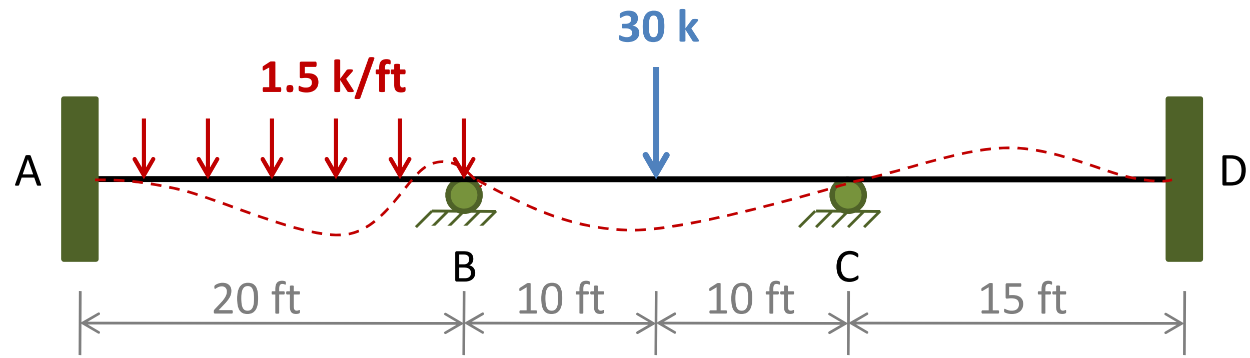
$$DM_{CB} = 0.429(+75) = +32.2\ \mathrm{k-ft}$$

$$DM_{CD} = 0.571(+75) = +42.8\ \mathrm{k-ft}$$



These distributed moments are recorded in line 2 of the MD Table, and a line is drawn beneath them to indicate that joint C is now balanced.

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors

1. Fixed-end Moments

2. Balance joint C and carryover

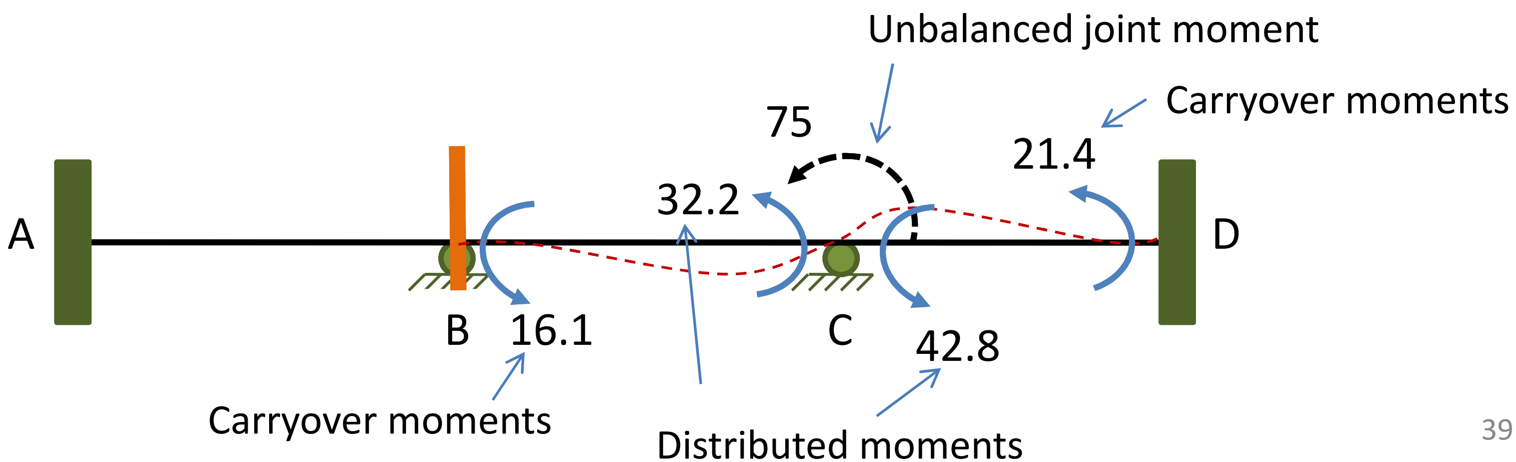
	0.5	0.5	0.429	0.571	
+50	-50	+75	-75		
			+32.2	+42.8	

The DM at end C of member BC induces a COM at the far end B, which can be determined by multiplying the DM by the COF of the member.

Since joint B remains clamped, the COF of the member BC is ½ (Eq.13). Thus, COM at the end B of member BC is

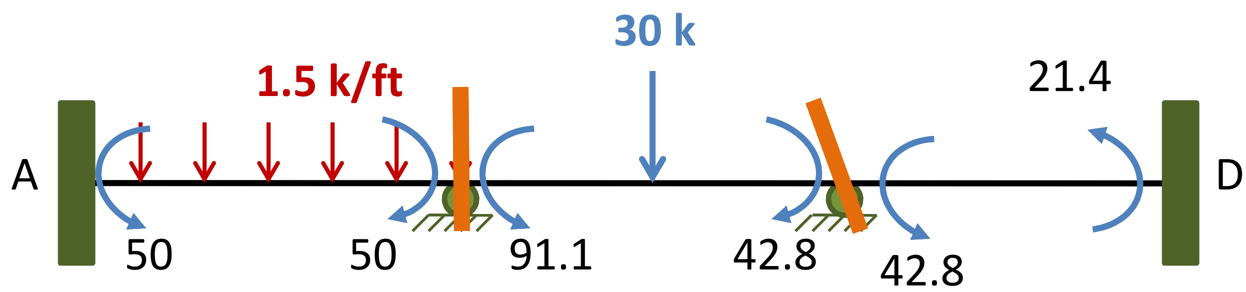
$$COM_{BC} = COF_{CB}\left(DM_{CB}\right) = \frac{1}{2}\left(+32.2\right) = +16.1 \ \text{k}-\text{ft}$$

$$COM_{DC} = COF_{CD}\left(DM_{CD}\right) = \frac{1}{2}\left(+42.8\right) = +21.4 \ \text{k}-\text{ft}$$



These COM are recorded on the same line of the MD Table as the DM, with a horizontal arrow from each DM to its COM.

The total member end moments at this point in this analysis are depicted in Figure.



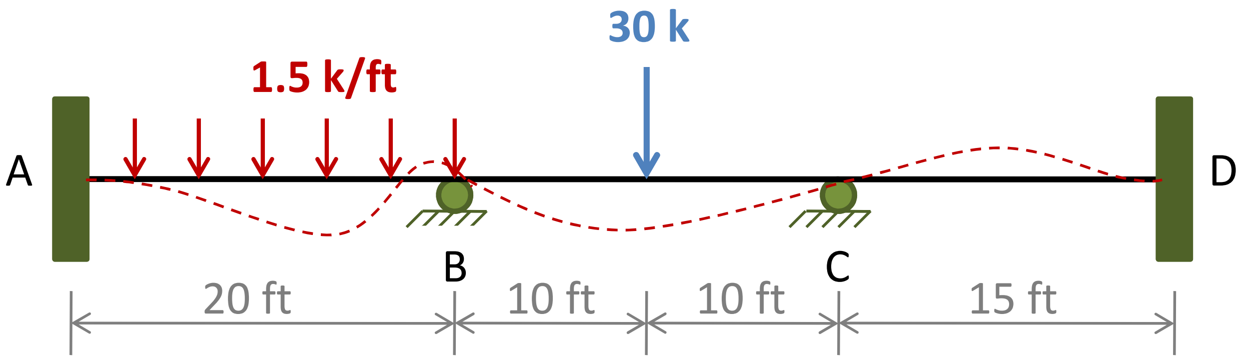
It can be seen that joint C is now in equilibrium, because it is subjected to two equal, but opposite moments.

Joint B, however, is not in equilibrium, and it needs to be balanced. Before we release joint B, an imaginary clamp is applied to joint C in its rotated position.

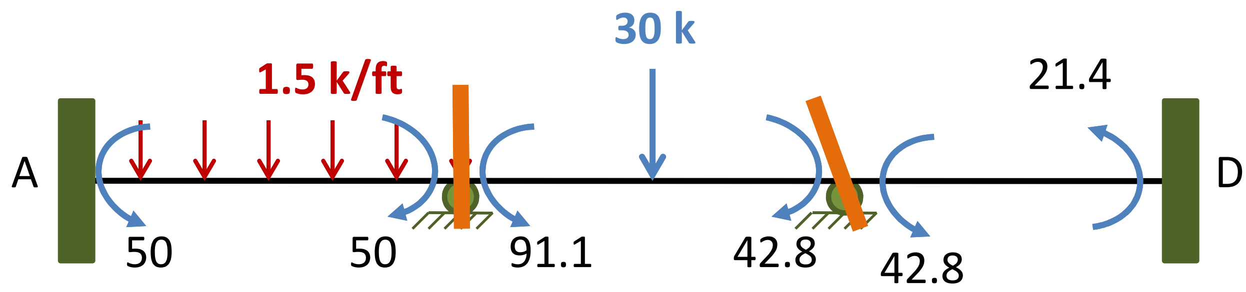
EI = constant

E = 29,000 ksi

I = 500 in^4



	A	B	B	C	C	D
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1	+32.2	+42.8	+24.1



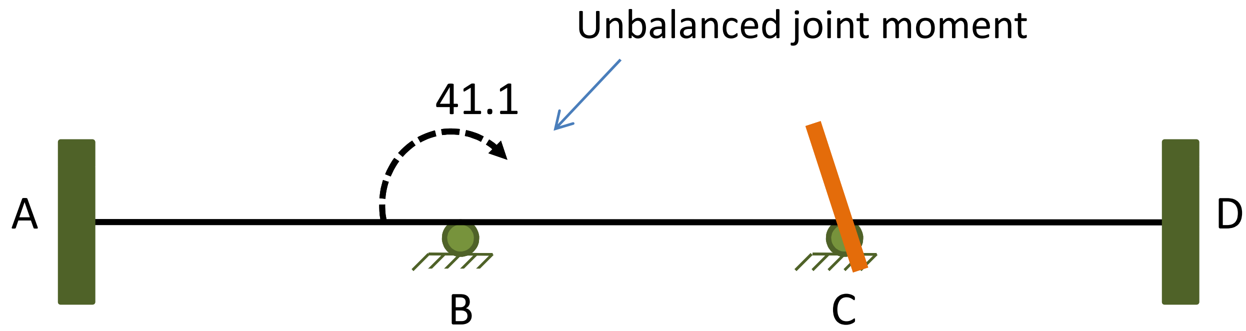
Balancing Joint *B*

Joint B is now released. The unbalanced moment at this joint is obtained by summing all the moments acting at the ends B of members AB and BC, which are rigidly connected to joint B.

From the MD Table (lines 1 & 2), we can see that there is a -50 k-ft FEM at end B of member AB, whereas the end B of member BC is subjected to a +75 k-ft FEM and a +16.1 k-ft COM. The unbalanced moment at joint B is

$$UM_B = -50 + 75 + 16.1 = +41.1 \text{ k} - \text{ft}$$

This UM causes joint B to rotate, as shown, and induces DM at ends B of member AB and BC.



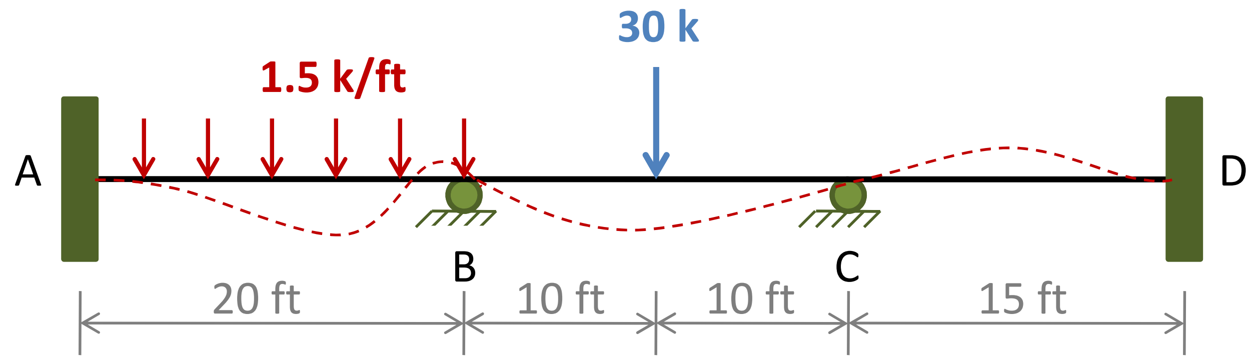
The DM are evaluated by multiplying the negative of the UM by the distribution factors:

$$DM_{BA} = 0.5(-41.1) = -20.6\ \mathrm{k-ft}$$

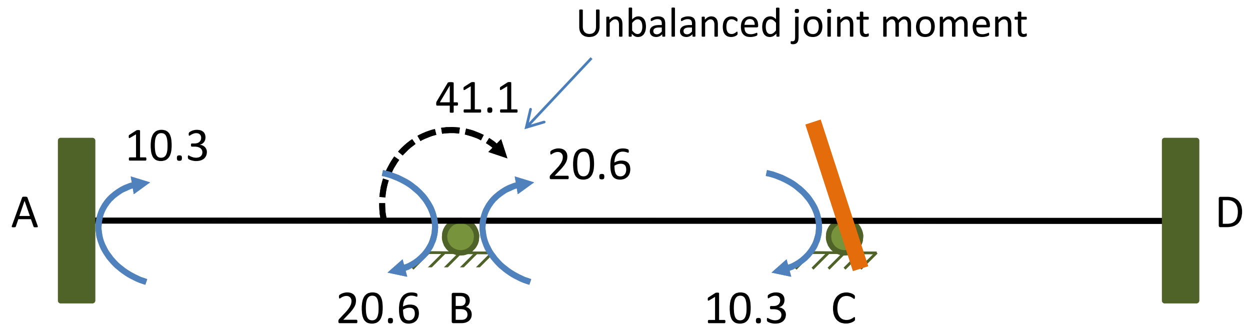
$$DM_{BC} = 0.5(-41.1) = -20.6\ \mathrm{k-ft}$$

These DM are recorded on line 3 of the MD Table and a line is drawn beneath them to indicate that joint B is now balanced.

EI = constant
E = 29,000 ksi
I = 500 in^4

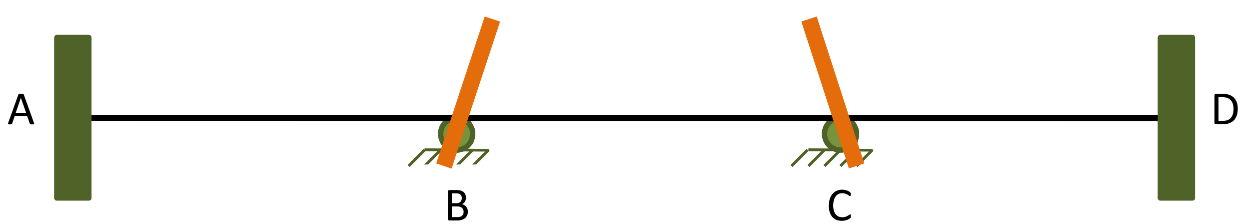


Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1	+32.2	+42.8	+24.1
3.Balance joint B and carryover	-10.3	-20.6	-20.6	-10.3		



One-half of the DM are then carried over to the far ends A and C of members AB and BC, respectively, as indicated by the horizontal arrows on line 3 of Table.

Joint B is then reclamped in its rotated position.



Balancing Joint *C*

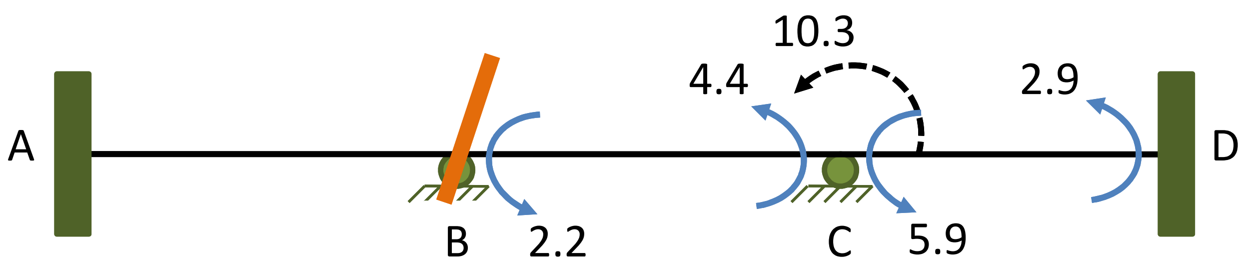
With joint B now balanced, we can see from the MD Table (line 3) that, due to the carryover effect, there is a -10.3 k-ft UM at joint C.

Recall that the moments above the horizontal line at joint C were balanced previously. Thus we release joint C again and distribute the UM to ends C of members BC and CD as

$$DM_{CB} = 0.429(+10.3) = +4.4 \text{ k} - \text{ft}$$

$$DM_{CD} = 0.571(+10.3) = +5.9 \text{ k} - \text{ft}$$

The DM are recorded on line 4 of the MD Table, and one-half of these moments are carried over to the ends B and D of members BC and CD, respectively. Joint C is then reclamped.



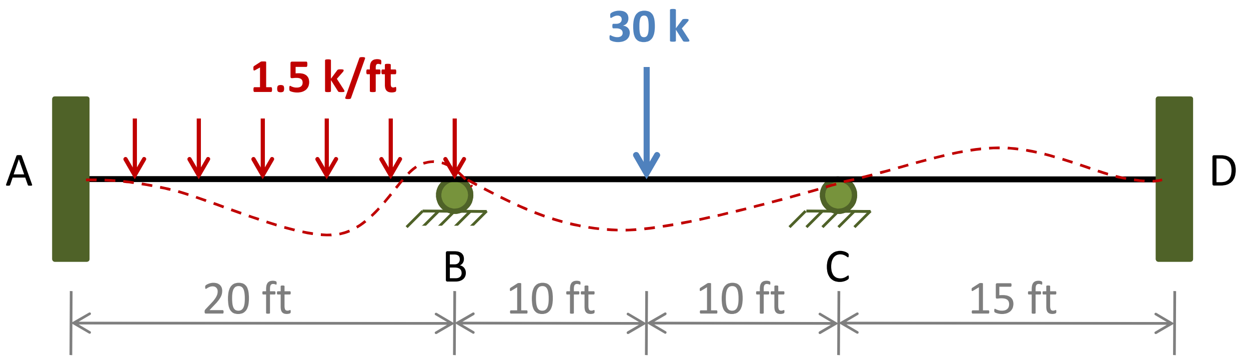
Balancing Joint *B*

The +2.2 k-ft UM at joint B (line 4) is balanced in a similar manner.

The DM and COM thus computed are shown on line 5 of the MD Table (slide 49).

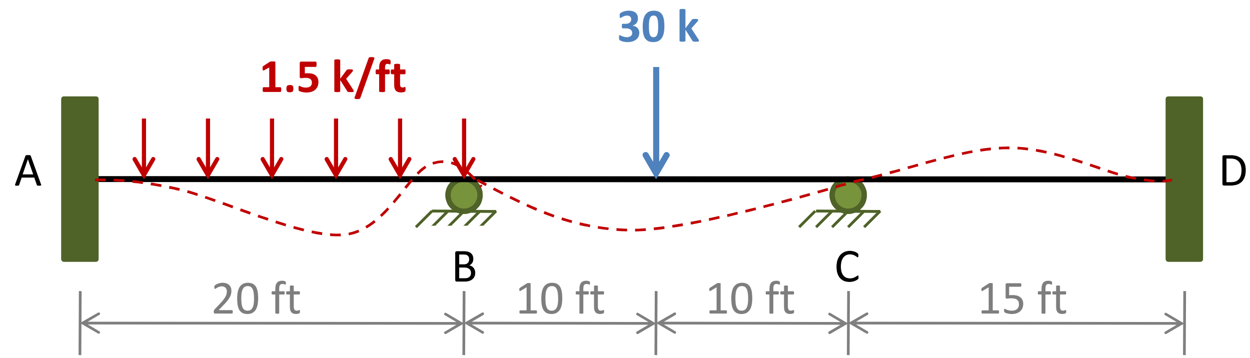
Joint B is then reclamped.

EI = constant
E = 29,000 ksi
I = 500 in^4



	A	BA	BC	CB	CD	D
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1	+32.2	+42.8	+24.1
3.Balance joint B and carryover	-10.3	-20.6	-20.6	-10.3		
4.Balance joint C and carryover			+2.2	+4.4	+5.9	+2.9

EI = constant
E = 29,000 ksi
I = 500 in^4



	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1 ←	+32.2	+42.8	→ +24.1
3.Balance joint B and carryover	-10.3 ←	-20.6	-20.6	→ -10.3		
4.Balance joint C and carryover			+2.2 ←	+4.4	+5.9	→ +2.9
5.Balance joint B and carryover	-0.6 ←	-1.1	-1.1	→ -0.6		

It can be seen from line 5 of the MD Table that the UM at joint C has now been reduced to only -0.6 k-ft.

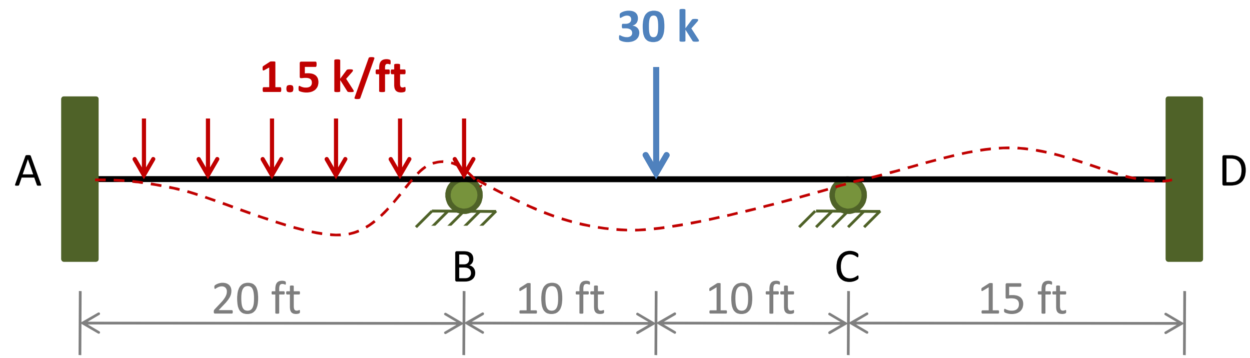
Another balancing of joint C produces an even smaller unbalanced moment of +0.2 k-ft at joint B, as shown on line 6 of the MD Table.

Since the DM induced by this unbalancing moment are negligibly small, we end the moment distribution process.

The final member end moments are obtained by algebraically summing the entries in each column of the MD Table.

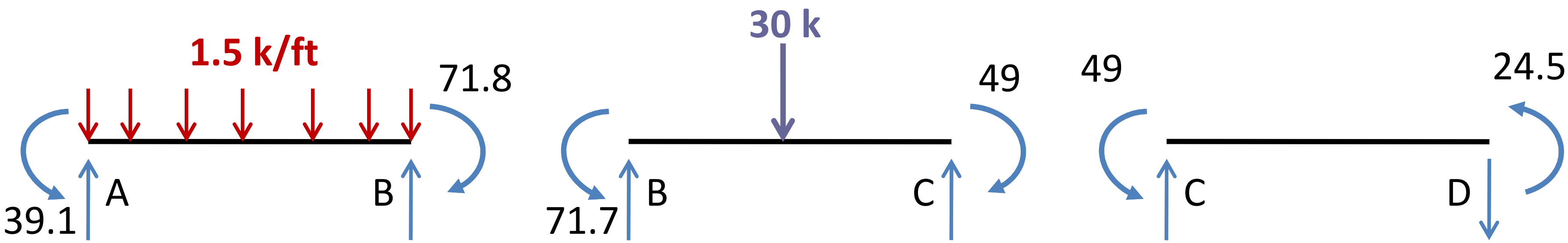
The final Moments are recorded on line 8 of The MD Table.

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1 ←	+32.2	+42.8	→ +24.1
3.Balance joint B and carryover	-10.3 ←	-20.6	-20.6	→ -10.3		
4.Balance joint C and carryover			+2.2 ←	+4.4	+5.9	→ +2.9
5.Balance joint B and carryover	-0.6 ←	-1.1	-1.1	→ -0.6		
6.Balance joint C and carryover			+0.2 ←	+0.3	+0.3	→ +0.2
7.Balance joint B		-0.1	-0.1			
8.Final Moments	+39.1	-71.8	+71.7	-49	+49	+24.5

The final moments are shown on the free body diagrams of members in Fig.



With the MEM known, member end shears and support reactions can now be determined by considering the equilibrium of members and joints.

SFD and BMD are same to those which are drawn in Slope Deflection Method for the same beam.

Practical Application of the MDM

The foregoing approach provides the clearer insight into the basic concept of the MDM.

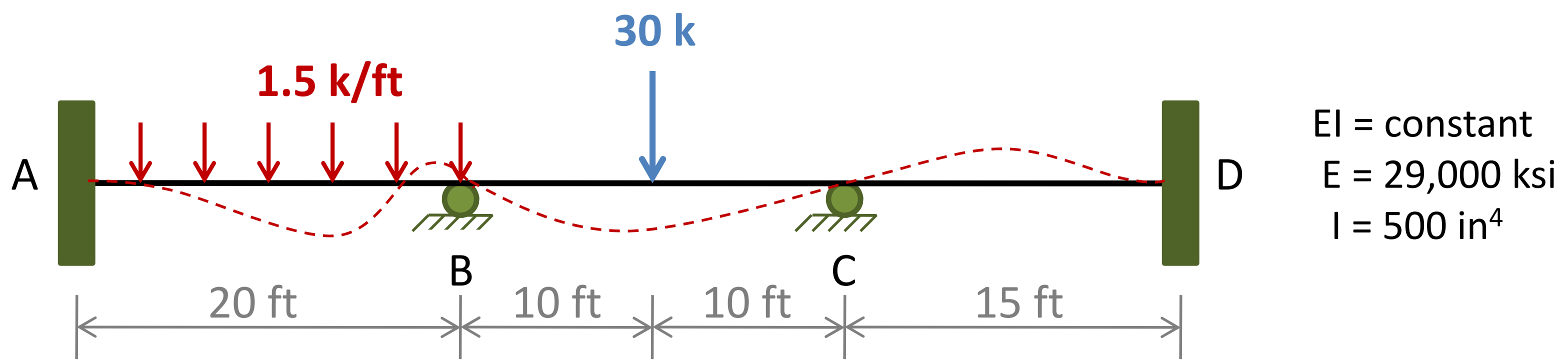
From a practical point of view, it is usually more convenient to use an alternative approach in which all the joints of the structure that are free to rotate are balanced simultaneously in the same step.

All the COMs that are induced at the far ends of the members are then computed simultaneously in the following step.

The process of balancing of joints and COMs is repeated until the UMs at the joints are negligibly small.

Practical Application of the MDM

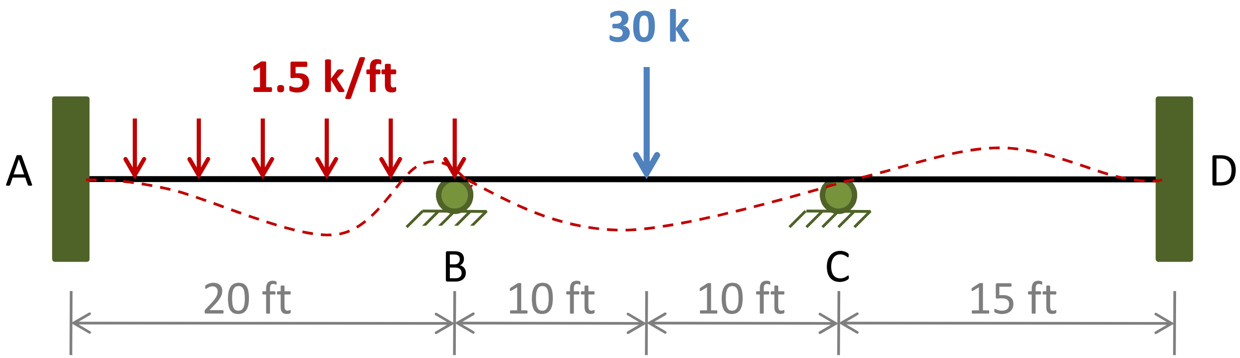
Consider again the three span continuous beam shown in figure.



The MD Table used for carrying out the computations is shown in the next slide.

The previously computed distribution factors and FEMs are recorded on the top and the first line, respectively of the table.

EI = constant
E = 29,000 ksi
I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		

The MD process is started by balancing joints B and C.

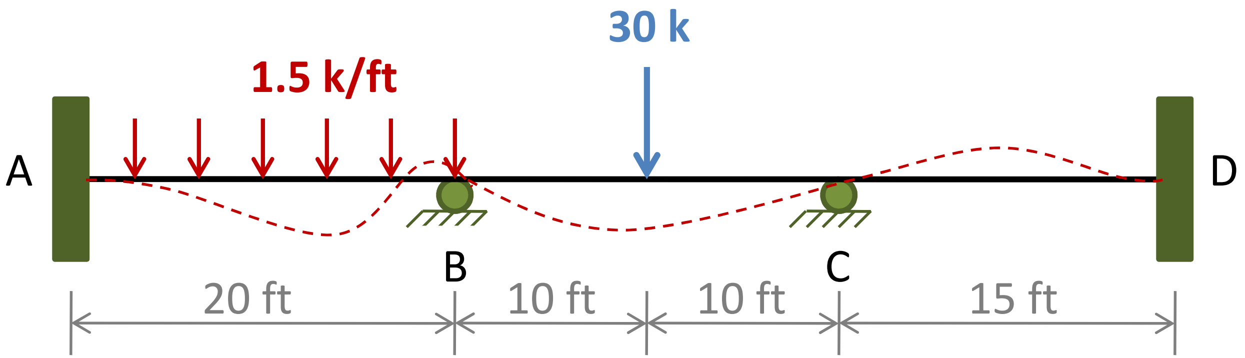
From line 1 of the MD Table we can see that the UM at joint B is

$$UM_B = -50 + 75 = +25 \text{ k}-\text{ft}$$

The balancing of joint B induces DMs at ends B of members AB and BC, which can be evaluated by multiplying the negative of the UM by the distribution factor.

$$DM_{BA} = 0.5(-25) = -12.5 \text{ k}-\text{ft}$$
$$DM_{BC} = 0.5(-25) = -12.5 \text{ k}-\text{ft}$$

EI = constant
E = 29,000 ksi
I = 500 in^4



	AB	BA	BC	CB	CD	DC
Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5			

Joint C is then balanced in a similar manner.

From line 1 of the MD Table, we can see that the UM at joint C is

$$UM_C = -75 \text{ k} - \text{ft}$$

The balancing of joint C induces the following DMs at ends C of members BC and CD, respectively

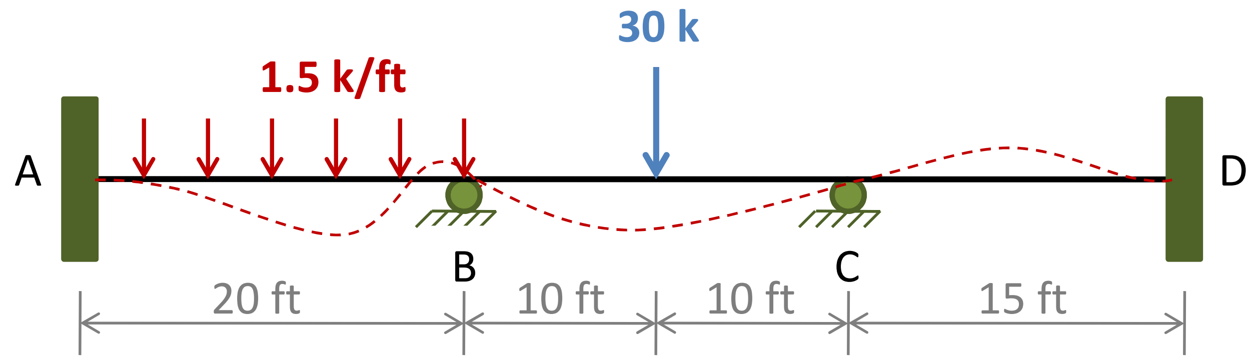
$$DM_{CB} = 0.429(+75) = +32.2 \text{ k} - \text{ft}$$
$$DM_{CD} = 0.571(+75) = +42.8 \text{ k} - \text{ft}$$

The four DMs are recorded on line 2 on the MD Table, and a line is drawn beneath them, across the entire width of the table, to indicate that all the joints are now balanced.

EI = constant

E = 29,000 ksi

I = 500 in^4



	AB	BA	BC	CB	CD	DC
Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	

In the next step of analysis, the COMs that develops at the far ends of the members are computed by multiplying the distributed moments by the COFs.

$$COM_{AB} = \frac{1}{2}\left(DM_{BA}\right) = \frac{1}{2}\left(-12.5\right) = -6.3 \text{ k} - \text{ft}$$

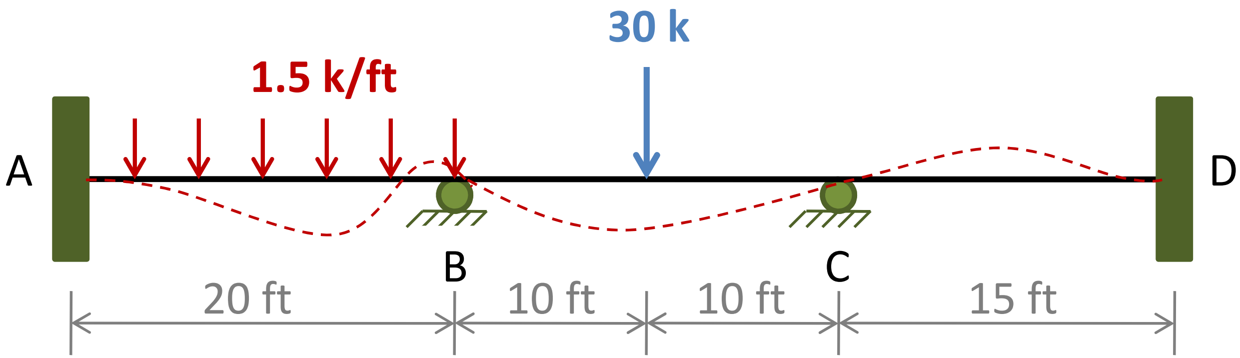
$$COM_{CB} = \frac{1}{2}\left(DM_{BC}\right) = \frac{1}{2}\left(-12.5\right) = -6.3 \text{ k} - \text{ft}$$

$$COM_{BC} = \frac{1}{2}\left(DM_{CB}\right) = \frac{1}{2}\left(+32.2\right) = +16.1 \text{ k} - \text{ft}$$

$$COM_{DC} = \frac{1}{2}\left(DM_{CD}\right) = \frac{1}{2}\left(+42.8\right) = +21.4 \text{ k} - \text{ft}$$

These COMs are recorded on the line 3 of the MD Table, with an inclined arrow pointing from each DM to its COM in the next slide.

EI = constant
E = 29,000 ksi
I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	
3.Carryover	-6.3		+16.1	-6.3		+21.4

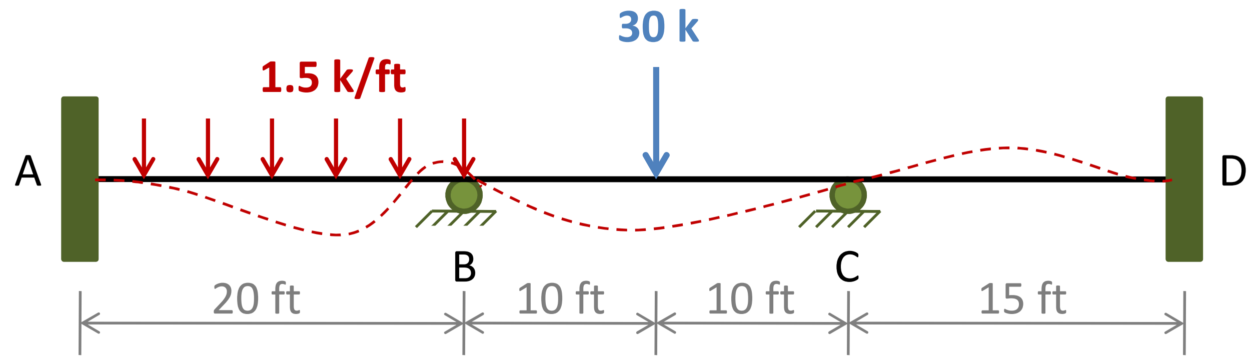
We can see from line 3 of MD Table that, due to the carryover effects, there are now +16.1 k-ft and -6.3 k-ft unbalanced moments at joints B and C, respectively.

Thus these joints are balanced again, and the DMs thus obtained are recorded on the line 4 of the MD Table.

One-half of the DMs are then carried over to the far ends of the members (line 5), and the process is continues until the UMs are negligibly small.

The final MEMs, obtained by algebraically summing the entries in each column of the MD Table, are recorded on line 11 of the table.

EI = constant
E = 29,000 ksi
I = 500 in^4

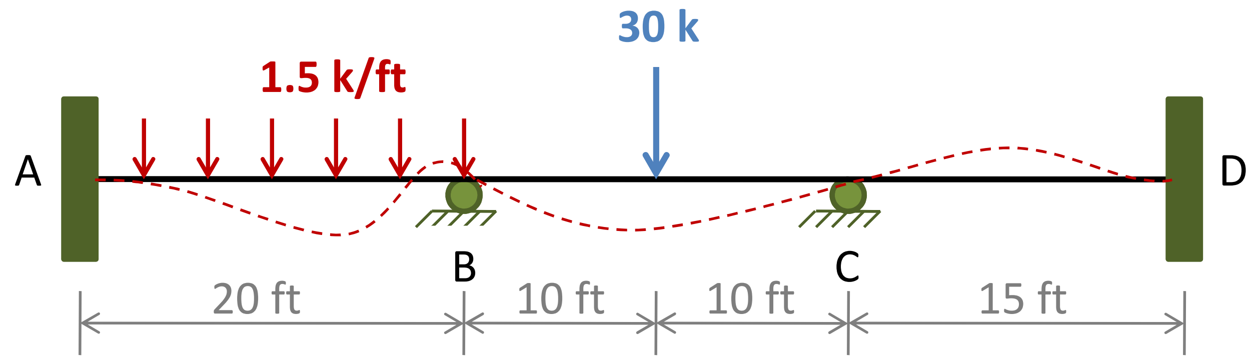


Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	
3.Carryover	-6.3		+16.1	-6.3		+21.4
4.Balance Joints		-8.1	-8.1	+2.7	+3.6	
5.Carryover	-4.1		+1.4	-4.1		+1.8
6.Balance Joints		-0.7	-0.7	+1.8	+2.3	
7.Carryover	-0.4		+0.9	-0.4		+1.2
8.Balance Joints		-0.5	-0.5	+0.2	+0.2	
9.Carryover	-0.3		+0.1	-0.3		+0.1
10.Balance Joints		-0.05	-0.05	+0.1	+0.2	
11.Final Moments	**+38.9**	**-71.8**	**+71.7**	**-49.1**	**+49.1**	**+24.5**

EI = constant

E = 29,000 ksi

I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	
3.Carryover	-6.3		+16.1	-6.3		+21.4
4.Balance Joints		-8.1	-8.1	+2.7	+3.6	
5.Carryover	-4.1		+1.4	-4.1		+1.8
6.Balance Joints		-0.7	-0.7	+1.8	+2.3	
7.Carryover	-0.4		+0.9	-0.4		+1.2
8.Balance Joints		-0.5	-0.5	+0.2	+0.2	
9.Carryover	-0.3		+0.1	-0.3		+0.1
10.Balance Joints		-0.05	-0.05	+0.1	+0.2	
11.Final Moments	**+38.9**	**-71.8**	**+71.7**	**-49.1**	**+49.1**	**+24.5**

Flow Chart for MDM

Calculate Distribution Factors, $DF = \dfrac{K}{\sum K}$

$\downarrow$

Calculate Fixed End Moments

$\downarrow$

Balance the Moments at All Joints Free to Rotate

$\downarrow$

Evaluate UMs and then Find DMs

$\downarrow$

Find Carryover Moments

$\downarrow$

Repeat the Above Two Steps Until the UMs are Negligibly Small

$\downarrow$

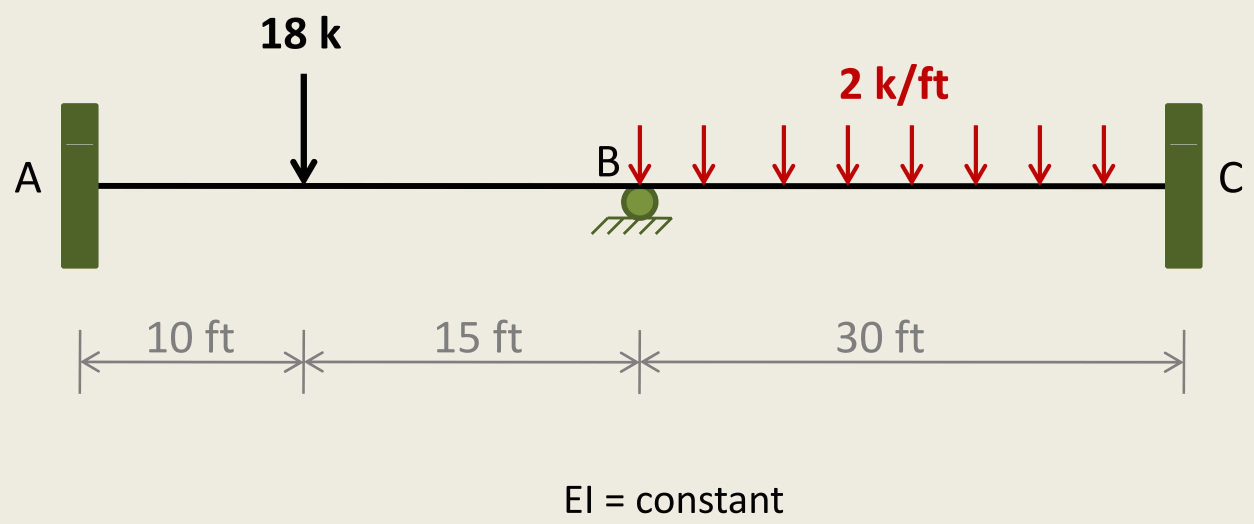
Determine the Final End Moments

$\downarrow$

Compute Member End Shears, Determine Support Reactions, and draw SFD & BMD

Example 1

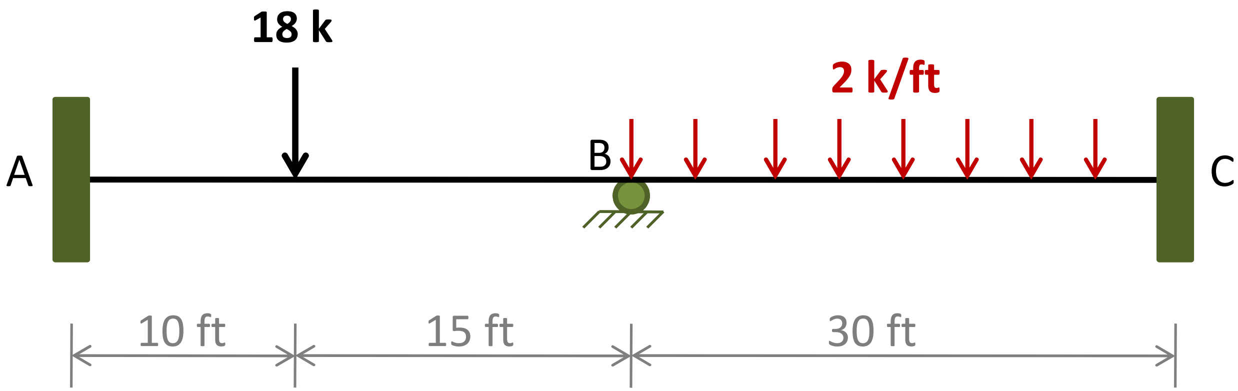
- Determine the reactions and draw the shear and bending moment diagrams for the two-span continuous beam shown in Figure.



Solution

1.Distribution Factors

Only joint B is free to rotate. The DFs at this joint are

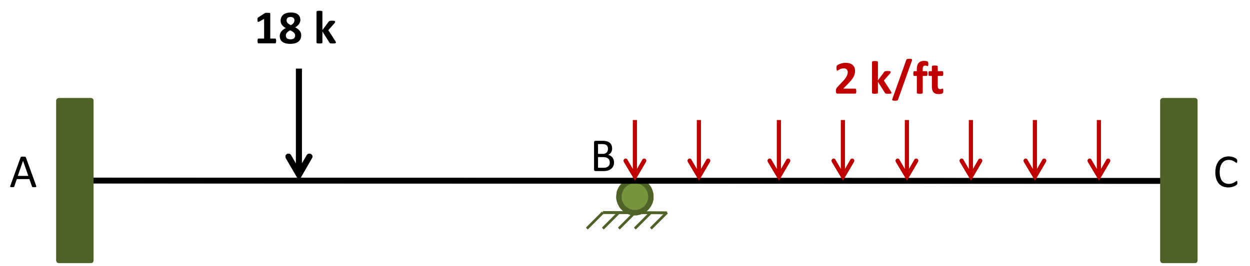


$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{I/25}{(I/25) + (I/30)} = 0.545$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{I/30}{(I/25) + (I/30)} = 0.455$$

$$DF_{BA} + DF_{BC} = 0.545 + 0.455 = 1 \qquad \text{Checks}$$

EI = constant

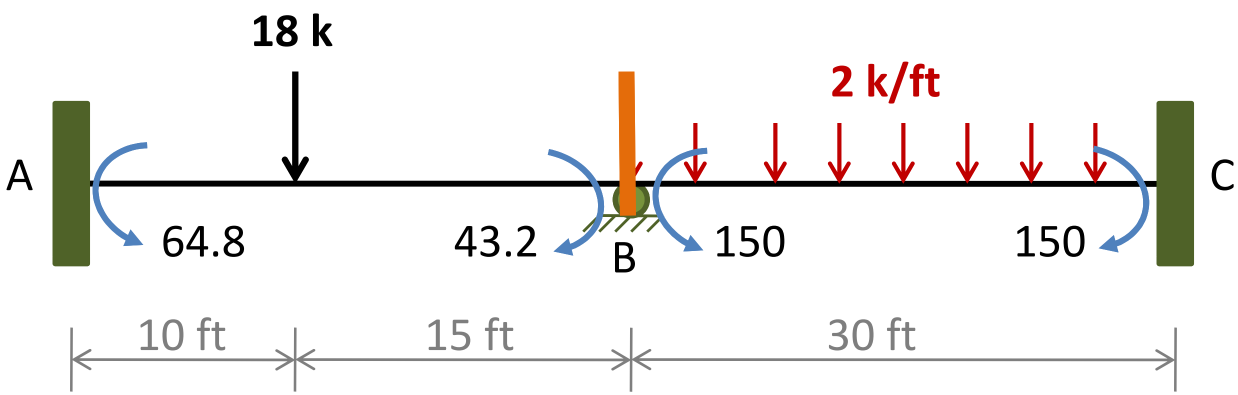


Distribution Factors

0.545	0.455

2.Fixed-End Moments (FEMs)

Assuming that joint B is clamped against rotation, we calculate the FEMs due to the external loads by using the FEM expressions



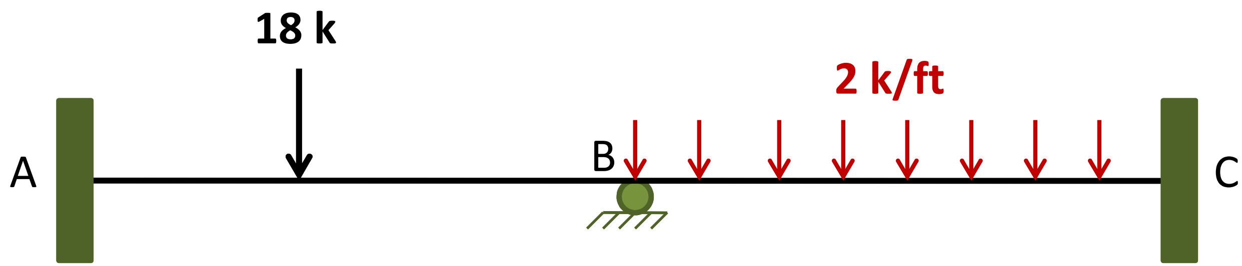
$$FEM_{AB} = \frac{18(10)(15)^2}{(25)^2} = 64.8\ \text{k}-\text{ft} \quad \text{or} \quad +64.8\ \text{k-ft}$$

$$FEM_{BA} = \frac{18(10)^2(15)}{(25)^2} = 43.2\ \text{k}-\text{ft} \quad \text{or} \quad -43.2\ \text{k-ft}$$

$$FEM_{BC} = \frac{2(30)^2}{12} = 150\ \text{k}-\text{ft} \quad \text{or} \quad +150\ \text{k-ft}$$

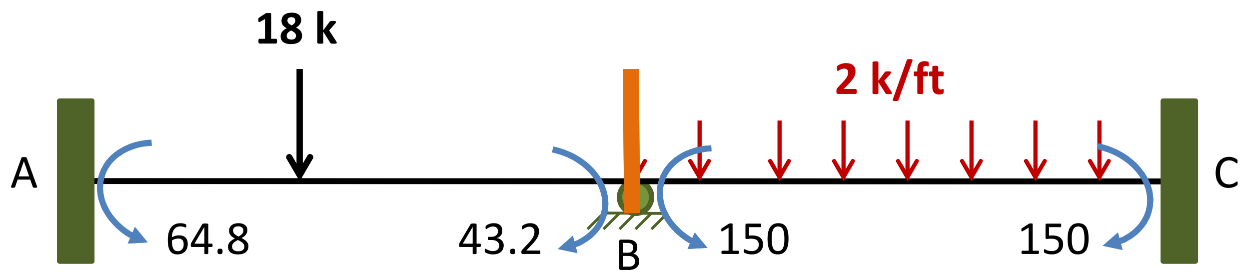
$$FEM_{CB} = \frac{2(30)^2}{12} = 150\ \text{k}-\text{ft} \quad \text{or} \quad -150\ \text{k-ft}$$

EI = constant



	AB	BA	BC	CB
Distribution Factors		0.545	0.455	
1.Fixed-end Moments	+64.8	-43.2	+150	-150

Since Joint B is actually not clamped, we release the joint and determine the unbalanced moment (UM) acting on it by summing the moments at ends B of members AB and BC



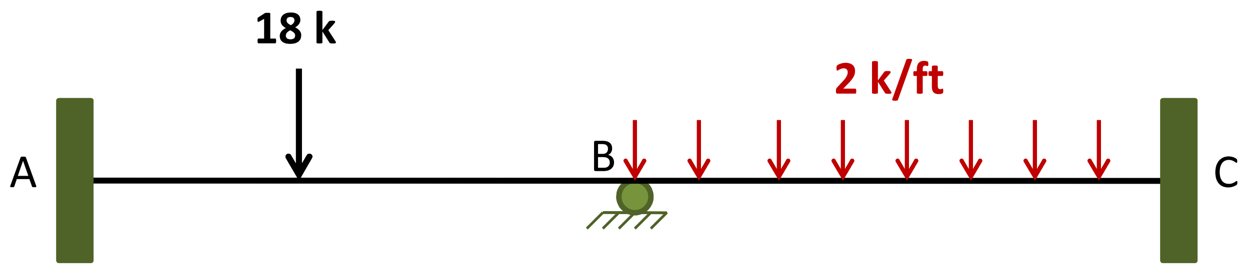
$$UM_B = -43.2 + 150 = +106.8 \ \text{k} - \text{ft}$$

The DMs due to these UMs at end B of member AB and BC are determined by multiplying the negative of the UM by the DF

$$DM_{BA} = DF_{BA}(-UM_B) = 0.545(-106.8) = -58.2 \ \text{k} - \text{ft}$$

$$DM_{BC} = DF_{BC}(-UM_B) = 0.455(-106.8) = -48.6 \ \text{k} - \text{ft}$$

EI = constant



Distribution Factors

1.Fixed-end Moments

2.Balance Joint B

AB	BA	BC	CB
	0.545	0.455	
+64.8	-43.2	+150	-150
	-58.2	-48.6	

3.Moment Distribution

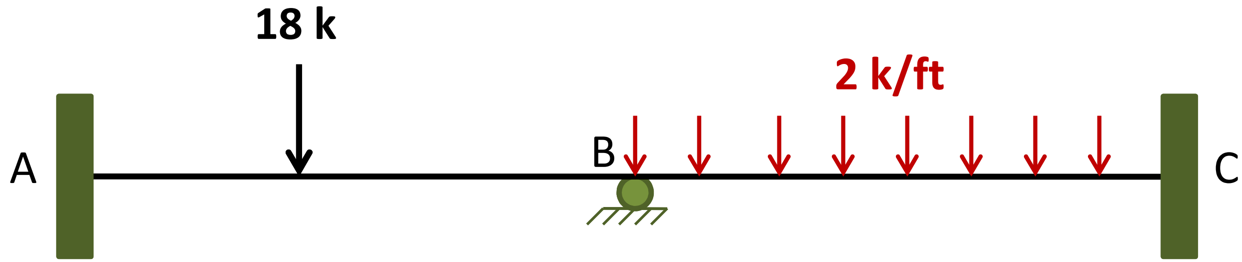
The COMs at the far ends A and C of members AB and BC, respectively, are then computed as

$$COM_{AB} = \frac{1}{2}\left(DM_{BA}\right) = \frac{1}{2}\left(-58.2\right) = -29.1 \text{ k} - \text{ft}$$

$$COM_{CB} = \frac{1}{2}\left(DM_{BC}\right) = \frac{1}{2}\left(-48.6\right) = -24.3 \text{ k} - \text{ft}$$

Joint B is the only joint of the structure that is free to rotate, and because it has been balanced, we end the moment distribution process.

EI = constant



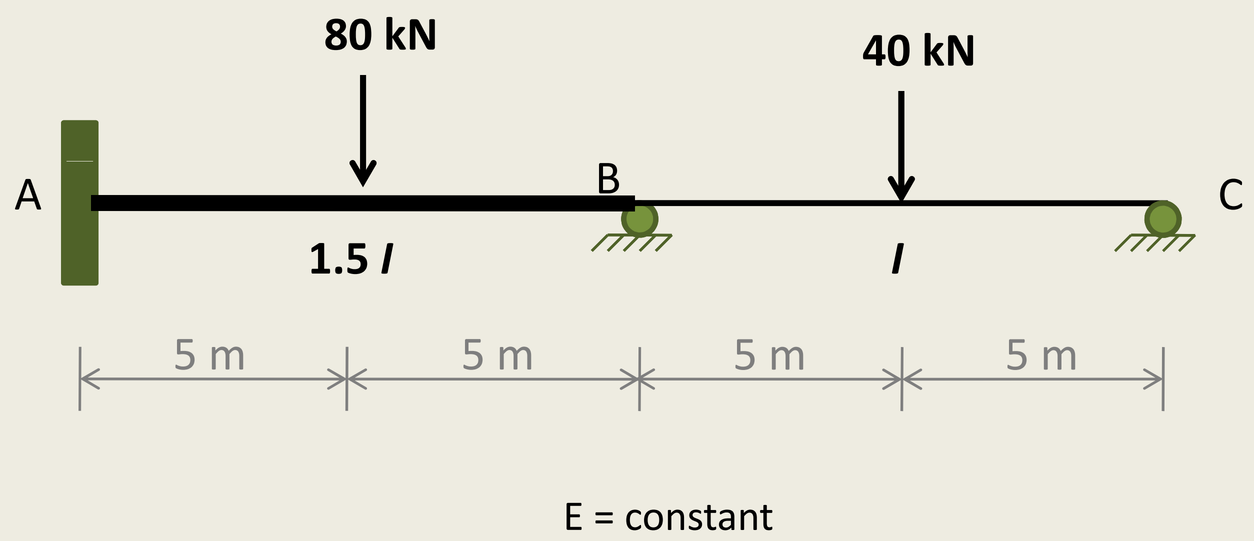
	AB	BA	BC	CB
Distribution Factors		0.545	0.455	
1.Fixed-end Moments	+64.8	-43.2	+150	-150
2.Balance Joint B		-58.2	-48.6	
3.Carryover	-29.1			-24.3
4.Final Moments	+35.7	-101.4	+101.4	-174.3

Member End Shears, Support Reactions, SFD & BMD

See Example 1 in Slope-Deflection Method

Example 2

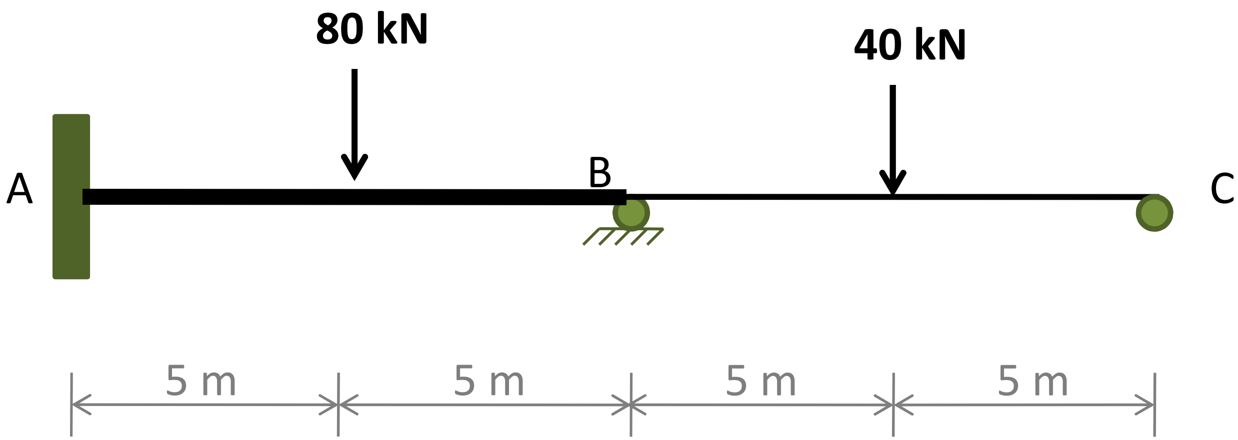
- Determine the reactions and draw the shear and bending moment diagrams for the two-span continuous beam shown in Figure.



Solution

1. Distribution Factors

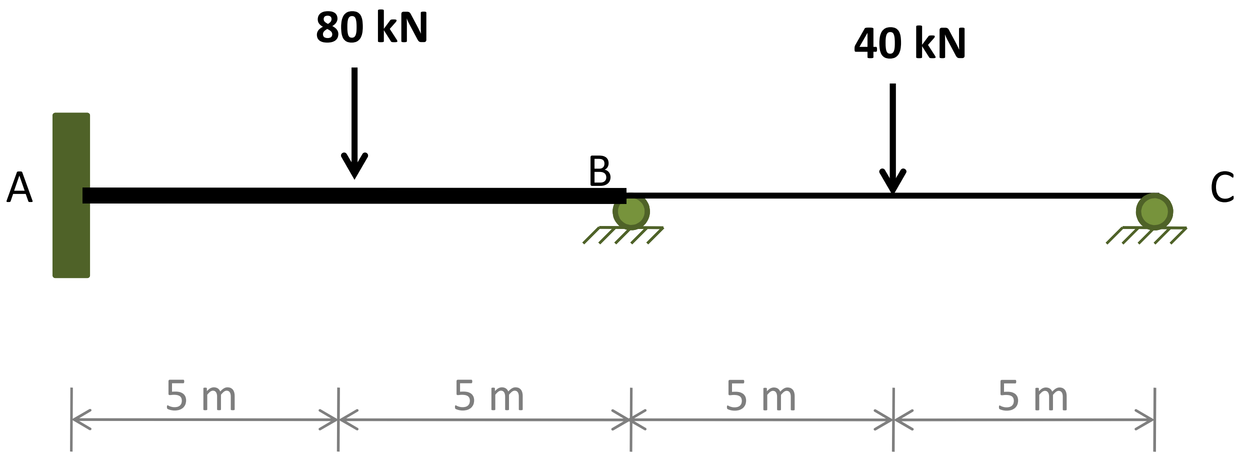
Joints B and C of the continuous beam are free to rotate. The DFs at joint B are



$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{1.5I/10}{(1.5I/10) + (I/10)} = 0.6$$

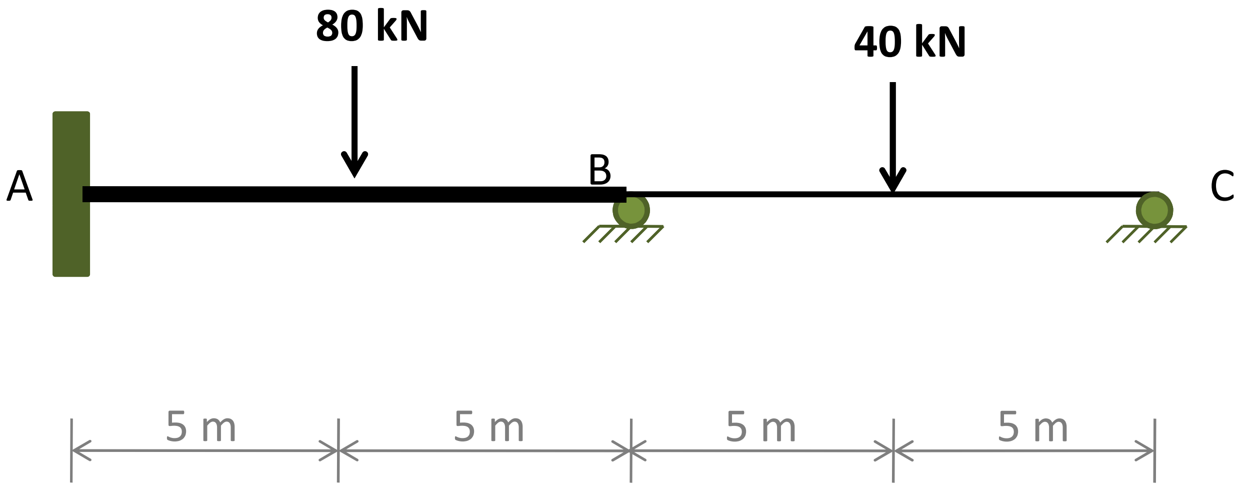
$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{I/10}{(1.5I/10) + (I/10)} = 0.4$$

Similarly, at joint C,



$$DF_{CB} = \frac{K_{CB}}{K_{CB}} = \frac{0.1I}{0.1I} = 1$$

2. Fixed-End Moments



$$FEM_{AB} = \frac{+80(10)}{8} = +100 \text{ kN.m}$$

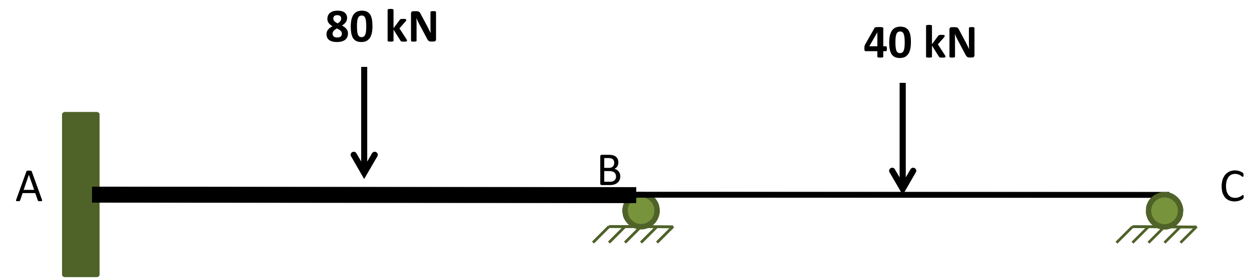
$$FEM_{BA} = -100 \text{ kN.m}$$

$$FEM_{AB} = \frac{+40(10)}{8} = +50 \text{ kN.m}$$

$$FEM_{BA} = -50 \text{ kN.m}$$

MD TABLE

E = constant



	AB	BA	BC	CB
Distribution Factors		0.6	0.4	1.0
1.Fixed-end Moments	+100	-100	+50	-50

3. Moment Distribution

After recording the DFs and the FEMs in the MD Table, we begin the MD process by balancing joints B and C.

The UM at joint B is equal to -100+50=-50 kN.m. Thus DMs at the ends B of members AB and BC are

$$DM_{BA} = DF_{BA}(-UM_B) = 0.6(+50) = +30 \ \text{kN.m}$$

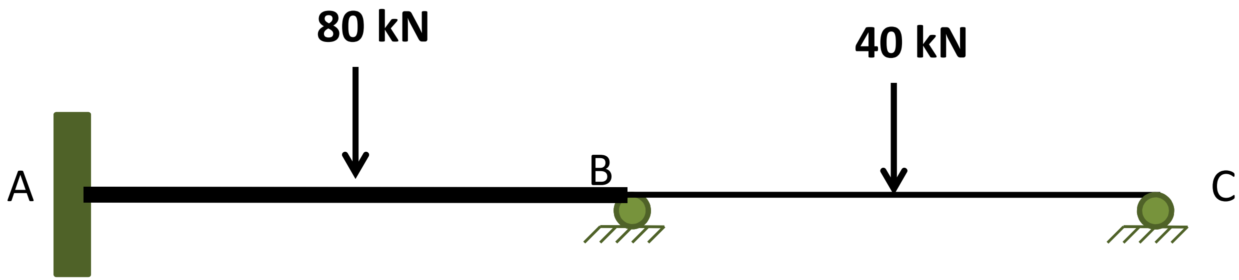
$$DM_{BC} = DF_{BC}(-UM_B) = 0.4(+50) = +20 \ \text{kN.m}$$

Similarly, the UM at joint C is -50 kN.m, we determine the DM at end C of member BC to be

$$DM_{CB} = DF_{CB}(-UM_C) = 1(+50) = +50 \ \text{kN.m}$$

MD TABLE

E = constant



Distribution Factors

1.Fixed-end Moments

2.Balance Joints B and C

	AB	BA	BC	CB
Distribution Factors		0.6	0.4	1.0
1.Fixed-end Moments	+100	-100	+50	-50
2.Balance Joints B and C		+30	+20	+50

3. Moment Distribution

One-half of these DMs are then carried over to the far ends of the members.

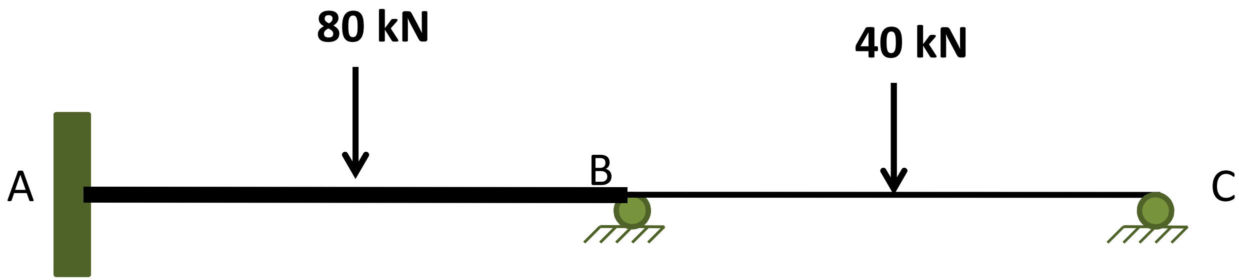
This process is repeated, until the UMs are negligibly small.

4. Final Moments

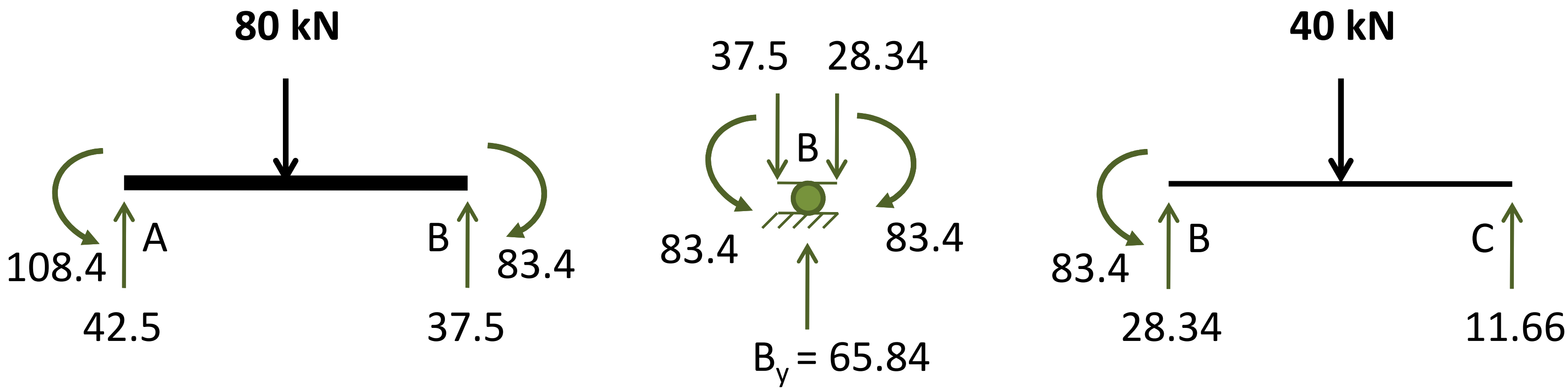
The final MEMs, obtained by summing the moments in each column of the MD Table, are recorded on the last line of the table.

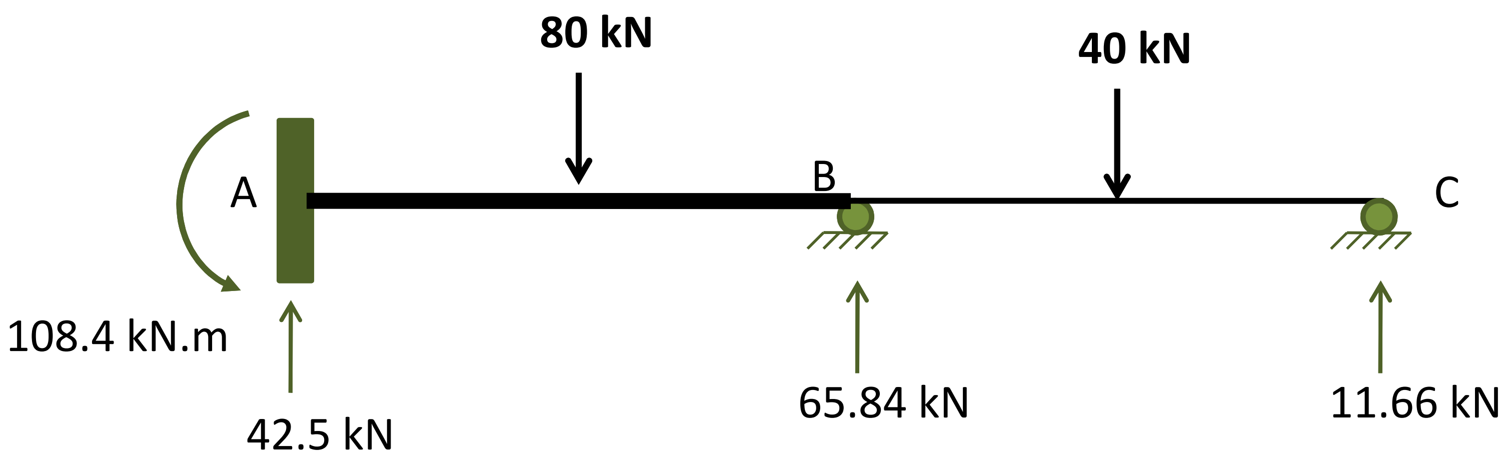
MD TABLE

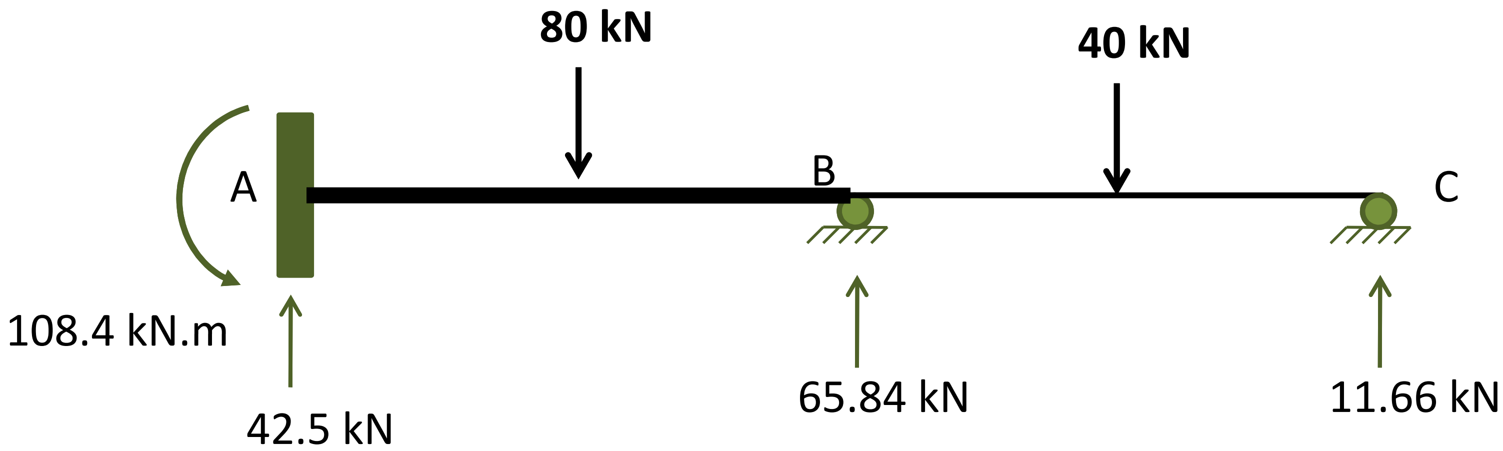
E = constant

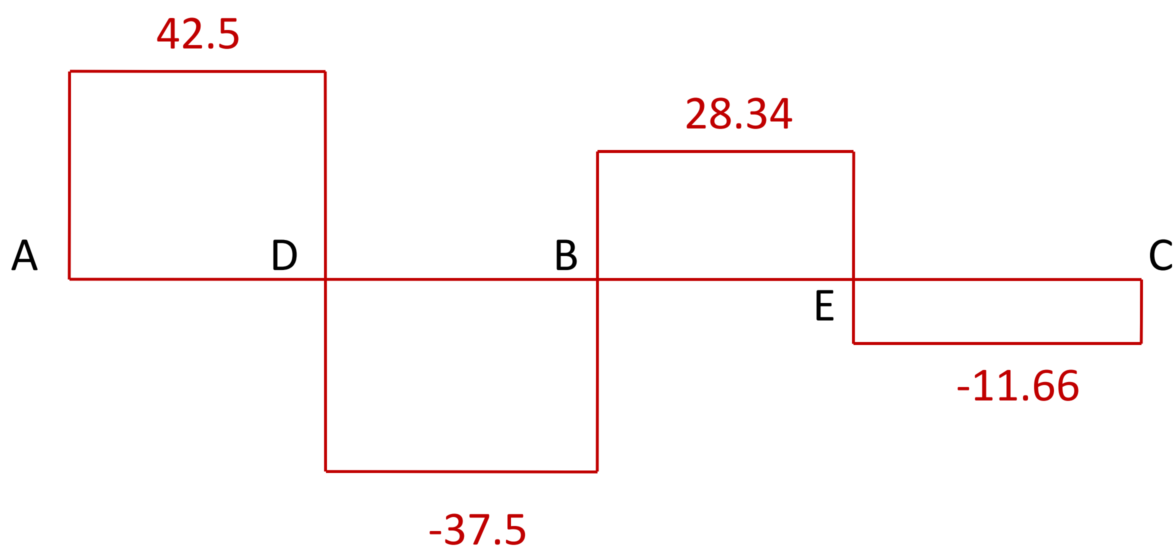


	AB	BA	BC	CB
Distribution Factors		0.6	0.4	1.0
1.Fixed-end Moments	+100	-100	+50	-50
2.Balance Joints B and C		+30	+20	+50
3.Carryover	+15		+25	+10
4.Balance Joints B and C		-15	-10	-10
5.Carryover	-7.5		-5	-5
6.Balance Joints B and C		+3	+2	+5
7.Carryover	+1.5		+2.5	+1
8.Balance Joints B and C		-1.5	-1	-1
9.Carryover	-0.8		-0.5	-0.5
10.Balance Joints B and C		+0.3	+0.2	+0.5
11.Carryover	+0.2		+0.3	+0.1
12.Balance Joints B and C		-0.2	-0.1	-0.1
13. Final Moments	**+108.4**	**-83.4**	**+83.4**	0

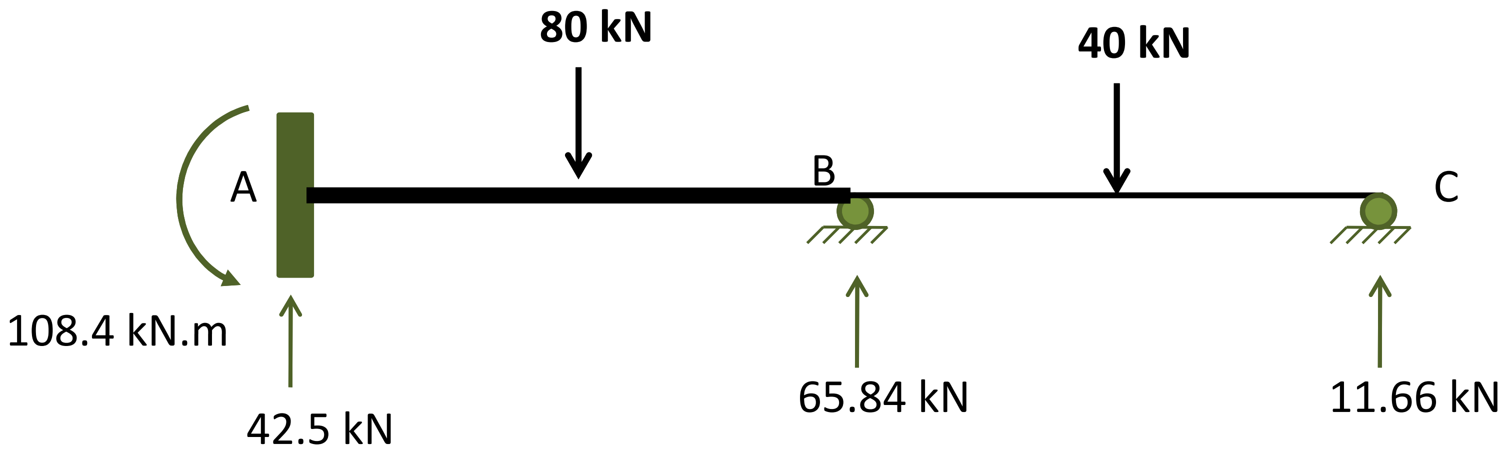

80 kN
108.4
A
42.5
B
83.4
37.5
37.5
28.34
B
83.4
83.4
B_y = 65.84
40 kN
83.4
B
C
28.34
11.66

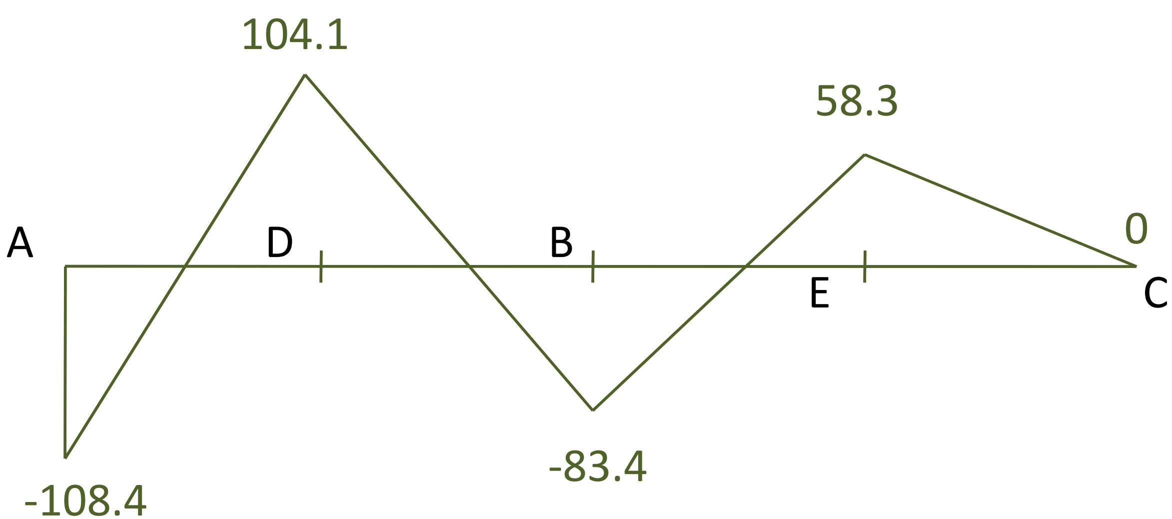

80 kN
40 kN
A
B
C
108.4 kN.m
42.5 kN
65.84 kN
11.66 kN





Shear Force Diagram (kN)

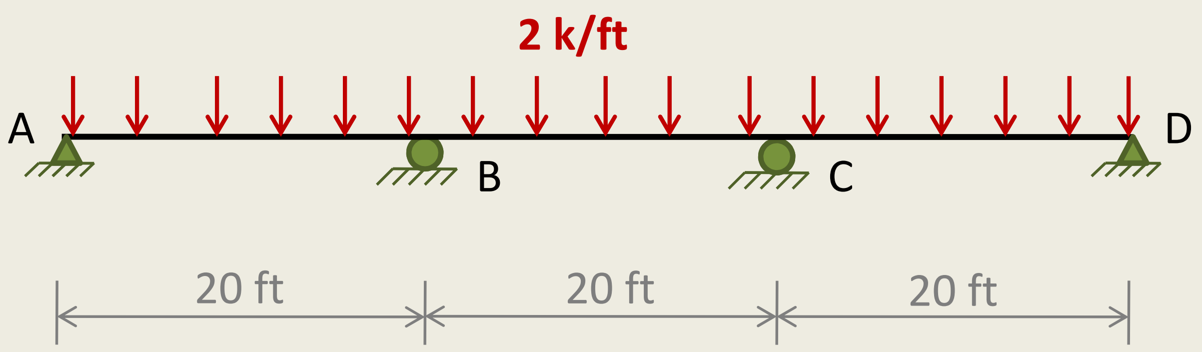




Bending Moment Diagram (kN . m)

Example 3

- Determine the member end moments and reactions for the three-span continuous beam shown, due to the uniformly distributed load and due to the support settlements of 5/8 in. at B, and 1.5 in. at C, and ¾ in. at D.

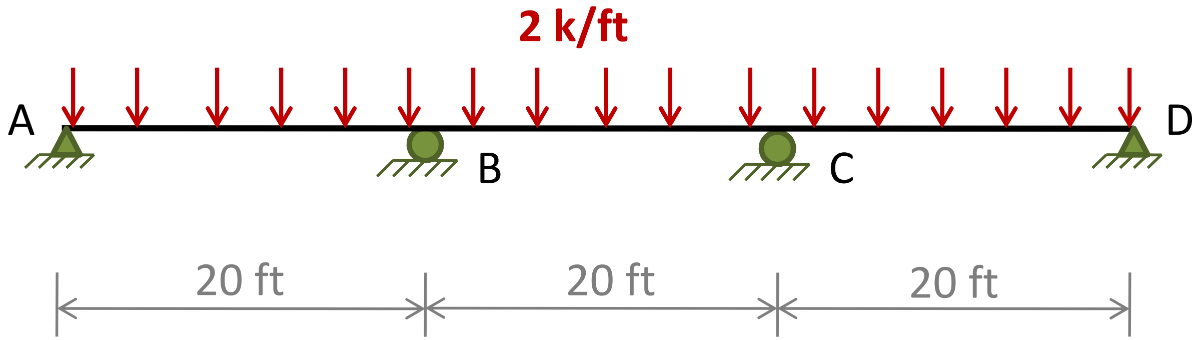


EI = 29,000 ksi

I = 7,800 in.^{4}

Solution

1. Distribution Factors



At Joint A

$$DF_{AB} = 1$$

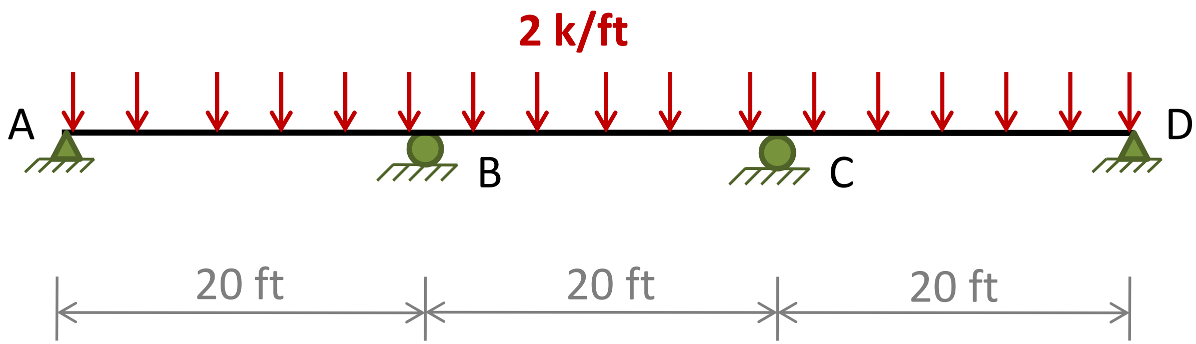
At Joint B

$$DF_{BA} = \frac{3I/80}{(3I/80)+(I/20)} = 0.429$$

$$DF_{BC} = \frac{I/20}{(3I/80)+(I/20)} = 0.571$$

Solution

1. Distribution Factors



At Joint C

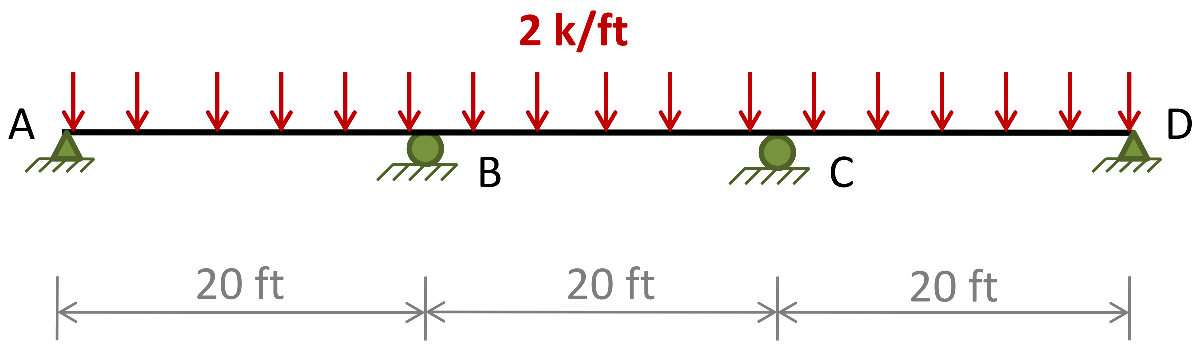
$$DF_{CB} = \frac{I/20}{(3I/80)+(I/20)} = 0.571$$

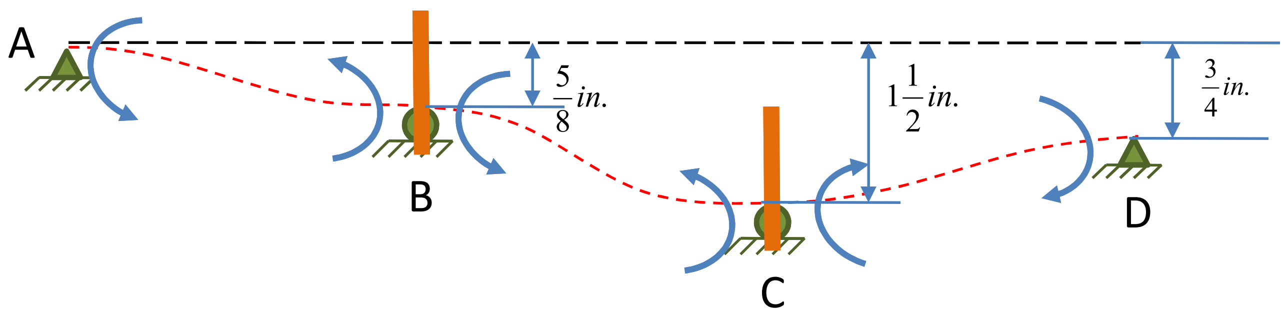
$$DF_{CD} = \frac{3I/80}{(3I/80)+(I/20)} = 0.429$$

At Joint D

$$DF_{DC} = 1$$

2. Fixed-End Moments



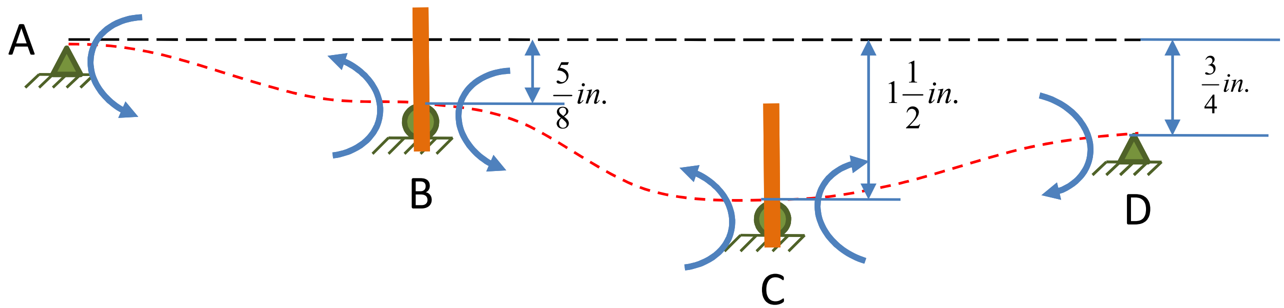


$$\Delta_{AB} = \tfrac{5}{8}\, in.$$

$$\Delta_{BC} = 1\tfrac{1}{2} - \tfrac{5}{8} = \tfrac{7}{8}\, in.$$

$$\Delta_{BC} = 1\tfrac{1}{2} - \tfrac{3}{4} = \tfrac{3}{4}\, in.$$

2. Fixed-End Moments

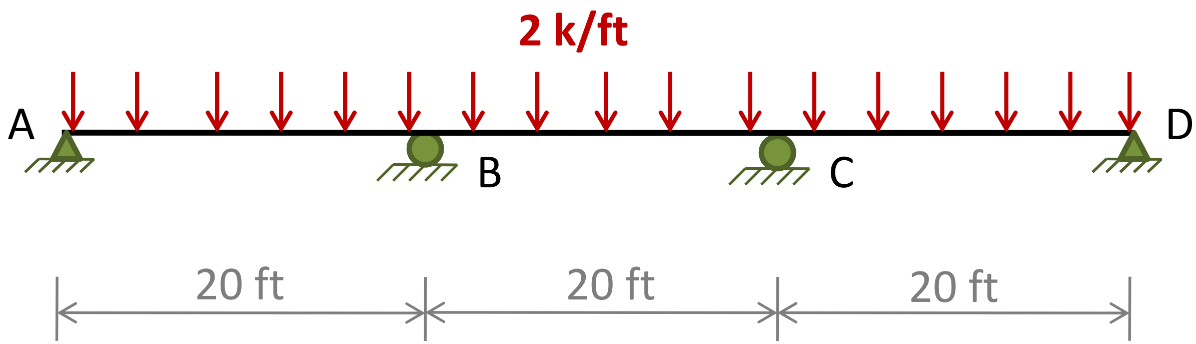


$$FEM_{AB} = FEM_{BA} = +\frac{6EI\Delta}{L^2} = +\frac{6(29{,}000)(7{,}800)\left(\frac{5}{8}\right)}{(20)^2(12)^3} = +1{,}227.2\ \text{k - ft}$$

$$FEM_{BC} = FEM_{CB} = +\frac{6EI\Delta}{L^2} = +\frac{6(29{,}000)(7{,}800)\left(\frac{7}{8}\right)}{(20)^2(12)^3} = +1{,}718.1\ \text{k - ft}$$

$$FEM_{CD} = FEM_{DC} = -\frac{6EI\Delta}{L^2} = +\frac{6(29{,}000)(7{,}800)\left(\frac{3}{4}\right)}{(20)^2(12)^3} = -1{,}472.7\ \text{k - ft}$$

2. Fixed-End Moments



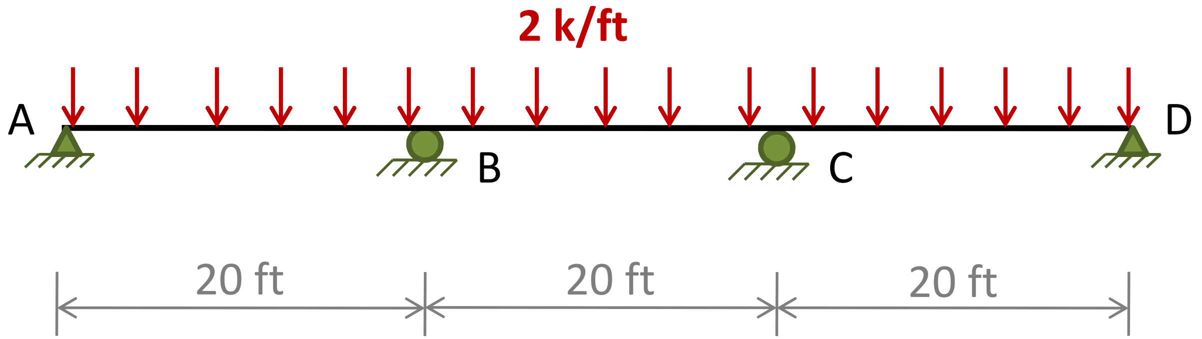
The FEMs due to the 2 k/ft external load are

$$FEM_{AB} = FEM_{BC} = FEM_{CD} = +\frac{2(20)^2}{12} = +66.7\ \text{k-ft}$$

$$FEM_{BA} = FEM_{CB} = FEM_{DC} = -\frac{2(20)^2}{12} = -66.7\ \text{k-ft}$$

Thus the FEMs due to the combined effect of the external load and the support settlements are

2. Fixed-End Moments



$$FEM_{AB} = +1{,}293.9\text{ k-ft} \qquad FEM_{BA} = +1{,}160.5\text{ k-ft}$$

$$FEM_{BC} = +1{,}784.8\text{ k-ft} \qquad FEM_{CB} = +1{,}651.4\text{ k-ft}$$

$$FEM_{CD} = -1{,}406\text{ k-ft} \qquad FEM_{DC} = -1{,}539.4\text{ k-ft}$$

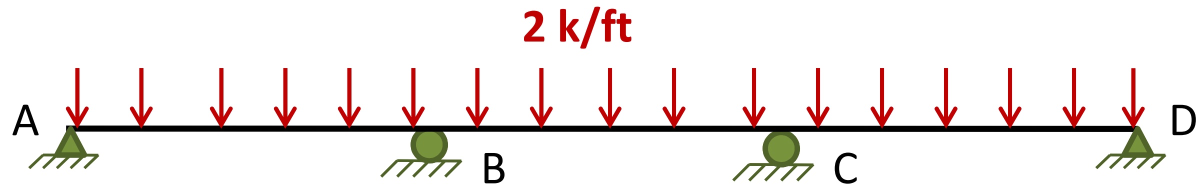
3. Moment Distribution

The MD is carried out in the usual manner, as shown in the MD Table.

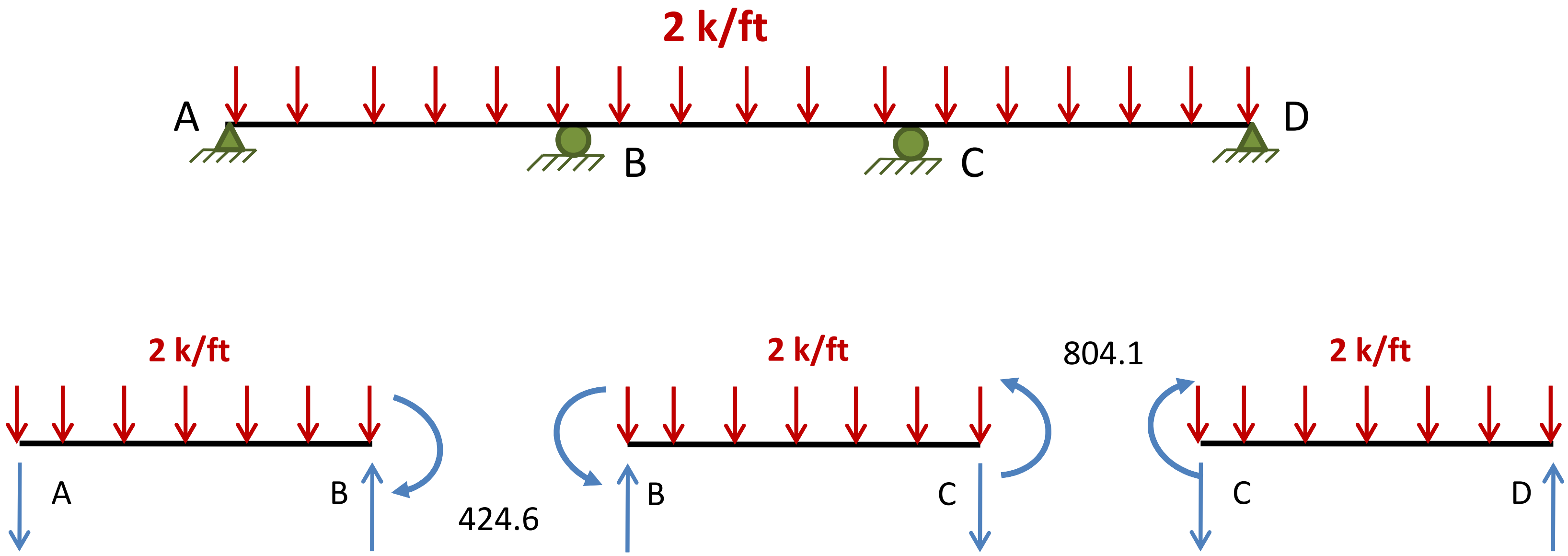
Note that the joints A and D at the simple end supports are balanced only once and that no moments are carried over to these joints.

4. Final Moments

See the MD Table and Figure on next slides.



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors	1	0.429	0.571	0.571	0.429	1
1.Fixed-end Moments	+1293.9	+1160.5	+1784.8	+1651.4	-1406	-1539.4
2.Balance Joints	-1293.9	-1263.5	-1681.8	-140.1	-105.3	+1539.4
3.Carryover		-647	-70.1	-840.9	+769.7	
4.Balance Joints		+307.6	+409.5	+40.7	+30.5	
5.Carryover			+20.4	+204.8		
6.Balance Joints		-8.8	-11.6	-116.9	-87.9	
7.Carryover			-58.5	-5.8		
8.Balance Joints		+25.1	+33.4	+3.3	+2.5	
9.Carryover			+1.7	+16.7		
10.Balance Joints		-0.7	-1.0	-9.5	-7.2	
11.Carryover			-4.8	-0.5		
12.Balance Joints		+2.1	+2.7	+0.3	+0.2	
13.Carryover			+0.2	+1.4		
14.Balance Joints		-0.1	-0.1	-0.8	-0.6	
13.Carryover			-0.4			
14.Balance Joints		+0.2	+0.2			
11.Final Moments	**0**	**-426.6**	**+426.6**	**+804.1**	**-804.1**	**0**


2 k/ft
A
B
C
D
2 k/ft
A
B
424.6
2 k/ft
B
C
804.1
2 k/ft
C
D