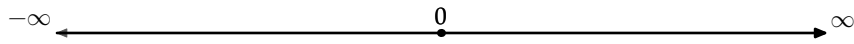Fixing this broken symmetry will make $a = b$ a natural easy case. One fully symmetric, dimensionless combination of a and b is $\log(a/b)$. This combination ranges between $-\infty$ and ∞:

$-\infty$ ⟷ 0 ∞

The special points in this range include the endpoints $-\infty$ and ∞ corresponding again to the easy cases $a = 0$ and $b = 0$ respectively; and the midpoint 0 corresponding to the third easy case $a = b$. In short, extreme cases are not the only easy cases; and easy cases also arise by equating symmetric quantities.


Problem 4.7   Other symmetric combinations

Invent other symmetric, dimensionless combinations of a and b – such as $(a - b)/(a + b)$. Investigate whether those combinations have $a = b$ as an interesting point.

4.3 Volume of a truncated pyramid

The two preceding examples – the Gaussian integral (**Section 4.1**) and the area of an ellipse (**Section 4.2**) – used easy cases to check proposed formulas, as a method of analysis. The next level of sophistication – the next level in Bloom's taxonomy [6] – is to use easy cases as a method of synthesis.

As an example, start with a pyramid with a square base and slice a piece from its top using a knife parallel to the base. This so-called frustum has a square base and square top parallel to the base. Let its height be h, the side length of the base be b, and the side length of the top be a. What is its volume?
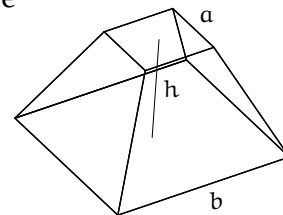
Since volume has dimensions of L^3, candidates such as ab, ab^3, and bh are impossible. However, dimensions cannot distinguish among choices with correct dimensions, such as ab^2, abh, or even a^2b^2/h. Further progress requires creating easy-cases tests.

4.3.1 Easy cases

The simplest test is $h = 0$: a pyramid with zero height and therefore zero volume. This test eliminates the candidate a^2b^2/h, which has correct dimensions but, when $h = 0$, incorrectly predicts infinite volume.

▶ *How should volume depend on height?*

Candidates that predict zero volume include products such as ha^2 or h^2a. To choose from such candidates, decide how the volume V should depend on h, the height of the solid. This decision is aided by a thought experiment. Chop the solid into vertical slivers, each like an oil-drilling core (see figure), then vary h. For example, doubling h doubles the height and volume of each sliver; therefore doubling h doubles V. The same thought experiment shows that tripling h triples V. In short: $V \propto h$, which rules out h^2a as a possible volume.
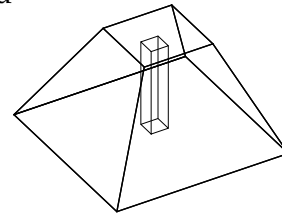
> **Problem 4.8   Another reason that $V \propto h$**
> Use the easy case of $a = b$ to argue that $V \propto h$.

The constraint $V \propto h$, together with the requirement of dimensional correctness, means that

$$V = h \times \text{function of } a \text{ and } b \text{ with dimensions of } L^2.$$

Further easy-cases tests help synthesize that function.

▶ *What are other easy cases?*

A second easy case is the extreme case $a = 0$, where the top surface shrinks to a point. The symmetry between a and b suggests the extreme case $b = 0$ as another easy case. The symmetry also suggests $a = b$ as a non-extreme easy case. Let's apply the three new tests in turn, developing formulas to synthesize a candidate that passes all the tests.

$\boxed{a = 0}$. In this extreme case, the truncated pyramid becomes an ordinary pyramid with height h and square base of side length b. So that its volume has dimensions of L^3 and is proportional to h, its volume must have the form $V \sim hb^2$. This form, which does not specify the dimensionless constant, stands for a family of candidates. Perhaps this family contains the correct volume for the truncated pyramid? Each family member passes the $a = 0$ test, by construction. How do they fare with the other easy cases?
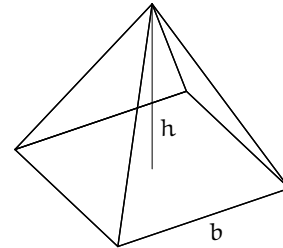
$\boxed{b = 0}$. In this extreme case, the truncated pyramid becomes an upside-down but otherwise ordinary pyramid. All the candidates $V \sim hb^2$ predict zero volume when $b = 0$, so all fail the $b = 0$ test. The symmetric alternatives $V \sim ha^2$ pass the $b = 0$ test; unfortunately, they fail the $a = 0$ test. Are we stuck?
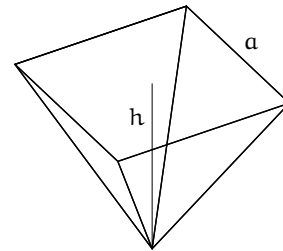
▶ *Invent a candidate that passes the $a = 0$ and $b = 0$ tests.*

To a family of candidates that pass the $a = 0$ and $b = 0$ tests, add the two families that pass each test to get $V \sim ha^2 + hb^2$ or

$$V \sim h(a^2 + b^2).$$

Two other families of candidates that pass both tests include

$$V \sim h(a + b)^2.$$

and

$$V \sim h(a - b)^2.$$

Choosing among them requires the last easy case: $a = b$.

$\boxed{a = b}$. When $a = b$ – the easiest of the last three cases – the truncated pyramid becomes a rectangular prism with height h, base area b^2 (or a^2), and volume hb^2. When $a = b$, the family of candidates $V \sim h(a^2 + b^2)$ predicts hb^2 when the dimensionless constant is $1/2$. So
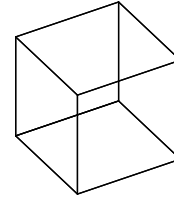
$$V = \frac{1}{2}h(a^2 + b^2)$$

passes the $a = b$ test. When $a = b$, the family of candidates $V \sim h(a+b)^2$ predicts the correct volume when the dimensionless constant is $1/4$. So

$$V = \frac{1}{4}h(a + b)^2$$

passes the $a = b$ test. However, the family of candidate $V \sim h(a - b)^2$ predict zero volume when $a = b$, so they all fail the $a = b$ test.

To decide between the survivors $V = h(a + b)^2/2$ and $V = h(a + b)^2/4$, return to the easy case $a = 0$. In that extreme case, the survivors predict $V = hb^2/2$ and $V = hb^2/4$, respectively. These predictions differ in the dimensionless constant. If we could somehow guess the correct dimensionless constant in the volume of an ordinary pyramid, we can decide between the surviving candidates for the volume of the truncated pyramid.

▶ *What is the dimensionless constant?*

### 4.3.2 Finding the dimensionless constant

Finding the dimensionless constant looks like a calculus problem: Slice an ordinary pyramid into thin horizontal sections, then add (integrate) their volumes. A simple but surprising alternative is the method of easy cases – surprising because easy cases only rarely determine a dimensionless constant.

The method is best created with an analogy: Rather than guessing the dimensionless constant in the volume of a pyramid, solve a similar but simpler problem – a method discussed in detail in **Chapter 8**. In this case, let's invent a two-dimensional shape and find the dimensionless constant in its area.

An analogous shape is a triangle with base b and height h. What is the dimensionless constant in its $A \sim bh$? To reap the full benefit of the analogy, answer that question using easy cases rather than calculus. So, choose b and h to make an easy triangle. The easiest triangle is a 45° right triangle with h = b. Two of these triangles form an easy shape – a square with area b^2 – so the area of one triangle is $A = b^2/2$ when h = b. Therefore, the dimensionless constant is $1/2$ and $A = bh/2$.
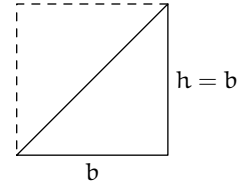
Now extend this reasoning to our three-dimensional solid: What square-based pyramid, combined with itself a few times, makes an easy solid? Choosing a pyramid means choosing its base length b and its height h. However, only the aspect ratio h/b matters in the analysis – as in the triangle example. So, our procedure will be to choose a goal solid and then to choose a convenient b and h so that such pyramids combine to form the goal shape.

▶ *What is a convenient goal solid?*

A convenient goal solid is suggested by the square pyramid base – perhaps one face of a cube? If so, the cube would be formed from six pyramids. To choose their b and h, imagine how the pyramids fit into a cube. With the base of each pyramid forming one face of the cube, the tips of the pyramids point inward and meet in the center of the cube. To make the tips meet, the pyramid's height must be b/2, and six of those pyramids make a cube of side length b and volume b^3.
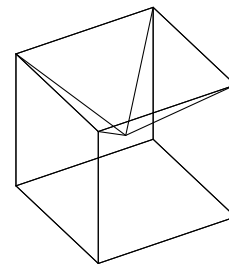
To keep b and h integers, choose b = 2 and h = 1. Then six pyramids make a cube with volume 8, and the volume of one pyramid is 4/3. Easy cases have now excavated sufficient information to determine the dimensionless constant: Since the volume of the pyramid is $V \sim hb^2$ and since $hb^2 = 4$ for these pyramids, the missing constant must be 1/3. Therefore

$$V = \frac{1}{3}hb^2.$$

Problem 4.9   Vertex location

The six pyramids do not make a cube unless each pyramid's top vertex is directly above the center of the base. So the result $V = hb^2/3$ might apply only in that special case. If instead the top vertex is above one of the base vertices, what is the volume?
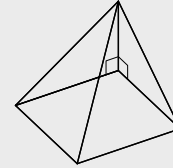
Problem 4.10   Triangular base

Guess the volume of a pyramid with a triangular base.

4.3.3   Using the magic factor of one-third

The purpose of the preceding easy-cases analysis for an ordinary pyramid was to decide between two candidates for the volume of a truncated pyramid: $V = h(a^2 + b^2)/2$ and $V = h(a + b)^2/4$. Unfortunately, neither candidate predicts the correct volume $V = hb^2/3$ for an ordinary pyramid ($a = 0$). Oh, no!

We need new candidates. One way to generate them is first to rewrite the two families of candidates that passed the $a = 0$ and $b = 0$ tests:

$$V \sim a^2 + b^2 = a^2 + b^2,$$
$$V \sim (a + b)^2 = a^2 + 2ab + b^2,$$
$$V \sim (a - b)^2 = a^2 - 2ab + b^2.$$

The expanded versions on the right have identical a^2 and b^2 terms but differ in the ab term. This variation suggests an idea: that by choosing the coefficient of ab, the volume might pass all easy-cases tests. Hence the following three-part divide-and-conquer procedure:

1. Choose the coefficient of a^2 to pass the $b = 0$ test.

2. Choose the coefficient of b^2 to pass the $a = 0$ test. Choosing this coefficient will not prejudice the already passed $b = 0$ test, because when $b = 0$ the b^2 term vanishes

3. Finally, choose the coefficient of ab to pass the $a = b$ test. Choosing this coefficient will not prejudice the already passed $b = 0$ and $a = 0$ tests, because in either case ab vanishes.

The result is a volume that passes the three easy-cases tests: $a = 0$, $b = 0$, and $a = b$.

To pass the $b = 0$ test, the coefficient of a^2 must be $1/3$ – the result of combining six pyramids into a cube. Similarly, to pass the $a = 0$ test, the coefficient of b^2 must also be $1/3$. The resulting family of candidates is:

$$V = \frac{1}{3}h(a^2 + nab + b^2).$$

Among this family, one must pass the $a = b$ test. When $a = b$, the candidates predict

$$V = \frac{2+n}{3}hb^2.$$

When $a = b$, the truncated pyramid becomes a rectangular prism with volume hb^2, so the coefficient $(2 + n)/3$ should be 1. Therefore $n = 1$, and the volume of the truncated pyramid is

$$V = \frac{1}{3}h(a^2 + ab + b^2).$$

Problem 4.11   Integration
Use integration to check that $V = h(a^2 + ab + b^2)/3$.

Problem 4.12   Triangular pyramid
Instead of a pyramid with a square base, start with a pyramid with an equilateral triangle of side length b as its base. Then make the usual truncated pyramid by slicing a piece off the top using a plane parallel to the base. If the top is an equilateral triangle of side length a, and the height is h, what is the volume of this truncated pyramid? (See also **Problem 4.10**.)

Problem 4.13   Truncated cone
What is the volume of a truncated cone with a circular base of radius r_1 and circular top of radius r_2 (with the top parallel to the base)?

4.4 Drag: Using dimensions and easy cases

Equations often exceed the capabilities of known mathematics, whereupon easy cases and other street-fighting tools are among the few ways